Journal of Inequalities in Pure and Applied Mathematics

ON NEIGHBORHOODS OF ANALYTIC FUNCTIONS HAVING POSITIVE REAL PART

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volume 7, issue 3, article 109, 2006.

Received 10 November, 2005; accepted 15 July, 2006.

Communicated by: G. Kohr

ISSN (electronic): 1443-5756
334-05

Abstract

Two subclasses $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$ and $\mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ of certain analytic functions having positive real part in the open unit disk $\mathbb{U}$ are introduced. In the present paper, several properties of the subclass $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$ of analytic functions with real part greater than $\frac{\alpha-m}{n}$ are derived. For $p(z) \in \mathcal{P}\left(\frac{\alpha-m}{n}\right)$ and $\delta \geq 0$, the $\delta-$neighborhood $\mathcal{N}_\delta(p(z))$ of $p(z)$ is defined. For $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$, $P'\left(\frac{\alpha-m}{n}\right)$, and $N_\delta(p(z))$, we prove that if $p(z) \in P'\left(\frac{\alpha-m}{n}\right)$, then $N_{\beta\delta}(p(z)) \subset P\left(\frac{\alpha-m}{n}\right)$.

2000 Mathematics Subject Classification: Primary 30C45.
Key words: Function with positive real part, subordinate function, $\delta-$neighborhood, convolution (Hadamard product).

Contents

1. Introduction

Let $\mathcal{T}$ be the class of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $p(z) \in \mathcal{T}$ is said to be in the class $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$ if it satisfies

$$\operatorname{Re}\{p(z)\} > \frac{\alpha - m}{n} \qquad (z \in \mathbb{U})$$

for some $m \leq \alpha < m + n$, $m \in \mathbb{N}_0 = 0, 1, 2, 3, \dots$, and $n \in \mathbb{N} = 1, 2, 3, \dots$. For any $p(z) \in \mathcal{P}\left(\frac{\alpha-m}{n}\right)$ and $\delta \geq 0$, we define the $\delta-$neighborhood $\mathcal{N}_\delta(p(z))$ of $p(z)$ by

$$\mathcal{N}_\delta(p(z)) = \left\{ q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k \in \mathcal{T} : \sum_{k=1}^{\infty} |p_k - q_k| \leq \delta \right\}.$$

The concept of $\delta-$neighborhoods $\mathcal{N}_\delta(f(z))$ of analytic functions $f(z)$ in $\mathbb{U}$ with $f(0) = f'(0) - 1 = 0$ was first introduced by Ruscheweyh [12] and was studied by Fournier [4, 6] and by Brown [2]. Walker has studied the δ_1-neighborhood $\mathcal{N}_{\delta_1}(p(z))$ of $p(z) \in \mathcal{P}_1(0)$ [13]. Later, Owa et al. [9] extended the result by Walker.

In this paper, we give some inequalities for the class $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$. Furthermore, we define a neighborhood of $p(z) \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ and determine $\delta > 0$ so that $\mathcal{N}_{\beta\delta}(p(z)) \subset \mathcal{P}\left(\frac{\alpha-m}{n}\right)$, where $\beta = \frac{m+n-\alpha}{n}$.

2. Some Inequalities for the Class $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$

Our first result for functions $p(z)$ in $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$ is contained in

Theorem 2.1. *Let* $p(z) \in \mathcal{P}\left(\frac{\alpha-m}{n}\right)$. *Then, for* $|z| = r < 1, m \leq \alpha < m + n, m \in \mathbb{N}_0$ *and* $n \in \mathbb{N}$,

$$|zp'(z)| \leq \frac{2r}{1-r^2} \operatorname{Re}\left\{p(z) - \frac{\alpha-m}{n}\right\}. \tag{2.1}$$

For each $m \leq \alpha < m+n$, *the equality is attained at* $z = r$ *for the function*

$$p(z) = \frac{\alpha-m}{n} + \left(1 - \frac{\alpha-m}{n}\right)\frac{1-z}{1+z} = 1 - \frac{2}{n}(n-\alpha+m)z + \cdots .$$

Proof. Let us consider the case of $p(z) \in \mathcal{P}(0)$. Then the function $k(z)$ defined by

$$k(z) = \frac{1-p(z)}{1+p(z)} = \eta_1 z + \eta_2 z^2 + \cdots$$

is analytic in $\mathbb{U}$ and $|k(z)| < 1\,(z \in \mathbb{U})$. Hence $k(z) = z\Phi(z)$, where $\Phi(z)$ is analytic in $\mathbb{U}$ and $|\Phi(z)| \leq 1\,(z \in \mathbb{U})$. For such a function $\Phi(z)$, we have

$$|\Phi'(z)| \leq \frac{\left(1-|\Phi(z)|^2\right)}{\left(1-|z|^2\right)} \qquad (z \in \mathbb{U}). \tag{2.2}$$

From $z\Phi(z) = \frac{1-p(z)}{1+p(z)}$, we obtain

(i)

$$|\Phi(z)|^2 = \frac{1}{r^2}\left|\frac{1-p(z)}{1+p(z)}\right|^2,$$

(ii)
$$|\Phi'(z)| = \frac{1}{r^2}\left|\frac{2zp'(z)+(1-p^2(z))}{(1+p(z))^2}\right|,$$

where $|z| = r$. Substituting (i) and (ii) into (2.2), and then multiplying by $|1+p(z)|^2$ we obtain

$$\left|2zp'(z)+(1-p^2(z))\right| \leq \frac{r^2\left|1+p(z)\right|^2-\left|1-p(z)\right|^2}{1-r^2},$$

which implies that

$$|2zp'(z)| \leq \left|(1-p^2(z))\right| + \frac{r^2\left|1+p(z)\right|^2-\left|1-p(z)\right|^2}{1-r^2}.$$

Thus, to prove (2.1) (with $\alpha = m$), it is sufficient to show that

$$\left|(1-p^2(z))\right| + \frac{r^2\left|1+p(z)\right|^2-\left|1-p(z)\right|^2}{1-r^2} \leq \frac{4r\operatorname{Re}p(z)}{1-r^2}. \tag{2.3}$$

Now we express $|1+p(z)|^2$, $|1-p(z)|^2$ and $\operatorname{Re}p(z)$ in terms of $|1-p^2(z)|$. From $z\Phi(z) = \frac{1-p(z)}{1+p(z)}$ we obtain that

(iii) $|1-p(z)|^2 = |1-p^2(z)|\,|z\Phi(z)|$

and

(iv) $|1+p(z)|^2\,|z\Phi(z)| = \left|1-\operatorname{Re}^2(z)\right|.$

From (iii) and (iv) we have
(v)

$$4\,\mathrm{Re}\,p(z) = |1+p(z)|^2 - |1-p(z)|^2 = \left|1-p^2(z)\right| \left[\frac{1-|z\Phi(z)|^2}{|z\Phi(z)|}\right].$$

Substituting (iii), (iv), and (v) into (2.3), and then cancelling $|1-p^2|$ we obtain

$$\begin{aligned} \left|(1-p^2(z))\right| + \frac{r^2\frac{\left|1-p^2(z)\right|}{|z\Phi(z)|} - \left|1-p^2(z)\right|\left|z\Phi(z)\right|}{1-r^2} &= \frac{4\,\mathrm{Re}\,p(z) + (1-r^2)\left|1-p^2(z)\right|\left(1-\frac{1}{|z\Phi(z)|}\right)}{1-r^2} \\ &\leq \frac{4r\,\mathrm{Re}\,p(z)}{1-r^2}, \end{aligned}$$

which gives us that the inequality (2.1) holds true when $\alpha = m$. Further, considering the function $w(z)$ defined by

$$w(z) = \frac{p(z) - \left(\frac{\alpha-m}{n}\right)}{1-\left(\frac{\alpha-m}{n}\right)},$$

in the case of $\alpha \neq m$, we complete the proof of the theorem. □

Remark 1. *The result obtained from Theorem 2.1 for $n = 1$ and $m = 0$ coincides with the result due to Bernardi [1].*

Lemma 2.2. *The function* $w(z)$ *defined by*

$$w(z) = \frac{1 - \frac{1}{n}\{2\alpha - (2m+n)\}z}{1-z}$$

is univalent in $\mathbb{U}$, $w(0) = 1$, *and* $\operatorname{Re} w(z) > \frac{\alpha - m}{n}$ *for* $m < \alpha < m+n$, $m \in \mathbb{N}_0$, *and* $n \in \mathbb{N}$ *for* $\mathbb{U}$.

Lemma 2.3. *Let* $p(z) \in \mathcal{P}\left(\frac{\alpha-m}{n}\right)$. *Then the disk* $|z| \leq r < 1$ *is mapped by* $p(z)$ *onto the disk* $|p(z) - \eta(A)| \leq \xi(A)$, *where*

$$\eta(A) = \frac{1+Ar^2}{1-r^2}, \quad \xi(A) = \frac{r(A+1)}{1-r^2}, \quad A = \frac{2m+n-2\alpha}{n}.$$

Now, we give general inequalities for the class $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$.

Theorem 2.4. *Let the function* $p(z)$ *be in the class* $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$, $k \geq 0$, *and* $r = |z| < 1$. *Then we have*

$$\begin{aligned} (2.4)\quad \operatorname{Re}\left\{p(z) + \frac{zp'(z)}{p(z)+k}\right\} \\ > \left(\frac{\alpha-m}{n}\right) + \frac{(k+1) + 2\left(2 - \frac{\alpha-m}{n}\right)r + \left((1-k) - 2\left(\frac{\alpha-m}{n}\right)\right)r^2}{(k+1) - 2\left(1 - \frac{\alpha-m}{n}\right)r + \left((1-k) - 2\left(\frac{\alpha-m}{n}\right)\right)r^2} \\ \times \operatorname{Re}\left[p(z) - \left(\frac{\alpha-m}{n}\right)\right]. \end{aligned}$$

Proof. With the help of Lemma 2.3, we observe that

$$|p(z)+k| \geq |\eta(A)+k| - \xi(A) = \frac{1+Ar^2}{1-r^2} + k - \frac{r(A+1)}{1-r^2}.$$

Therefore, an application of Theorem 2.1 yields that

$$
\begin{aligned}
&\operatorname{Re}\left\{p(z)+\frac{zp'(z)}{p(z)+k}\right\}\\
&\quad\geq \operatorname{Re}\{p(z)\}-\left|\frac{zp'(z)}{p(z)+k}\right|\\
&\quad\geq \operatorname{Re}\{p(z)\}-\frac{\frac{2r}{1-r^2}}{\frac{1+Ar^2+k(1-r^2)-r(A+1)}{1-r^2}}\operatorname{Re}\left[p(z)-\left(\frac{\alpha-m}{n}\right)\right]\\
&\quad> \left(\frac{\alpha-m}{n}\right)-\left\{1-\frac{\frac{2r}{1-r^2}}{\frac{1+Ar^2+k(1-r^2)-r(A+1)}{1-r^2}}\right\}\operatorname{Re}\left[p(z)-\frac{\alpha-m}{n}\right],
\end{aligned}
$$

which proves the assertion (2.4). □

Remark 2. *The result obtained from this theorem for* $n = 1$*, and* $m = 0$ *coincides with the result by Pashkouleva [10].*

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3. Preliminary Results

Let the functions $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then $f(z)$ is said to be subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| \leq |z| < 1$ such that $f(z) = g(w(z))$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and

$$f(\mathbb{U}) \subset g(\mathbb{U}) \quad \text{(cf. [11, p. 36, Lemma 2.1]).}$$

For $f(z)$ and $g(z)$ given by

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k. \tag{3.1}$$

Further, let $\mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ be the subclass of $\mathcal{T}$ consisting of functions $p(z)$ defined by (1.1) which satisfy

$$\operatorname{Re}\{(zp(z))'\} > \frac{\alpha - m}{n} \quad (z \in \mathbb{U}) \tag{3.2}$$

for some $m \leq \alpha < m + n$, $m \in \mathbb{N}_0$, and $n \in \mathbb{N}$. It follows from the definitions of $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$ and $\mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ that

$$p(z) \in P\left(\frac{\alpha - m}{n}\right) \Leftrightarrow p(z) \prec \frac{1 - \frac{1}{n}\{2\alpha - (2m+n)\}z}{1 - z} \quad (z \in \mathbb{U}) \tag{3.3}$$

and that

$$p(z) \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right) \tag{3.4}$$
$$\Leftrightarrow (zp(z))' \prec \frac{1-\frac{1}{n}\{2\alpha-(2m+n)\}z}{1-z} \qquad (z\in\mathbb{U})$$
$$\Leftrightarrow \frac{(zp(z))'}{(z)'} \prec \frac{1-\frac{1}{n}\{2\alpha-(2m+n)\}z}{1-z} \qquad (z\in\mathbb{U}).$$

Applying the result by Miller and Mocanu [7, p. 301, Theorem 10] for (3.4), we see that if $p(z)\in\mathcal{P}'\left(\frac{\alpha-m}{n}\right)$, then

$$p(z) \prec \frac{1-\frac{1}{n}\{2\alpha-(2m+n)\}z}{1-z} \qquad (z\in\mathbb{U}), \tag{3.5}$$

which implies that $\mathcal{P}'\left(\frac{\alpha-m}{n}\right)\subset\mathcal{P}\left(\frac{\alpha-m}{n}\right)$. Noting that the function

$$\frac{1-\frac{1}{n}\{2\alpha-(2m+n)\}z}{1-z}$$

is univalent in $\mathbb{U}$, we have that $q(z)\in\mathcal{P}\left(\frac{\alpha-m}{n}\right)$ if and only if

$$q(z) \neq \frac{1-\frac{1}{n}\{2\alpha-(2m+n)\}e^{i\theta}}{1-e^{i\theta}} \qquad (0<\theta<2\pi;\, z\in\mathbb{U}) \tag{3.6}$$

or

$$\left(1-e^{i\theta}\right)q(z) - \left\{1-\frac{1}{n}\left(2\alpha-(2m+n)\right)e^{i\theta}\right\} \neq 0 \tag{3.7}$$
$$(0<\theta<2\pi;\, z\in\mathbb{U}).$$

Further, using the convolutions, we obtain that

$$(3.8)\quad \left(1-e^{i\theta}\right)q(z)-\left\{1-\frac{1}{n}\left(2\alpha-(2m+n)\right)e^{i\theta}\right\}$$
$$=\left(1-e^{i\theta}\right)\left(\frac{1}{1-z}*q(z)\right)-\left\{1-\frac{1}{n}\left[2\alpha-(2m+n)\right]e^{i\theta}\right\}*q(z)$$
$$=\left\{\frac{1-e^{i\theta}}{1-z}-\left[1-\frac{1}{n}(2\alpha-(2m+n))e^{i\theta}\right]\right\}*q(z).$$

Therefore, if we define the function $h_\theta(z)$ by

$$(3.9)\quad h_\theta(z)=\frac{n}{2(\alpha-m-n)e^{i\theta}}\left\{\frac{1-e^{i\theta}}{1-z}-\left[1-\frac{1}{n}(2\alpha-(2m+n))e^{i\theta}\right]\right\},$$

then $h_\theta(0)=1\ (0<\theta<2\pi)$. This gives us that

$$(3.10)\quad q(z)\in\mathcal{P}\left(\frac{\alpha-m}{n}\right)$$
$$(3.11)\quad \Leftrightarrow \frac{2}{n}\left(\alpha-m-n\right)e^{i\theta}\left\{h_\theta(z)*q(z)\right\}\neq 0\quad (0<\theta<2\pi;\, z\in\mathbb{U})$$
$$(3.12)\quad \Leftrightarrow h_\theta(z)*q(z)\neq 0(0<\theta<2\pi;\, z\in D).$$

4. Main Results

In order to derive our main result, we need the following lemmas.

Lemma 4.1. *If* $p(z) \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ *with* $m \leq \alpha < m+n; m \in \mathbb{N}_0, n \in \mathbb{N}$, *then* $z(p(z) * h_\theta(z))$ *is univalent for each* θ $(0 < \theta < 2\pi)$ *.*

Proof. For fixed θ $(0 < \theta < 2\pi)$, we have

$$\begin{aligned}
&[z(p(z) * h_\theta(z))]' \\
&= \left[\frac{zn}{2(\alpha-m-n)e^{i\theta}}\left(\frac{1-e^{i\theta}}{1-z} - \left\{1 - \frac{1}{n}(2\alpha-(2m+n))e^{i\theta}\right\}\right) * p(z)\right]' \\
&= \left[\frac{zn}{2(\alpha-m-n)e^{i\theta}}\left((1-e^{i\theta})p(z) - \left\{1 - \frac{1}{n}(2\alpha-(2m+n))e^{i\theta}\right\}\right)\right]' \\
&= \left[\frac{zn}{2(\alpha-m-n)e^{i\theta}}(1-e^{i\theta})\left(p(z) - \frac{\left\{1 - \frac{1}{n}(2\alpha-(2m+n))e^{i\theta}\right\}}{1-e^{i\theta}}\right)\right]' \\
&= \frac{(1-e^{i\theta})}{e^{i\theta}}\left[\frac{n}{2(\alpha-m-n)}\left(zp(z) - \frac{\left\{1 - \frac{1}{n}(2\alpha-(2m+n))e^{i\theta}\right\}}{1-e^{i\theta}}z\right)\right]' \\
&= \frac{n}{2(\alpha-m-n)}\left\{(zp(z))' - \frac{\left\{1 - \frac{1}{n}(2\alpha-(2m+n))e^{i\theta}\right\}}{1-e^{i\theta}}\right\}\frac{1-e^{i\theta}}{e^{i\theta}}.
\end{aligned}$$

By the definition of $\mathcal{P}'(\frac{\alpha-m}{n})$, the range of $(zp(z))'$ for $|z| < 1$ lies in $\mathrm{Re}(w) > \frac{\alpha-m}{n}$. On the other hand

$$\mathrm{Re}\left\{\frac{1 - \frac{1}{n}\left\{2\alpha-(2m+n)\right\}e^{i\theta}}{1-e^{i\theta}}\right\} = \frac{1 + \frac{1}{n}\left\{2\alpha-(2m+n)\right\}}{2}.$$

Thus, we write

$$[z(p(z) * h_\theta(z))]' \tag{4.1}$$
$$= \frac{n}{2(\alpha - m - n)} \cdot \frac{e^{-i\phi}}{K} \left\{ (zp(z))' - \frac{\{1 - \frac{1}{n}(2\alpha - (2m+n))e^{i\theta}\}}{1 - e^{i\theta}} \right\},$$

where

$$K = \left| \frac{e^{i\theta}}{e^{i\theta} - 1} \right| = \frac{1}{\sqrt{2(1 - \cos\theta)}}$$

and

$$\phi = \arg \left\{ \frac{e^{i\theta}}{e^{i\theta} - 1} \right\} = \theta - \tan^{-1} \left(\frac{\sin\theta}{\cos\theta - 1} \right).$$

Consequently, we obtain that

$$\operatorname{Re}\left\{ K e^{i\phi} (z(p(z) * h_\theta(z)))' \right\} > 0 \qquad (z \in \mathbb{U}),$$

because $p(z) \in \mathcal{P}'\left(\frac{\alpha - m}{n}\right)$. An application of the Noshiro-Warschawski theorem (cf. [3, p. 47]) gives that $z(p(z) * h_\theta(z))$ is univalent for each θ $(0 < \theta < 2\pi)$. □

Lemma 4.2. *If* $p(z) \in \mathcal{P}'\left(\frac{\alpha - m}{n}\right)$ *with* $m \leq \alpha < m + n, m \in \mathbb{N}_0$, *and* $n \in \mathbb{N}$, *then*

$$\left| \{z(p(z) * h_\theta(z))\}' \right| \geq \frac{1 - r}{1 + r} \tag{4.2}$$

for $|z| = r < 1$ *and* $0 < \theta < 2\pi$.

Proof. Using the expression (4.1) for $\left|\{z(p(z) * h_\theta(z))\}'\right|$, we define

$$F(w) = e^{-i\theta}(1 - e^{i\theta}) \left\{ \frac{1 + \frac{1}{n}(2m + n - 2\alpha)e^{i\theta}}{1 - e^{i\theta}} - w \right\},$$

where

$$w = \frac{1 + \frac{1}{n}\left[2m + n - 2\alpha\right] re^{it}}{1 - re^{it}} \qquad (0 \le t \le 2\pi).$$

Then the function $F(w)$ may be rewritten as

$$\begin{aligned} F(w) &= e^{-i\theta} \left\{ \left(1 + \frac{1}{n}(2m + n - 2\alpha)e^{i\theta} - (1 - e^{i\theta})w \right) \right\} \\ &= e^{-i\theta} \left\{ (1 - w) + \left[\frac{1}{n}(2m + n - 2\alpha) + w \right] e^{i\theta} \right\} \\ &= \left[\frac{1}{n}(2m + n - 2\alpha) + w \right] e^{-i\theta} \left\{ \frac{1 - w}{\frac{1}{n}(2m + n - 2\alpha) + w} + e^{i\theta} \right\} \end{aligned}$$

for $0 < \theta < 2\pi$. Thus we see that

$$\begin{aligned} |F(w)| &= \left| \frac{1}{n}(2m + n - 2\alpha) + w \right| \left| \frac{1 - w}{\frac{1}{n}(2m + n - 2\alpha) + w} + e^{i\theta} \right| \\ &= \left| \frac{1}{n}(2m + n - 2\alpha) + w \right| \left| e^{i\theta} - re^{it} \right| \\ &= \left| \frac{1}{n}(2m + n - 2\alpha) + w \right| \left| 1 - re^{i(t-\theta)} \right| \\ &\ge \left| \frac{1}{n}(2m + n - 2\alpha) + w \right| (1 - r). \end{aligned}$$

Since

$$\left|\frac{1}{n}(2m+n-2\alpha)+w\right| = \left|\frac{1}{n}(2m+n-2\alpha)+\frac{1+\frac{1}{n}(2m+n-2\alpha)re^{it}}{1-re^{it}}\right|$$
$$= \left|\frac{1+\frac{1}{n}(2m+n-2\alpha)}{1-re^{it}}\right|$$
$$\geq \frac{1+\frac{1}{n}(2m+n-2\alpha)}{1+r},$$

it is clear that

$$|F(w)| \geq \frac{(1-r)}{(1+r)}\left[1+\frac{1}{n}(2m+n-2\alpha)\right].$$

Since $p \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ and (4.1) holds, by letting $w=[zp(z)]'$, we get the desired inequality, That is,

$$\left|[zp(z)]'\right| \geq \frac{n}{2(m+n-\alpha)}\cdot\frac{1+\frac{1}{n}(2m+n-2\alpha)}{1+r}(1-r)$$
$$= \frac{(1-r)}{(1+r)}.$$

Therefore, the lemma is proved. □

Further, we need the following lemma.

Lemma 4.3. *If* $p(z) \in \mathcal{P}'(\frac{\alpha-m}{n})$ *with* $m \leq \alpha < m+n$, $m \in \mathbb{N}_0$, *and* $n \in \mathbb{N}$, *then*

(4.3) $$|p(z) * h_\theta(z)| \geq \delta \quad (0<\theta<2\pi; z \in \mathbb{U}),$$

where

$$\delta = \int_0^1 \frac{2}{1+t}dt - 1 = 2\ln 2 - 1.$$

Proof. Since Lemma 4.1 shows that $z(p(z) * h_\theta(z))$ is univalent for each θ $(0 < \theta < 2\pi)$ for $p(z)$ belonging to the class $\mathcal{P}'\left(\frac{\alpha-m}{n}\right)$, we can choose a point $z_0 \in \mathbb{U}$ with $|z_0| = r < 1$ such that

$$\min_{|z|=r} |z(p(z) * h_\theta(z))| = |z_0(p(z_0) * h_\theta(z_0))|$$

for fixed r $(0 < r < 1)$. Then the pre-image γ of the line segment from 0 to $z_0(p(z_0) * h_\theta(z_0))$ is an arc inside $|z| \leq r$. Hence, for $|z| \leq r$, we have that

$$\begin{aligned} |z(p(z) * h_\theta(z))| &\geq |z_0(p(z_0) * h_\theta(z_0))| \\ &= \int_\gamma |(z(p(z) * h_\theta(z)))'| \, |dz| \,. \end{aligned}$$

An application of Lemma 4.2 leads us to

$$|p(z) * h_\theta(z)| \geq \frac{1}{r}\int_0^r \frac{1-t}{1+t}dt = \frac{1}{r}\int_0^r \frac{2}{1+t}dt - 1.$$

Note that the function $\Omega(r)$ defined by

$$\Omega(r) = \frac{1}{r}\int_0^r \frac{2}{1+t}dt - 1$$

is decreasing for r $(0 < r < 1)$. Therefore, we have

$$|p(z) * h_\theta(z)| \geq \delta = \int_0^1 \frac{2}{1+t}dt - 1 = 2\ln 2 - 1,$$

which completes the proof of Lemma 4.3. □

Now, we give the statement and the proof of our main result.

Theorem 4.4. *If* $p(z) \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$ *with* $m \leq \alpha < m+n, m \in \mathbb{N}_0$, *and* $n \in \mathbb{N}$, *then*

$$\mathcal{N}_{\beta\delta}(p(z)) \subset \mathcal{P}\left(\frac{\alpha-m}{n}\right),$$

where $\beta = \frac{m+n-\alpha}{n}$ *and*

$$\delta = \int_0^1 \frac{2}{1+t}dt - 1 = 2\ln 2 - 1. \tag{4.4}$$

The result is sharp.

Proof. Let $q(z) = 1 + \sum_{k=1}^{\infty} q_k z^k$. Then, by the definition of neighborhoods, we have to prove that if $q(z) \in \mathcal{N}_{\beta\delta}(p(z))$ for $p(z) \in \mathcal{P}'\left(\frac{\alpha-m}{n}\right)$, then $q(z)$ belongs to the class $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$. Using Lemma 4.3 and the inequality

$$\sum_{k=1}^{\infty} |p_k - q_k| \leq \delta,$$

we get

$$\begin{aligned}
|h_\theta(z) * q(z)| &\geq |h_\theta(z) * p(z)| - |h_\theta(z) * (p(z) - q(z))| \\
&\geq \delta - \left| \sum_{k=1}^{\infty} \frac{n(1-e^{i\theta})}{2(\alpha-m-n)e^{i\theta}} (p_k - q_k) z^k \right| \\
&> \delta - \frac{n}{m+n-\alpha} \sum_{k=1}^{\infty} |p_k - q_k|
\end{aligned}$$

$$> \delta - \frac{n}{m+n-\alpha}\left\{\frac{m+n-\alpha}{n}\right\}\delta$$
$$\geq \delta - \delta = 0.$$

Since $h_\theta(z) * q(z) \neq 0$ for $0 < \theta < 2\pi$ and $z \in \mathbb{U}$, we conclude that $q(z)$ belongs to the class $\mathcal{P}\left(\frac{\alpha-m}{n}\right)$, that is, that $\mathcal{N}_{\beta\delta}(p(z)) \subset P(\frac{\alpha-m}{n})$.

Further, taking the function $p(z)$ defined by

$$(zp(z))' = \frac{1 - \frac{1}{n}\left\{2\alpha - (2m+n)\right\} z}{1-z},$$

we have

$$p(z) = \frac{1}{n}(2\alpha - (2m+n)) + \frac{\frac{2}{n}(m+n-\alpha)}{z}\left\{\int_0^z \frac{1}{1-t} dt\right\}.$$

If we define the function $q(z)$ by

$$q(z) = p(z) + \left(\frac{m+n-\alpha}{n}\right)\delta z,$$

then $q(z) \in \mathcal{N}_{\beta\delta}(p(z))$. Letting $z = e^{i\pi}$, we see that $q(z) = q(e^{i\pi}) = \frac{\alpha-m}{n}$. This implies that if

$$\delta > \int_0^1 \frac{2}{1+t} dt - 1,$$

then $q(e^{i\pi}) < \frac{\alpha-m}{n}$. Therefore, $\operatorname{Re}\{q(z)\} < \frac{\alpha-m}{n}$ for z near $e^{i\pi}$, which contradicts $q(z) \in \mathcal{P}(\frac{\alpha-m}{n})$ (otherwise $\operatorname{Re}\{q(z)\} > \frac{\alpha-m}{n}$; $z \in \mathbb{U}$). Consequently, the result of the theorem is sharp. $\square$

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