

# Journal of Inequalities in Pure and Applied Mathematics

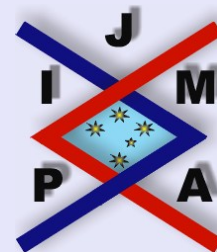
## ON NEIGHBORHOODS OF ANALYTIC FUNCTIONS HAVING POSITIVE REAL PART

SHIGEYOSHI OWA, NIGAR YILDIRIM AND MUHAMMET KAMALI

Department of Mathematics  
Kinki University  
Higashi-Osaka, Osaka 577-8502, Japan.  
*EMail:* [owa@math.kindai.ac.jp](mailto:owa@math.kindai.ac.jp)

Kafkas Üniversitesi, Fen-Edebiyat Fakültesi  
Matematik Bölümü,  
Kars, Turkey.

Atatürk Üniversitesi, Fen-Edebiyat Fakültesi,  
Matematik Bölümü,  
25240 Erzurum Turkey.  
*EMail:* [mkamali@atauni.edu.tr](mailto:mkamali@atauni.edu.tr)



---

volume 7, issue 3, article 109,  
2006.

*Received 10 November, 2005;  
accepted 15 July, 2006.*

*Communicated by: G. Kohr*

---

[Abstract](#)

[Contents](#)



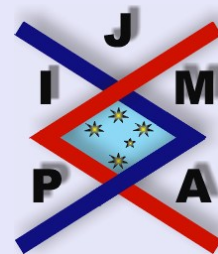
[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)





## Abstract

Two subclasses  $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$  and  $\mathcal{P}'\left(\frac{\alpha-m}{n}\right)$  of certain analytic functions having positive real part in the open unit disk  $\mathbb{U}$  are introduced. In the present paper, several properties of the subclass  $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$  of analytic functions with real part greater than  $\frac{\alpha-m}{n}$  are derived. For  $p(z) \in \mathcal{P}\left(\frac{\alpha-m}{n}\right)$  and  $\delta \geq 0$ , the  $\delta$ -neighborhood  $\mathcal{N}_\delta(p(z))$  of  $p(z)$  is defined. For  $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$ ,  $\mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ , and  $\mathcal{N}_\delta(p(z))$ , we prove that if  $p(z) \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ , then  $\mathcal{N}_{\beta\delta}(p(z)) \subset \mathcal{P}\left(\frac{\alpha-m}{n}\right)$ .

*2000 Mathematics Subject Classification:* Primary 30C45.

*Key words:* Function with positive real part, subordinate function,  $\delta$ -neighborhood, convolution (Hadamard product).

## Contents

1	Introduction .....	3
2	Some Inequalities for the Class $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$ .....	4
3	Preliminary Results .....	9
4	Main Results .....	12
	References	

**On Neighborhoods of Analytic Functions having Positive Real Part**

Shigeyoshi Owa, Nigar Yildirim  
and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 2 of 20

# 1. Introduction

Let  $\mathcal{T}$  be the class of functions of the form

$$(1.1) \quad p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

which are analytic in the open unit disk  $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ . A function  $p(z) \in \mathcal{T}$  is said to be in the class  $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$  if it satisfies

$$\operatorname{Re}\{p(z)\} > \frac{\alpha - m}{n} \quad (z \in \mathbb{U})$$

for some  $m \leq \alpha < m + n$ ,  $m \in \mathbb{N}_0 = 0, 1, 2, 3, \dots$ , and  $n \in \mathbb{N} = 1, 2, 3, \dots$ . For any  $p(z) \in \mathcal{P}\left(\frac{\alpha-m}{n}\right)$  and  $\delta \geq 0$ , we define the  $\delta$ -neighborhood  $\mathcal{N}_\delta(p(z))$  of  $p(z)$  by

$$\mathcal{N}_\delta(p(z)) = \left\{ q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k \in \mathcal{T} : \sum_{k=1}^{\infty} |p_k - q_k| \leq \delta \right\}.$$

The concept of  $\delta$ -neighborhoods  $\mathcal{N}_\delta(f(z))$  of analytic functions  $f(z)$  in  $\mathbb{U}$  with  $f(0) = f'(0) - 1 = 0$  was first introduced by Ruscheweyh [12] and was studied by Fournier [4, 6] and by Brown [2]. Walker has studied the  $\delta_1$ -neighborhood  $\mathcal{N}_{\delta_1}(p(z))$  of  $p(z) \in \mathcal{P}_1(0)$  [13]. Later, Owa et al. [9] extended the result by Walker.

In this paper, we give some inequalities for the class  $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$ . Furthermore, we define a neighborhood of  $p(z) \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$  and determine  $\delta > 0$  so that  $\mathcal{N}_{\beta\delta}(p(z)) \subset \mathcal{P}\left(\frac{\alpha-m}{n}\right)$ , where  $\beta = \frac{m+n-\alpha}{n}$ .



---

On Neighborhoods of Analytic Functions having Positive Real Part

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

---

Title Page

Contents



Go Back

Close

Quit

Page 3 of 20

## 2. Some Inequalities for the Class $\mathcal{P} \left( \frac{\alpha-m}{n} \right)$

Our first result for functions  $p(z)$  in  $\mathcal{P} \left( \frac{\alpha-m}{n} \right)$  is contained in

**Theorem 2.1.** *Let  $p(z) \in \mathcal{P} \left( \frac{\alpha-m}{n} \right)$ . Then, for  $|z| = r < 1, m \leq \alpha < m + n, m \in \mathbb{N}_0$  and  $n \in \mathbb{N}$ ,*

$$(2.1) \quad |zp'(z)| \leq \frac{2r}{1-r^2} \operatorname{Re} \left\{ p(z) - \frac{\alpha-m}{n} \right\}.$$

For each  $m \leq \alpha < m + n$ , the equality is attained at  $z = r$  for the function

$$p(z) = \frac{\alpha-m}{n} + \left( 1 - \frac{\alpha-m}{n} \right) \frac{1-z}{1+z} = 1 - \frac{2}{n} (n - \alpha + m) z + \dots$$

*Proof.* Let us consider the case of  $p(z) \in \mathcal{P}(0)$ . Then the function  $k(z)$  defined by

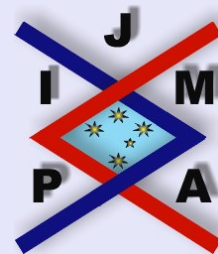
$$k(z) = \frac{1-p(z)}{1+p(z)} = \eta_1 z + \eta_2 z^2 + \dots$$

is analytic in  $\mathbb{U}$  and  $|k(z)| < 1$  ( $z \in \mathbb{U}$ ). Hence  $k(z) = z\Phi(z)$ , where  $\Phi(z)$  is analytic in  $\mathbb{U}$  and  $|\Phi(z)| \leq 1$  ( $z \in \mathbb{U}$ ). For such a function  $\Phi(z)$ , we have

$$(2.2) \quad |\Phi'(z)| \leq \frac{(1 - |\Phi(z)|^2)}{(1 - |z|^2)} \quad (z \in \mathbb{U}).$$

From  $z\Phi(z) = \frac{1-p(z)}{1+p(z)}$ , we obtain

$$(i) \quad |\Phi(z)|^2 = \frac{1}{r^2} \left| \frac{1-p(z)}{1+p(z)} \right|^2,$$



On Neighborhoods of Analytic Functions having Positive Real Part

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

Title Page

Contents

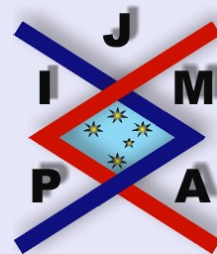


Go Back

Close

Quit

Page 4 of 20



Title Page

Contents



Go Back

Close

Quit

Page 5 of 20

(ii)

$$|\Phi'(z)| = \frac{1}{r^2} \left| \frac{2zp'(z) + (1 - p^2(z))}{(1 + p(z))^2} \right|,$$

where  $|z| = r$ . Substituting (i) and (ii) into (2.2), and then multiplying by  $|1 + p(z)|^2$  we obtain

$$|2zp'(z) + (1 - p^2(z))| \leq \frac{r^2 |1 + p(z)|^2 - |1 - p(z)|^2}{1 - r^2},$$

which implies that

$$|2zp'(z)| \leq |(1 - p^2(z))| + \frac{r^2 |1 + p(z)|^2 - |1 - p(z)|^2}{1 - r^2}.$$

Thus, to prove (2.1) (with  $\alpha = m$ ), it is sufficient to show that

$$(2.3) \quad |(1 - p^2(z))| + \frac{r^2 |1 + p(z)|^2 - |1 - p(z)|^2}{1 - r^2} \leq \frac{4r \operatorname{Re} p(z)}{1 - r^2}.$$

Now we express  $|1 + p(z)|^2$ ,  $|1 - p(z)|^2$  and  $\operatorname{Re} p(z)$  in terms of  $|1 - p^2(z)|$ . From  $z\Phi(z) = \frac{1-p(z)}{1+p(z)}$  we obtain that

$$(iii) \quad |1 - p(z)|^2 = |1 - p^2(z)| |z\Phi(z)|$$

and

$$(iv) \quad |1 + p(z)|^2 |z\Phi(z)| = |1 - \operatorname{Re}^2(z)|.$$

From (iii) and (iv) we have

(v)

$$4 \operatorname{Re} p(z) = |1 + p(z)|^2 - |1 - p(z)|^2 = |1 - p^2(z)| \left[ \frac{1 - |z\Phi(z)|^2}{|z\Phi(z)|} \right].$$

Substituting (iii), (iv), and (v) into (2.3), and then cancelling  $|1 - p^2|$  we obtain

$$\begin{aligned} |(1 - p^2(z))| &+ \frac{r^2 \frac{|1 - p^2(z)|}{|z\Phi(z)|} - |1 - p^2(z)| |z\Phi(z)|}{1 - r^2} \\ &= \frac{4 \operatorname{Re} p(z) + (1 - r^2) |1 - p^2(z)| \left(1 - \frac{1}{|z\Phi(z)|}\right)}{1 - r^2} \\ &\leq \frac{4r \operatorname{Re} p(z)}{1 - r^2}, \end{aligned}$$

which gives us that the inequality (2.1) holds true when  $\alpha = m$ . Further, considering the function  $w(z)$  defined by

$$w(z) = \frac{p(z) - \left(\frac{\alpha - m}{n}\right)}{1 - \left(\frac{\alpha - m}{n}\right)},$$

in the case of  $\alpha \neq m$ , we complete the proof of the theorem.  $\square$

**Remark 1.** *The result obtained from Theorem 2.1 for  $n = 1$  and  $m = 0$  coincides with the result due to Bernardi [1].*



On Neighborhoods of Analytic Functions having Positive Real Part

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 6 of 20

**Lemma 2.2.** The function  $w(z)$  defined by

$$w(z) = \frac{1 - \frac{1}{n} \{2\alpha - (2m + n)\} z}{1 - z}$$

is univalent in  $\mathbb{U}$ ,  $w(0) = 1$ , and  $\operatorname{Re} w(z) > \frac{\alpha - m}{n}$  for  $m < \alpha < m + n$ ,  $m \in \mathbb{N}_0$ , and  $n \in \mathbb{N}$  for  $\mathbb{U}$ .

**Lemma 2.3.** Let  $p(z) \in \mathcal{P} \left( \frac{\alpha - m}{n} \right)$ . Then the disk  $|z| \leq r < 1$  is mapped by  $p(z)$  onto the disk  $|p(z) - \eta(A)| \leq \xi(A)$ , where

$$\eta(A) = \frac{1 + Ar^2}{1 - r^2}, \quad \xi(A) = \frac{r(A + 1)}{1 - r^2}, \quad A = \frac{2m + n - 2\alpha}{n}.$$

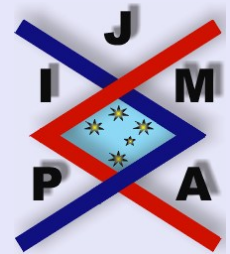
Now, we give general inequalities for the class  $\mathcal{P} \left( \frac{\alpha - m}{n} \right)$ .

**Theorem 2.4.** Let the function  $p(z)$  be in the class  $\mathcal{P} \left( \frac{\alpha - m}{n} \right)$ ,  $k \geq 0$ , and  $r = |z| < 1$ . Then we have

$$(2.4) \quad \operatorname{Re} \left\{ p(z) + \frac{zp'(z)}{p(z) + k} \right\} > \left( \frac{\alpha - m}{n} \right) + \frac{(k + 1) + 2 \left( 2 - \frac{\alpha - m}{n} \right) r + \left( (1 - k) - 2 \left( \frac{\alpha - m}{n} \right) \right) r^2}{(k + 1) - 2 \left( 1 - \frac{\alpha - m}{n} \right) r + \left( (1 - k) - 2 \left( \frac{\alpha - m}{n} \right) \right) r^2} \times \operatorname{Re} \left[ p(z) - \left( \frac{\alpha - m}{n} \right) \right].$$

*Proof.* With the help of Lemma 2.3, we observe that

$$|p(z) + k| \geq |\eta(A) + k| - \xi(A) = \frac{1 + Ar^2}{1 - r^2} + k - \frac{r(A + 1)}{1 - r^2}.$$



On Neighborhoods of Analytic Functions having Positive Real Part

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

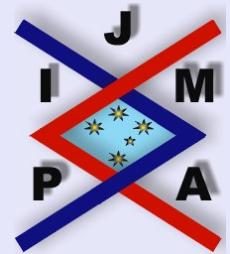
Page 7 of 20

Therefore, an application of Theorem 2.1 yields that

$$\begin{aligned}
 & \operatorname{Re} \left\{ p(z) + \frac{zp'(z)}{p(z) + k} \right\} \\
 & \geq \operatorname{Re} \{p(z)\} - \left| \frac{zp'(z)}{p(z) + k} \right| \\
 & \geq \operatorname{Re} \{p(z)\} - \frac{\frac{2r}{1-r^2}}{1+Ar^2+k(1-r^2)-r(A+1)} \operatorname{Re} \left[ p(z) - \left( \frac{\alpha - m}{n} \right) \right] \\
 & > \left( \frac{\alpha - m}{n} \right) - \left\{ 1 - \frac{\frac{2r}{1-r^2}}{1+Ar^2+k(1-r^2)-r(A+1)} \right\} \operatorname{Re} \left[ p(z) - \frac{\alpha - m}{n} \right],
 \end{aligned}$$

which proves the assertion (2.4).  $\square$

**Remark 2.** The result obtained from this theorem for  $n = 1$ , and  $m = 0$  coincides with the result by Pashkouleva [10].



On Neighborhoods of Analytic Functions having Positive Real Part

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 8 of 20



### 3. Preliminary Results

Let the functions  $f(z)$  and  $g(z)$  be analytic in  $\mathbb{U}$ . Then  $f(z)$  is said to be subordinate to  $g(z)$ , written  $f(z) \prec g(z)$ , if there exists an analytic function  $w(z)$  in  $\mathbb{U}$  with  $w(0) = 0$  and  $|w(z)| \leq |z| < 1$  such that  $f(z) = g(w(z))$ . If  $g(z)$  is univalent in  $\mathbb{U}$ , then the subordination  $f(z) \prec g(z)$  is equivalent to  $f(0) = g(0)$  and

$$f(\mathbb{U}) \subset g(\mathbb{U}) \quad (\text{cf. [11, p. 36, Lemma 2.1]}).$$

For  $f(z)$  and  $g(z)$  given by

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of  $f(z)$  and  $g(z)$  is defined by

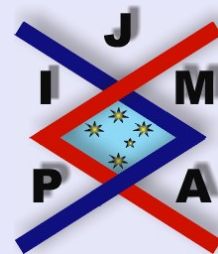
$$(3.1) \quad (f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Further, let  $\mathcal{P}'\left(\frac{\alpha-m}{n}\right)$  be the subclass of  $\mathcal{T}$  consisting of functions  $p(z)$  defined by (1.1) which satisfy

$$(3.2) \quad \operatorname{Re} \{(z p(z))'\} > \frac{\alpha - m}{n} \quad (z \in \mathbb{U})$$

for some  $m \leq \alpha < m + n$ ,  $m \in \mathbb{N}_0$ , and  $n \in \mathbb{N}$ . It follows from the definitions of  $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$  and  $\mathcal{P}'\left(\frac{\alpha-m}{n}\right)$  that

$$(3.3) \quad p(z) \in \mathcal{P}\left(\frac{\alpha - m}{n}\right) \Leftrightarrow p(z) \prec \frac{1 - \frac{1}{n}\{2\alpha - (2m + n)\}z}{1 - z} \quad (z \in \mathbb{U})$$



On Neighborhoods of Analytic Functions having Positive Real Part

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 9 of 20

and that

$$\begin{aligned}
 (3.4) \quad p(z) &\in \mathcal{P}' \left( \frac{\alpha - m}{n} \right) \\
 &\Leftrightarrow (zp(z))' \prec \frac{1 - \frac{1}{n} \{2\alpha - (2m + n)\} z}{1 - z} \quad (z \in \mathbb{U}) \\
 &\Leftrightarrow \frac{(zp(z))'}{(z)'} \prec \frac{1 - \frac{1}{n} \{2\alpha - (2m + n)\} z}{1 - z} \quad (z \in \mathbb{U}).
 \end{aligned}$$

Applying the result by Miller and Mocanu [7, p. 301, Theorem 10] for (3.4), we see that if  $p(z) \in \mathcal{P}' \left( \frac{\alpha - m}{n} \right)$ , then

$$(3.5) \quad p(z) \prec \frac{1 - \frac{1}{n} \{2\alpha - (2m + n)\} z}{1 - z} \quad (z \in \mathbb{U}),$$

which implies that  $\mathcal{P}' \left( \frac{\alpha - m}{n} \right) \subset \mathcal{P} \left( \frac{\alpha - m}{n} \right)$ . Noting that the function

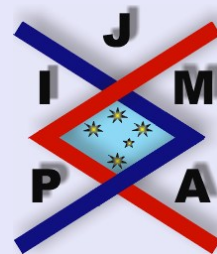
$$\frac{1 - \frac{1}{n} \{2\alpha - (2m + n)\} z}{1 - z}$$

is univalent in  $\mathbb{U}$ , we have that  $q(z) \in \mathcal{P} \left( \frac{\alpha - m}{n} \right)$  if and only if

$$(3.6) \quad q(z) \neq \frac{1 - \frac{1}{n} \{2\alpha - (2m + n)\} e^{i\theta}}{1 - e^{i\theta}} \quad (0 < \theta < 2\pi; z \in \mathbb{U})$$

or

$$(3.7) \quad (1 - e^{i\theta}) q(z) - \left\{ 1 - \frac{1}{n} (2\alpha - (2m + n)) e^{i\theta} \right\} \neq 0 \\
 (0 < \theta < 2\pi; z \in \mathbb{U}).$$



On Neighborhoods of Analytic Functions having Positive Real Part

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 10 of 20

Further, using the convolutions, we obtain that

$$\begin{aligned}
 (3.8) \quad & (1 - e^{i\theta})q(z) - \left\{ 1 - \frac{1}{n} (2\alpha - (2m + n)) e^{i\theta} \right\} \\
 &= (1 - e^{i\theta}) \left( \frac{1}{1-z} * q(z) \right) - \left\{ 1 - \frac{1}{n} [2\alpha - (2m + n)] e^{i\theta} \right\} * q(z) \\
 &= \left\{ \frac{1 - e^{i\theta}}{1 - z} - \left[ 1 - \frac{1}{n} (2\alpha - (2m + n)) e^{i\theta} \right] \right\} * q(z).
 \end{aligned}$$

Therefore, if we define the function  $h_\theta(z)$  by

$$(3.9) \quad h_\theta(z) = \frac{n}{2(\alpha - m - n)e^{i\theta}} \left\{ \frac{1 - e^{i\theta}}{1 - z} - \left[ 1 - \frac{1}{n} (2\alpha - (2m + n)) e^{i\theta} \right] \right\},$$

then  $h_\theta(0) = 1$  ( $0 < \theta < 2\pi$ ). This gives us that

$$(3.10) \quad q(z) \in \mathcal{P} \left( \frac{\alpha - m}{n} \right)$$

$$(3.11) \quad \Leftrightarrow \frac{2}{n} (\alpha - m - n) e^{i\theta} \{h_\theta(z) * q(z)\} \neq 0 \quad (0 < \theta < 2\pi; z \in \mathbb{U})$$

$$(3.12) \quad \Leftrightarrow h_\theta(z) * q(z) \neq 0 \quad (0 < \theta < 2\pi; z \in D).$$



**On Neighborhoods of Analytic Functions having Positive Real Part**

Shigeyoshi Owa, Nigar Yildirim  
and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 11 of 20

## 4. Main Results

In order to derive our main result, we need the following lemmas.

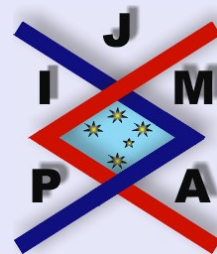
**Lemma 4.1.** *If  $p(z) \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$  with  $m \leq \alpha < m+n$ ;  $m \in \mathbb{N}_0, n \in \mathbb{N}$ , then  $z(p(z) * h_\theta(z))$  is univalent for each  $\theta$  ( $0 < \theta < 2\pi$ ).*

*Proof.* For fixed  $\theta$  ( $0 < \theta < 2\pi$ ), we have

$$\begin{aligned}
 & [z(p(z) * h_\theta(z))]' \\
 &= \left[ \frac{zn}{2(\alpha-m-n)e^{i\theta}} \left( \frac{1-e^{i\theta}}{1-z} - \left\{ 1 - \frac{1}{n}(2\alpha - (2m+n))e^{i\theta} \right\} \right) * p(z) \right]' \\
 &= \left[ \frac{zn}{2(\alpha-m-n)e^{i\theta}} \left( (1-e^{i\theta})p(z) - \left\{ 1 - \frac{1}{n}(2\alpha - (2m+n))e^{i\theta} \right\} \right) \right]' \\
 &= \left[ \frac{zn}{2(\alpha-m-n)e^{i\theta}} (1-e^{i\theta}) \left( p(z) - \frac{\left\{ 1 - \frac{1}{n}(2\alpha - (2m+n))e^{i\theta} \right\}}{1-e^{i\theta}} \right) \right]' \\
 &= \frac{(1-e^{i\theta})}{e^{i\theta}} \left[ \frac{n}{2(\alpha-m-n)} \left( zp(z) - \frac{\left\{ 1 - \frac{1}{n}(2\alpha - (2m+n))e^{i\theta} \right\}}{1-e^{i\theta}} z \right) \right]' \\
 &= \frac{n}{2(\alpha-m-n)} \left\{ (zp(z))' - \frac{\left\{ 1 - \frac{1}{n}(2\alpha - (2m+n))e^{i\theta} \right\}}{1-e^{i\theta}} \right\} \frac{1-e^{i\theta}}{e^{i\theta}}.
 \end{aligned}$$

By the definition of  $\mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ , the range of  $(zp(z))'$  for  $|z| < 1$  lies in  $\text{Re}(w) > \frac{\alpha-m}{n}$ . On the other hand

$$\text{Re} \left\{ \frac{1 - \frac{1}{n} \{2\alpha - (2m+n)\} e^{i\theta}}{1 - e^{i\theta}} \right\} = \frac{1 + \frac{1}{n} \{2\alpha - (2m+n)\}}{2}.$$



On Neighborhoods of Analytic Functions having Positive Real Part

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 12 of 20

Thus, we write

$$(4.1) \quad [z(p(z) * h_\theta(z))]' \\ = \frac{n}{2(\alpha - m - n)} \cdot \frac{e^{-i\phi}}{K} \left\{ (zp(z))' - \frac{\left\{1 - \frac{1}{n}(2\alpha - (2m + n))e^{i\theta}\right\}}{1 - e^{i\theta}} \right\},$$

where

$$K = \left| \frac{e^{i\theta}}{e^{i\theta} - 1} \right| = \frac{1}{\sqrt{2(1 - \cos \theta)}}$$

and

$$\phi = \arg \left\{ \frac{e^{i\theta}}{e^{i\theta} - 1} \right\} = \theta - \tan^{-1} \left( \frac{\sin \theta}{\cos \theta - 1} \right).$$

Consequently, we obtain that

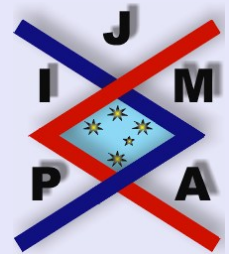
$$\operatorname{Re} \left\{ K e^{i\phi} (z(p(z) * h_\theta(z)))' \right\} > 0 \quad (z \in \mathbb{U}),$$

because  $p(z) \in \mathcal{P}' \left( \frac{\alpha - m}{n} \right)$ . An application of the Noshiro-Warschawski theorem (cf. [3, p. 47]) gives that  $z(p(z) * h_\theta(z))$  is univalent for each  $\theta$  ( $0 < \theta < 2\pi$ ).  $\square$

**Lemma 4.2.** *If  $p(z) \in \mathcal{P}' \left( \frac{\alpha - m}{n} \right)$  with  $m \leq \alpha < m + n$ ,  $m \in \mathbb{N}_0$ , and  $n \in \mathbb{N}$ , then*

$$(4.2) \quad \left| \{z(p(z) * h_\theta(z))\}' \right| \geq \frac{1 - r}{1 + r}$$

for  $|z| = r < 1$  and  $0 < \theta < 2\pi$ .



On Neighborhoods of Analytic Functions having Positive Real Part

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 13 of 20

*Proof.* Using the expression (4.1) for  $|\{z(p(z) * h_\theta(z))\}'|$ , we define

$$F(w) = e^{-i\theta}(1 - e^{i\theta}) \left\{ \frac{1 + \frac{1}{n}(2m + n - 2\alpha)e^{i\theta}}{1 - e^{i\theta}} - w \right\},$$

where

$$w = \frac{1 + \frac{1}{n}[2m + n - 2\alpha]re^{it}}{1 - re^{it}} \quad (0 \leq t \leq 2\pi).$$

Then the function  $F(w)$  may be rewritten as

$$\begin{aligned} F(w) &= e^{-i\theta} \left\{ \left( 1 + \frac{1}{n}(2m + n - 2\alpha)e^{i\theta} - (1 - e^{i\theta})w \right) \right\} \\ &= e^{-i\theta} \left\{ (1 - w) + \left[ \frac{1}{n}(2m + n - 2\alpha) + w \right] e^{i\theta} \right\} \\ &= \left[ \frac{1}{n}(2m + n - 2\alpha) + w \right] e^{-i\theta} \left\{ \frac{1 - w}{\frac{1}{n}(2m + n - 2\alpha) + w} + e^{i\theta} \right\} \end{aligned}$$

for  $0 < \theta < 2\pi$ . Thus we see that

$$\begin{aligned} |F(w)| &= \left| \frac{1}{n}(2m + n - 2\alpha) + w \right| \left| \frac{1 - w}{\frac{1}{n}(2m + n - 2\alpha) + w} + e^{i\theta} \right| \\ &= \left| \frac{1}{n}(2m + n - 2\alpha) + w \right| |e^{i\theta} - re^{it}| \\ &= \left| \frac{1}{n}(2m + n - 2\alpha) + w \right| |1 - re^{i(t-\theta)}| \\ &\geq \left| \frac{1}{n}(2m + n - 2\alpha) + w \right| (1 - r). \end{aligned}$$



**On Neighborhoods of Analytic Functions having Positive Real Part**

Shigeyoshi Owa, Nigar Yildirim  
and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 14 of 20

Since

$$\begin{aligned} \left| \frac{1}{n}(2m+n-2\alpha) + w \right| &= \left| \frac{1}{n}(2m+n-2\alpha) + \frac{1 + \frac{1}{n}(2m+n-2\alpha)re^{it}}{1-re^{it}} \right| \\ &= \left| \frac{1 + \frac{1}{n}(2m+n-2\alpha)}{1-re^{it}} \right| \\ &\geq \frac{1 + \frac{1}{n}(2m+n-2\alpha)}{1+r}, \end{aligned}$$

it is clear that

$$|F(w)| \geq \frac{(1-r)}{(1+r)} \left[ 1 + \frac{1}{n}(2m+n-2\alpha) \right].$$

Since  $p \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$  and (4.1) holds, by letting  $w = [zp(z)]'$ , we get the desired inequality. That is,

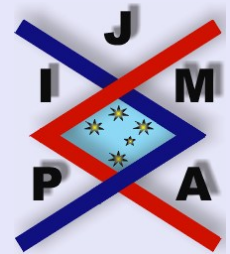
$$\begin{aligned} |[zp(z)]'| &\geq \frac{n}{2(m+n-\alpha)} \cdot \frac{1 + \frac{1}{n}(2m+n-2\alpha)}{1+r} (1-r) \\ &= \frac{(1-r)}{(1+r)}. \end{aligned}$$

Therefore, the lemma is proved.  $\square$

Further, we need the following lemma.

**Lemma 4.3.** *If  $p(z) \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$  with  $m \leq \alpha < m+n$ ,  $m \in \mathbb{N}_0$ , and  $n \in \mathbb{N}$ , then*

$$(4.3) \quad |p(z) * h_\theta(z)| \geq \delta \quad (0 < \theta < 2\pi; z \in \mathbb{U}),$$



On Neighborhoods of Analytic Functions having Positive Real Part

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 15 of 20

where

$$\delta = \int_0^1 \frac{2}{1+t} dt - 1 = 2 \ln 2 - 1.$$

*Proof.* Since Lemma 4.1 shows that  $z(p(z) * h_\theta(z))$  is univalent for each  $\theta$  ( $0 < \theta < 2\pi$ ) for  $p(z)$  belonging to the class  $\mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ , we can choose a point  $z_0 \in \mathbb{U}$  with  $|z_0| = r < 1$  such that

$$\min_{|z|=r} |z(p(z) * h_\theta(z))| = |z_0(p(z_0) * h_\theta(z_0))|$$

for fixed  $r$  ( $0 < r < 1$ ). Then the pre-image  $\gamma$  of the line segment from 0 to  $z_0(p(z_0) * h_\theta(z_0))$  is an arc inside  $|z| \leq r$ . Hence, for  $|z| \leq r$ , we have that

$$\begin{aligned} |z(p(z) * h_\theta(z))| &\geq |z_0(p(z_0) * h_\theta(z_0))| \\ &= \int_\gamma |(z(p(z) * h_\theta(z)))'| |dz|. \end{aligned}$$

An application of Lemma 4.2 leads us to

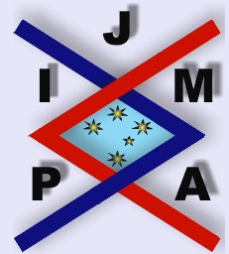
$$|p(z) * h_\theta(z)| \geq \frac{1}{r} \int_0^r \frac{1-t}{1+t} dt = \frac{1}{r} \int_0^r \frac{2}{1+t} dt - 1.$$

Note that the function  $\Omega(r)$  defined by

$$\Omega(r) = \frac{1}{r} \int_0^r \frac{2}{1+t} dt - 1$$

is decreasing for  $r$  ( $0 < r < 1$ ). Therefore, we have

$$|p(z) * h_\theta(z)| \geq \delta = \int_0^1 \frac{2}{1+t} dt - 1 = 2 \ln 2 - 1,$$



On Neighborhoods of Analytic Functions having Positive Real Part

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 16 of 20



which completes the proof of Lemma 4.3. □

Now, we give the statement and the proof of our main result.

**Theorem 4.4.** *If  $p(z) \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$  with  $m \leq \alpha < m+n$ ,  $m \in \mathbb{N}_0$ , and  $n \in \mathbb{N}$ , then*

$$\mathcal{N}_{\beta\delta}(p(z)) \subset \mathcal{P}\left(\frac{\alpha-m}{n}\right),$$

where  $\beta = \frac{m+n-\alpha}{n}$  and

$$(4.4) \quad \delta = \int_0^1 \frac{2}{1+t} dt - 1 = 2 \ln 2 - 1.$$

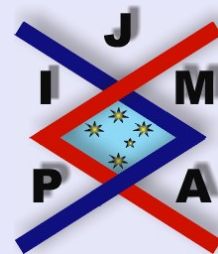
*The result is sharp.*

*Proof.* Let  $q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k$ . Then, by the definition of neighborhoods, we have to prove that if  $q(z) \in \mathcal{N}_{\beta\delta}(p(z))$  for  $p(z) \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ , then  $q(z)$  belongs to the class  $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$ . Using Lemma 4.3 and the inequality

$$\sum_{k=1}^{\infty} |p_k - q_k| \leq \delta,$$

we get

$$\begin{aligned} |h_{\theta}(z) * q(z)| &\geq |h_{\theta}(z) * p(z)| - |h_{\theta}(z) * (p(z) - q(z))| \\ &\geq \delta - \left| \sum_{k=1}^{\infty} \frac{n(1 - e^{i\theta})}{2(\alpha - m - n)e^{i\theta}} (p_k - q_k) z^k \right| \\ &> \delta - \frac{n}{m + n - \alpha} \sum_{k=1}^{\infty} |p_k - q_k| \end{aligned}$$



**On Neighborhoods of Analytic Functions having Positive Real Part**

Shigeyoshi Owa, Nigar Yildirim  
and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 17 of 20

$$\begin{aligned}
&> \delta - \frac{n}{m+n-\alpha} \left\{ \frac{m+n-\alpha}{n} \right\} \delta \\
&\geq \delta - \delta = 0.
\end{aligned}$$

Since  $h_\theta(z) * q(z) \neq 0$  for  $0 < \theta < 2\pi$  and  $z \in \mathbb{U}$ , we conclude that  $q(z)$  belongs to the class  $\mathcal{P}(\frac{\alpha-m}{n})$ , that is, that  $\mathcal{N}_{\beta\delta}(p(z)) \subset P(\frac{\alpha-m}{n})$ .

Further, taking the function  $p(z)$  defined by

$$(zp(z))' = \frac{1 - \frac{1}{n} \{2\alpha - (2m+n)\} z}{1-z},$$

we have

$$p(z) = \frac{1}{n}(2\alpha - (2m+n)) + \frac{\frac{2}{n}(m+n-\alpha)}{z} \left\{ \int_0^z \frac{1}{1-t} dt \right\}.$$

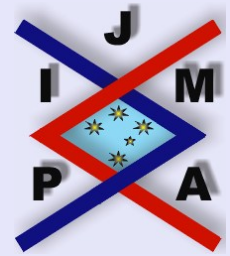
If we define the function  $q(z)$  by

$$q(z) = p(z) + \left( \frac{m+n-\alpha}{n} \right) \delta z,$$

then  $q(z) \in \mathcal{N}_{\beta\delta}(p(z))$ . Letting  $z = e^{i\pi}$ , we see that  $q(z) = q(e^{i\pi}) = \frac{\alpha-m}{n}$ . This implies that if

$$\delta > \int_0^1 \frac{2}{1+t} dt - 1,$$

then  $q(e^{i\pi}) < \frac{\alpha-m}{n}$ . Therefore,  $\operatorname{Re} \{q(z)\} < \frac{\alpha-m}{n}$  for  $z$  near  $e^{i\pi}$ , which contradicts  $q(z) \in \mathcal{P}(\frac{\alpha-m}{n})$  (otherwise  $\operatorname{Re} \{q(z)\} > \frac{\alpha-m}{n}$ ;  $z \in \mathbb{U}$ ). Consequently, the result of the theorem is sharp.  $\square$



**On Neighborhoods of Analytic Functions having Positive Real Part**

Shigeyoshi Owa, Nigar Yildirim and Muhammet Kamali

Title Page

Contents



Go Back

Close

Quit

Page 18 of 20

## References

- [1] S.D. BERNARDI, New distortion theorems for functions of positive real part and applications to the partial sums of univalent convex functions, *Proc. Amer. Math. Soc.*, **45** (1974), 113–118.
- [2] J.E. BROWN, Some sharp neighborhoods of univalent functions, *Trans. Amer. Math. Soc.*, **287** (1985), 475–482.
- [3] P.L. DUREN, *Univalent Functions*, Springer-Verlag, New York, 1983.
- [4] R. FOURNIER, A note on neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **87** (1983), 117–120.
- [5] R. FOURNIER, On neighborhoods of univalent starlike functions, *Ann. Polon. Math.*, **47** (1986), 189–202.
- [6] R. FOURNIER, On neighborhoods of univalent convex functions, *Rocky Mount. J. Math.*, **16** (1986), 579–589.
- [7] S.S. MILLER AND P.T. MOCANU, Second order differential inequalities in the complex plane, *J. Math. Anal. Appl.*, **65** (1978), 289–305.
- [8] Z. NEHARI, *Conformal Mapping*, McGraw-Hill, New York, 1952.
- [9] S. OWA, H. SAITOH AND M. NUNOKAWA, Neighborhoods of certain analytic functions, *Appl. Math. Lett.*, **6** (1993), 73–77.
- [10] D.Z. PASHKOULEVA, The starlikeness and spiral-convexity of certain subclasses of analytic functions, *Current Topics in Analytic Function The-*



---

On Neighborhoods of Analytic  
Functions having Positive Real  
Part

Shigeyoshi Owa, Nigar Yildirim  
and Muhammet Kamali

---

Title Page

Contents

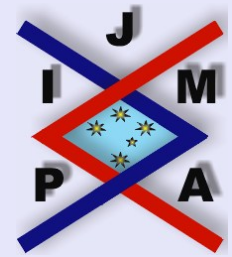


Go Back

Close

Quit

Page 19 of 20



---

**On Neighborhoods of Analytic Functions having Positive Real Part**

Shigeyoshi Owa, Nigar Yildirim  
and Muhammet Kamali

---

Title Page

Contents

◀◀

▶▶

◀

▶

Go Back

Close

Quit

Page 20 of 20

ory (H.M. Srivastava and S. Owa (Editors)), World Scientific, Singapore, New Jersey, London and Hong Kong (1992), 266–273.

- [11] Ch. POMMERENKE, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
- [12] St. RUSCHEWEYH, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **81** (1981), 521–527.
- [13] J.B. WALKER, A note on neighborhoods of analytic functions having positive real part, *Internat. J. Math. Math. Sci.*, **13** (1990), 425–430.