

# BEURLING-HÖRMANDER UNCERTAINTY PRINCIPLE FOR THE SPHERICAL MEAN OPERATOR



Uncertainty Principle for  
the Spherical Mean Operator

N. Msehli and L.T. Rachdi

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*Abstract:* We establish the Beurling-Hörmander theorem for the Fourier transform connected with the spherical mean operator. Applying this result, we prove the Gelfand-Shilov and Cowling-Price type theorems for this transform.

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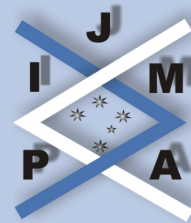
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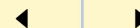
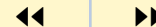
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## 1. Introduction

Uncertainty principles play an important role in harmonic analysis and have been studied by many authors, from many points of view [13, 19]. These principles state that a function  $f$  and its Fourier transform  $\hat{f}$  cannot be simultaneously sharply localized. Many aspects of such principles have been studied, for example the Heisenberg-Pauli-Weyl inequality [16] has been established for various Fourier transforms [26, 31, 32] and several generalized forms of this inequality are given in [28, 29, 30]. See also the theorems of Hardy, Morgan, Beurling and Gelfand-Shilov [7, 15, 23, 25, 26]. The most recent Beurling-Hörmander theorem has been proved by Hörmander [20] using an idea of Beurling [3]. This theorem states that if  $f$  is an integrable function on  $\mathbb{R}$  with respect to the Lebesgue measure and if

$$\iint_{\mathbb{R}^2} |f(x)||\hat{f}(y)|e^{|xy|} dx dy < +\infty,$$

then  $f = 0$  almost everywhere.

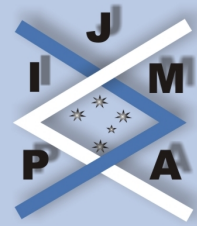
A strong multidimensional version of this theorem has been established by Bonami, Demange and Jaming [4] (see also [19]) who have showed that if  $f$  is a square integrable function on  $\mathbb{R}^n$  with respect to the Lebesgue measure, then

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x)||\hat{f}(y)|}{(1 + |x| + |y|)^d} e^{\langle x/y \rangle} dx dy < +\infty, \quad d \geq 0;$$

if and only if  $f$  can be written as

$$f(x) = P(x)e^{-\langle Ax/x \rangle},$$

where  $A$  is a real positive definite symmetric matrix and  $P$  is a polynomial with  $\text{degree}(P) < \frac{d-n}{2}$ .



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In particular for  $d \leq n$ ;  $f$  is identically zero.

The Beurling-Hörmander uncertainty principle has been studied by many authors for various Fourier transforms. In particular, Trimèche [33] has shown this uncertainty principle for the Dunkl transform, Kamoun and Trimèche [21] have proved an analogue of the Beurling-Hörmander theorem for some singular partial differential operators, Bouattour and Trimèche [5] have shown this theorem for the hypergroup of Chébli-Trimèche. We cite also Yakubovich [37], who has established the same result for the Kontorovich-Lebedev transform.

Many authors are interested in the Beurling-Hörmander uncertainty principle because this principle implies other well known quantitative uncertainty principles such as those of Gelfand-Shilov [14], Cowling Price [7], Morgan [2, 23], and the one of Hardy [15].

On the other hand, the spherical mean operator is defined on  $\mathcal{C}_*(\mathbb{R} \times \mathbb{R}^n)$  (the space of continuous functions on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable) by

$$\mathcal{R}(f)(r, x) = \int_{S^n} f(r\eta, x + r\xi) d\sigma_n(\eta, \xi),$$

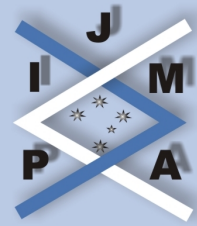
where  $S^n$  is the unit sphere  $\{(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n; \eta^2 + |\xi|^2 = 1\}$  in  $\mathbb{R} \times \mathbb{R}^n$  and  $\sigma_n$  is the surface measure on  $S^n$  normalized to have total measure one.

The dual operator  ${}^t\mathcal{R}$  of  $\mathcal{R}$  is defined by

$${}^t\mathcal{R}(g)(r, x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} g\left(\sqrt{r^2 + |x - y|^2}, y\right) dy,$$

where  $dy$  is the Lebesgue measure on  $\mathbb{R}^n$ .

The spherical mean operator  $\mathcal{R}$  and its dual  ${}^t\mathcal{R}$  play an important role and have many applications, for example; in the image processing of so-called synthetic aperture radar (SAR) data [17, 18], or in the linearized inverse scattering problem in



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acoustics [11]. These operators have been studied by many authors from many points of view [1, 8, 11, 24, 27].

In [24] (see also [8, 27]); the second author with others, associated to the spherical mean operator  $\mathcal{R}$  the Fourier transform  $\mathcal{F}$  defined by

$$\mathcal{F}(f)(\mu, \lambda) = \int_0^\infty \int_{\mathbb{R}^n} f(r, x) \varphi_{\mu, \lambda}(r, x) d\nu_n(r, x),$$

where

- $\varphi_{\mu, \lambda}(r, x) = \mathcal{R}(\cos(\mu \cdot) e^{-i\langle \lambda / \cdot \rangle})(r, x)$
- $d\nu_n$  is the measure defined on  $[0, +\infty[ \times \mathbb{R}^n$  by

$$d\nu_n(r, x) = \frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})} r^n dr \otimes \frac{dx}{(2\pi)^{\frac{n}{2}}}.$$

They have constructed the harmonic analysis related to the transform  $\mathcal{F}$  (inversion formula, Plancherel formula, Paley-Wiener theorem, Plancherel theorem).

Our purpose in the present work is to study the Beurling-Hörmander uncertainty principle for the Fourier transform  $\mathcal{F}$ , from which we derive the Gelfand-Shilov and Cowling-Price type theorems for this transform.

More precisely, we collect some basic harmonic analysis results for the Fourier transform  $\mathcal{F}$ .

In the third section, we establish the main result of this paper, that is, from the Beurling Hörmander theorem:

- Let  $f$  be a measurable function on  $\mathbb{R} \times \mathbb{R}^n$ ; even with respect to the first variable and such that  $f \in L^2(d\nu_n)$ . If

$$\iint_{\Gamma^+} \int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| \frac{|\mathcal{F}(f)(\mu, \lambda)| e^{|\langle r, x \rangle| |\theta(\mu, \lambda)|}}{(1 + |\langle r, x \rangle| + |\theta(\mu, \lambda)|)^d} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda) < +\infty; \quad d \geq 0,$$



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then

- i. For  $d \leq n + 1$ ;  $f = 0$ ;
- ii. For  $d > n + 1$ ; there exists a positive constant  $a$  and a polynomial  $P$  on  $\mathbb{R} \times \mathbb{R}^n$  even with respect to the first variable, such that

$$f(r, x) = P(r, x)e^{-a(r^2+|x|^2)}$$

with  $\text{degree}(P) < \frac{d-(n+1)}{2}$ ;

where

- $\Gamma_+$  is the set given by

$$\Gamma_+ = [0, +\infty[\times\mathbb{R}^n \cup \{(i\mu, \lambda); (\mu, \lambda) \in \mathbb{R} \times \mathbb{R}^n; 0 \leq \mu \leq |\lambda|\}$$

- $\theta$  is the bijective function defined on  $\Gamma_+$  by

$$\theta(\mu, \lambda) = \left( \sqrt{\mu^2 + |\lambda|^2}, \lambda \right)$$

- $d\tilde{\gamma}_n$  is the measure defined on  $\Gamma_+$  by

$$\begin{aligned} \iint_{\Gamma_+} g(\mu, \lambda) d\tilde{\gamma}_n(\mu, \lambda) &= \sqrt{\frac{2}{\pi}} \frac{1}{(2\pi)^{\frac{n}{2}}} \\ &\times \left[ \int_0^\infty \int_{\mathbb{R}^n} g(\mu, \lambda) \frac{\mu d\mu d\lambda}{\sqrt{\mu^2 + \lambda^2}} + \int_{\mathbb{R}^n} \int_0^{|\lambda|} g(i\mu, \lambda) \frac{\mu d\mu d\lambda}{\sqrt{\lambda^2 - \mu^2}} \right]. \end{aligned}$$

The last section of this paper is devoted to the Gelfand-Shilov and Cowling Price theorems for the transform  $\mathcal{F}$ .



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- Let  $p, q$  be two conjugate exponents;  $p, q \in ]1, +\infty[$ . Let  $\eta, \xi$  be two positive real numbers such that  $\xi\eta \geq 1$ . Let  $f$  be a measurable function on  $\mathbb{R} \times \mathbb{R}^n$ ; even with respect to the first variable such that  $f \in L^2(d\nu_n)$ .

If

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{\frac{\xi^p |(r, x)|^p}{p}}}{(1 + |(r, x)|)^d} d\nu_n(r, x) < +\infty$$

and

$$\iint_{\Gamma^+} \frac{|\mathcal{F}(f)(\mu, \lambda)| e^{\frac{\xi^q |(\mu, \lambda)|^q}{q}}}{(1 + |(\mu, \lambda)|)^d} d\tilde{\gamma}_n(\mu, \lambda) < +\infty; \quad d \geq 0,$$

then

- For  $d \leq \frac{n+1}{2}$ ;  $f = 0$ .
  - For  $d > \frac{n+1}{2}$ ; we have
    - $f = 0$  for  $\xi\eta > 1$
    - $f = 0$  for  $\xi\eta = 1$  and  $p \neq 2$
    - $f(r, x) = P(r, x)e^{-a(r^2+|x|^2)}$  for  $\xi\eta = 1$  and  $p = q = 2$ , where  $a > 0$  and  $P$  is a polynomial on  $\mathbb{R} \times \mathbb{R}^n$  even with respect to the first variable, with degree  $(P) < d - \frac{n+1}{2}$ .
- Let  $\eta, \xi, w_1$  and  $w_2$  be non negative real numbers such that  $\eta\xi \geq \frac{1}{4}$ . Let  $p, q$  be two exponents,  $p, q \in [1, +\infty]$  and let  $f$  be a measurable function on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable such that  $f \in L^2(d\nu_n)$ .

If

$$\left\| \frac{e^{\xi|(\cdot, \cdot)|^2}}{(1 + |(\cdot, \cdot)|)^{w_1}} \right\|_{p, \nu_n} < +\infty$$

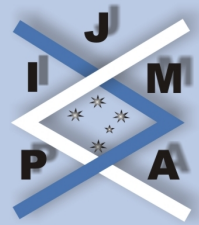
and

$$\left\| \frac{e^{\eta|\theta(\cdot, \cdot)|^2}}{(1 + |\theta(\cdot, \cdot)|)^{w_2}} \mathcal{F}(f) \right\|_{q, \tilde{\gamma}_n} < +\infty,$$

then

- i. For  $\xi\eta > \frac{1}{4}$ ;  $f = 0$ .
- ii. For  $\xi\eta = \frac{1}{4}$ ; there exists a positive constant  $a$  and a polynomial  $P$  on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable such that

$$f(r, x) = P(r, x)e^{-a(r^2 + |x|^2)}.$$



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## 2. The Spherical Mean Operator

For all  $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n$ ; if we denote by  $\varphi_{\mu, \lambda}$  the function defined by

$$\varphi_{\mu, \lambda}(r, x) = \mathcal{R} \left( \cos(\mu \cdot) e^{-i\langle \lambda / \cdot \rangle} \right) (r, x),$$

then we have

$$(2.1) \quad \varphi_{\mu, \lambda}(r, x) = j_{\frac{n-1}{2}} \left( r \sqrt{\mu^2 + \lambda^2} \right) e^{-i\langle \lambda / x \rangle},$$

where

- $\lambda^2 = \lambda_1^2 + \dots + \lambda_n^2$ ;  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ ;
- $\langle \lambda / x \rangle = \lambda_1 x_1 + \dots + \lambda_n x_n$ ;  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ ;
- $j_{\frac{n-1}{2}}$  is the modified Bessel function given by

$$(2.2) \quad \begin{aligned} j_{\frac{n-1}{2}}(s) &= 2^{\frac{n-1}{2}} \Gamma \left( \frac{n+1}{2} \right) \frac{J_{\frac{n-1}{2}}(s)}{s^{\frac{n-1}{2}}} \\ &= \Gamma \left( \frac{n+1}{2} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k + \frac{n+1}{2})} \left( \frac{s}{2} \right)^{2k}; \end{aligned}$$

and  $J_{\frac{n-1}{2}}$  is the usual Bessel function of first kind and order  $\frac{n-1}{2}$  [9, 10, 22, 36].

Also, the modified Bessel function  $j_{\frac{n-1}{2}}$  has the following integral representation, for all  $z \in \mathbb{C}$ :

$$j_{\frac{n-1}{2}}(z) = \frac{2\Gamma(\frac{n+1}{2})}{\sqrt{\pi} \Gamma(\frac{n}{2})} \int_0^1 (1-t^2)^{\frac{n}{2}-1} \cos(zt) dt.$$

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Thus, for all  $z \in \mathbb{C}$ ; we have

$$(2.3) \quad \left| j_{\frac{n-1}{2}}(z) \right| \leq e^{|\operatorname{Im} z|}.$$

Using the relation (2.1) and the properties of the function  $j_{\frac{n-1}{2}}$ , we deduce that the function  $\varphi_{\mu,\lambda}$  satisfies the following properties [24, 27]:

•

$$(2.4) \quad \sup_{(r,x) \in \mathbb{R} \times \mathbb{R}^n} |\varphi_{\mu,\lambda}(r, x)| = 1$$

if and only if  $(\mu, \lambda)$  belongs to the set  $\Gamma$  defined by

$$(2.5) \quad \Gamma = \mathbb{R} \times \mathbb{R}^n \cup \{(i\mu, \lambda); (\mu, \lambda) \in \mathbb{R} \times \mathbb{R}^n; |\mu| \leq |\lambda|\}.$$

• For all  $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n$ ; the function  $\varphi_{\mu,\lambda}$  is a unique solution of the system

$$\begin{cases} \frac{\partial u}{\partial x_j}(r, x) = -i \lambda_j u(r, x); 1 \leq j \leq n \\ Lu(r, x) = -\mu^2 u(r, x) \\ u(0, 0) = 1; \frac{\partial u}{\partial r}((0, x_1, \dots, x_n) = 0; \forall (x_1, \dots, x_n) \in \mathbb{R}^n \end{cases}$$

where

$$L = \frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r} - \sum_{j=1}^n \left( \frac{\partial}{\partial x_j} \right)^2.$$

In the following, we denote by

•  $dm_{n+1}$  the measure defined on  $[0, +\infty[ \times \mathbb{R}^n$ ; by

$$dm_{n+1}(r, x) = \sqrt{\frac{2}{\pi}} \frac{1}{(2\pi)^{\frac{n}{2}}} dr \otimes dx,$$



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where  $dx$  is the Lebesgue measure on  $\mathbb{R}^n$ .

- $L^p(dm_{n+1})$ ;  $p \in [1, +\infty]$ , the space of measurable functions  $f$  on  $[0, +\infty[ \times \mathbb{R}^n$  satisfying

$$\|f\|_{p, m_{n+1}} = \begin{cases} \left( \int_0^\infty \int_{\mathbb{R}^n} |f(r, x)|^p dm_{n+1}(r, x) \right)^{\frac{1}{p}} < +\infty, & \text{if } 1 \leq p < +\infty; \\ \text{ess sup}_{(r, x) \in [0, +\infty[ \times \mathbb{R}^n} |f(r, x)| < +\infty, & \text{if } p = +\infty. \end{cases}$$

- $d\nu_n$  the measure defined on  $[0, +\infty[ \times \mathbb{R}^n$  by

$$d\nu_n(r, x) = \frac{r^n dr}{2^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})} \otimes \frac{dx}{(2\pi)^{\frac{n}{2}}}.$$

- $L^p(d\nu_n)$ ,  $p \in [1, +\infty]$ , the space of measurable functions  $f$  on  $[0, +\infty[ \times \mathbb{R}^n$  such that  $\|f\|_{p, \nu_n} < +\infty$ .
- $\Gamma_+$  the subset of  $\Gamma$ , given by

$$\Gamma_+ = [0, +\infty[ \times \mathbb{R}^n \cup \{(i\mu, \lambda); (\mu, \lambda) \in \mathbb{R} \times \mathbb{R}^n; 0 \leq \mu \leq |\lambda|\}.$$

- $\mathcal{B}_{\Gamma_+}$  the  $\sigma$ -algebra defined on  $\Gamma_+$  by

$$(2.6) \quad \mathcal{B}_{\Gamma_+} = \{\theta^{-1}(B); B \in \mathcal{B}or([0, +\infty[ \times \mathbb{R}^n)\},$$

where  $\theta$  is the bijective function defined on  $\Gamma_+$  by

$$\theta(\mu, \lambda) = \left( \sqrt{\mu^2 + |\lambda|^2}, \lambda \right).$$



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- $d\gamma_n$  the measure defined on  $\mathcal{B}_{\Gamma_+}$  by

$$\forall A \in \mathcal{B}_{\Gamma_+}; \gamma_n(A) = \nu_n(\theta(A)).$$

- $L^p(d\gamma_n)$ ,  $p \in [1, +\infty]$ , the space of measurable functions  $g$  on  $\Gamma_+$  such that  $\|g\|_{p, \gamma_n} < +\infty$ .

- $d\tilde{\gamma}_n$  the measure defined on  $\mathcal{B}_{\Gamma_+}$  by

$$d\tilde{\gamma}_n(\mu, \lambda) = \frac{2^{\frac{n}{2}} \Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}} \frac{d\gamma_n(\mu, \lambda)}{(\mu^2 + |\lambda|^2)^{\frac{n}{2}}}.$$

- $L^p(d\tilde{\gamma}_n)$ ,  $p \in [1, +\infty]$ , the space of measurable functions  $g$  on  $\Gamma_+$  such that  $\|g\|_{p, \tilde{\gamma}_n} < +\infty$ .

- $S_*(\mathbb{R} \times \mathbb{R}^n)$  the Schwarz space formed by the infinitely differentiable functions on  $\mathbb{R} \times \mathbb{R}^n$ , rapidly decreasing together with all their derivatives, and even with respect to the first variable.

### Proposition 2.1.

- i. For all non negative measurable functions  $g$  on  $\Gamma_+$  (respectively integrable on  $\Gamma_+$  with respect to the measure  $d\gamma_n$ ), we have*

$$\begin{aligned} & \iint_{\Gamma_+} g(\mu, \lambda) d\gamma_n(\mu, \lambda) \\ &= \frac{1}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) (2\pi)^{\frac{n}{2}}} \left( \int_0^\infty \int_{\mathbb{R}^n} g(\mu, \lambda) (\mu^2 + |\lambda|^2)^{\frac{n-1}{2}} \mu d\mu d\lambda \right. \\ & \quad \left. + \int_{\mathbb{R}^n} \int_0^\infty g(i\mu, \lambda) (|\lambda|^2 - \mu^2)^{\frac{n-1}{2}} \mu d\mu d\lambda \right). \end{aligned}$$



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ii. For all non negative measurable functions  $f$  on  $[0, +\infty[\times\mathbb{R}^n$  (respectively integrable on  $[0, +\infty[\times\mathbb{R}^n$  with respect to the measure  $dm_{n+1}$ ), the function  $f \circ \theta$  is measurable on  $\Gamma_+$  (respectively integrable on  $\Gamma_+$  with respect to the measure  $d\gamma_n$ ) and we have

$$\iint_{\Gamma_+} f \circ \theta(\mu, \lambda) d\gamma_n(\mu, \lambda) = \int_0^\infty \int_{\mathbb{R}^n} f(r, x) d\nu_n(r, x).$$

iii. For all non negative measurable functions  $f$  on  $[0, +\infty[\times\mathbb{R}^n$  (respectively integrable on  $[0, +\infty[\times\mathbb{R}^n$  with respect to the measure  $dm_{n+1}$ ), we have

$$(2.7) \quad \iint_{\Gamma_+} f \circ \theta(\mu, \lambda) d\tilde{\gamma}_n(\mu, \lambda) = \int_0^\infty \int_{\mathbb{R}^n} f(r, x) dm_{n+1}(r, x),$$

where  $\theta$  is the function given by the relation (2.6).

In the sequel, we shall define the Fourier transform associated with the spherical mean operator and give some properties.

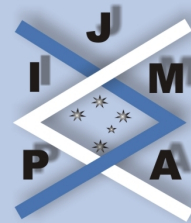
**Definition 2.2.** The Fourier transform  $\mathcal{F}$  associated with the spherical mean operator is defined on  $L^1(d\nu_n)$  by

$$\forall(\mu, \lambda) \in \Gamma; \quad \mathcal{F}(f)(\mu, \lambda) = \int_0^\infty \int_{\mathbb{R}^n} f(r, x) \varphi_{\mu, \lambda}(r, x) d\nu_n(r, x),$$

where  $\varphi_{\mu, \lambda}$  is the function given by the relation (2.1) and  $\Gamma$  is the set defined by (2.5).

**Remark 1.** For all  $(\mu, \lambda) \in \Gamma$ , we have

$$(2.8) \quad \mathcal{F}(f)(\mu, \lambda) = \tilde{\mathcal{F}}(f) \circ \theta(\mu, \lambda),$$



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where

$$(2.9) \quad \tilde{\mathcal{F}}(f)(\mu, \lambda) = \int_0^\infty \int_{\mathbb{R}^n} f(r, x) j_{\frac{n-1}{2}}(r\mu) e^{-i\langle \lambda/x \rangle} d\nu_n(r, x)$$

and  $j_{\frac{n-1}{2}}$  is the modified Bessel function given by the relation (2.2).

Moreover, by the relation (2.4), the Fourier transform  $\mathcal{F}$  is a bounded linear operator from  $L^1(d\nu_n)$  into  $L^\infty(d\gamma_n)$  and for all  $f \in L^1(d\nu_n)$ :

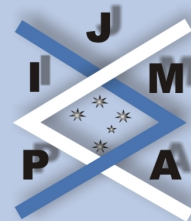
$$(2.10) \quad \|\mathcal{F}(f)\|_{\infty, \gamma_n} \leq \|f\|_{1, \nu_n}.$$

**Theorem 2.3 (Inversion formula).** *Let  $f \in L^1(d\nu_n)$  such that  $\mathcal{F}(f) \in L^1(d\gamma_n)$ , then for almost every  $(r, x) \in [0, +\infty[\times\mathbb{R}^n$ , we have*

$$(2.11) \quad \begin{aligned} f(r, x) &= \iint_{\Gamma^+} \mathcal{F}(f)(\mu, \lambda) \overline{\varphi_{\mu, \lambda}(r, x)} d\gamma_n(\mu, \lambda) \\ &= \int_0^\infty \int_{\mathbb{R}^n} \tilde{\mathcal{F}}(f)(\mu, \lambda) j_{\frac{n-1}{2}}(r\mu) e^{i\langle \lambda/x \rangle} d\nu_n(\mu, \lambda). \end{aligned}$$

**Lemma 2.4.** *Let  $\mathcal{R}_{\frac{n-1}{2}}$  be the mapping defined for all non negative measurable functions  $g$  on  $[0, +\infty[\times\mathbb{R}^n$  by*

$$(2.12) \quad \begin{aligned} \mathcal{R}_{\frac{n-1}{2}}(g)(r, x) &= \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)} r^{1-n} \int_0^r (r^2 - t^2)^{\frac{n}{2}-1} g(t, x) dt \\ &= \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)} \int_0^1 (1 - t^2)^{\frac{n}{2}-1} g(tr, x) dt, \end{aligned}$$



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then for all non negative measurable functions  $f, g$  on  $[0, +\infty[ \times \mathbb{R}^n$ , we have

$$(2.13) \quad \int_0^\infty \int_{\mathbb{R}^n} \mathcal{R}_{\frac{n-1}{2}}(g)(r, x) f(r, x) d\nu_n(r, x) \\ = \int_0^\infty \int_{\mathbb{R}^n} g(t, x) \mathcal{W}_{\frac{n-1}{2}}(f)(t, x) dm_{n+1}(t, x)$$

where  $\mathcal{W}_{\frac{n-1}{2}}$  is the classical Weyl transform defined for all non negative measurable functions  $g$  on  $[0, +\infty[ \times \mathbb{R}^n$  by

$$(2.14) \quad \mathcal{W}_{\frac{n-1}{2}}(f)(t, x) = \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \int_t^\infty (r^2 - t^2)^{\frac{n}{2}-1} f(r, x) 2r dr.$$

**Proposition 2.5.** For all  $f \in L^1(d\nu_n)$ , the function  $\mathcal{W}_{\frac{n-1}{2}}(f)$  given by the relation (2.14) is defined almost every where, belongs to the space  $L^1(dm_{n+1})$  and we have

$$(2.15) \quad \left\| \mathcal{W}_{\frac{n-1}{2}}(f) \right\|_{1, m_{n+1}} \leq \|f\|_{1, \nu_n}.$$

Moreover,

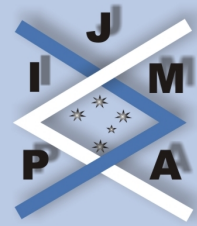
$$(2.16) \quad \tilde{\mathcal{F}}(f)(\mu, \lambda) = \Lambda_{n+1} \circ \mathcal{W}_{\frac{n-1}{2}}(f)(\mu, \lambda),$$

where  $\Lambda_{n+1}$  is the usual Fourier cosine transform defined on  $L^1(dm_{n+1})$  by

$$\Lambda_{n+1}(g)(\mu, \lambda) = \int_0^\infty \int_{\mathbb{R}^n} g(r, x) \cos(r\mu) e^{-i\langle \lambda, x \rangle} dm_{n+1}(r, x).$$

and  $\tilde{\mathcal{F}}$  is the Fourier-Bessel transform defined by the relation (2.9).

**Remark 2.** It is well known [34, 35] that the Fourier transforms  $\tilde{\mathcal{F}}$  and  $\Lambda_{n+1}$  are topological isomorphisms from  $S_*(\mathbb{R} \times \mathbb{R}^n)$  onto itself. Then, by the relation (2.16),



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we deduce that the classical Weyl transform  $\mathscr{W}_{\frac{n-1}{2}}$  is also a topological isomorphism from  $S_*(\mathbb{R} \times \mathbb{R}^n)$  onto itself, and the inverse isomorphism is given by [24]

$$(2.17) \quad \mathscr{W}_{\frac{n-1}{2}}^{-1}(f)(r, x) = (-1)^{[\frac{n}{2}]+1} F_{[\frac{n}{2}]-\frac{n}{2}+1} \left( \left( \frac{\partial}{\partial t^2} \right)^{[\frac{n}{2}]+1} f \right) (r, x),$$

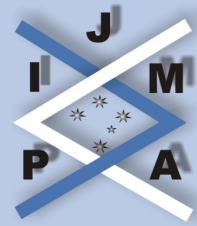
where  $F_a$ ;  $a > 0$  is the mapping defined on  $S_*(\mathbb{R} \times \mathbb{R}^n)$  by

$$(2.18) \quad F_a(f)(r, x) = \frac{1}{2^a \Gamma(a)} \int_r^\infty (t^2 - r^2)^{a-1} f(t, x) 2t dt$$

and  $\frac{\partial}{\partial r^2}$  is the singular partial differential operator defined by

$$\left( \frac{\partial}{\partial r^2} \right) f(r, x) = \frac{1}{r} \frac{\partial f(r, x)}{\partial r}.$$





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### 3. The Beurling-Hörmander Theorem for the Spherical Mean Operator

This section contains the main result of this paper, that is the Beurling-Hörmander theorems for the Fourier transform  $\mathcal{F}$  associated with the spherical mean operator.

We firstly recall the following result that has been established by Bonami, Demange and Jaming [4].

**Theorem 3.1.** *Let  $f$  be a measurable function on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable such that  $f \in L^2(dm_{n+1})$  and let  $d$  be a real number,  $d \geq 0$ . If*

$$\int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| |\Lambda_{n+1}(f)(s, y)|}{(1 + |(r, x)| + |(s, y)|)^d} e^{i|(r, x)||s, y|} dm_{n+1}(r, x) dm_{n+1}(s, y) < +\infty,$$

*then there exist a positive constant  $a$  and a polynomial  $P$  on  $\mathbb{R} \times \mathbb{R}^n$  even with respect to the first variable, such that*

$$f(r, x) = P(r, x) e^{-a(r^2 + |x|^2)},$$

*with  $\text{degree}(P) < \frac{d-(n+1)}{2}$ .*

*In particular,  $f = 0$  for  $d \leq (n + 1)$ .*

**Lemma 3.2.** *Let  $f \in L^2(d\nu_n)$  and let  $d$  be a real number,  $d \geq 0$ . If*

$$\iint_{\Gamma^+} \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| |\mathcal{F}(f)(\mu, \lambda)| e^{i|(r, x)||\theta(\mu, \lambda)|}}{(1 + |(r, x)| + |\theta(\mu, \lambda)|)^d} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda) < +\infty,$$

*then the function  $f$  belongs to the space  $L^1(d\nu_n)$ .*



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*Proof.* Let  $f \in L^2(d\nu_n)$ ,  $f \neq 0$ . From the relations (2.7) and (2.8), we obtain

$$\begin{aligned} & \iint_{\Gamma^+} \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| |\mathcal{F}(f)(\mu, \lambda)|}{(1 + |(r, x)| + |\theta(\mu, \lambda)|)^d} e^{|\theta(r, x)| |\theta(\mu, \lambda)|} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda) \\ &= \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| |\tilde{\mathcal{F}}(f)(\mu, \lambda)|}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} e^{|\theta(r, x)| |(\mu, \lambda)|} d\nu_n(r, x) dm_{n+1}(\mu, \lambda) < +\infty. \end{aligned}$$

Then for almost every  $(\mu, \lambda) \in [0, +\infty[ \times \mathbb{R}^n$ ,

$$\left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{|\theta(r, x)| |(\mu, \lambda)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} d\nu_n(r, x) < +\infty.$$

In particular, there exists  $(\mu_0, \lambda_0) \in [0, +\infty[ \times \mathbb{R}^n \setminus \{(0, 0)\}$  such that

$$\tilde{\mathcal{F}}(f)(\mu_0, \lambda_0) \neq 0 \text{ and } \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{|\theta(r, x)| |(\mu_0, \lambda_0)|}}{(1 + |(r, x)| + |(\mu_0, \lambda_0)|)^d} d\nu_n(r, x) < +\infty.$$

Let  $h$  be the function defined on  $[0, +\infty[$  by

$$h(s) = \frac{e^{s|(\mu_0, \lambda_0)|}}{(1 + s + |(\mu_0, \lambda_0)|)^d},$$

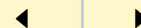
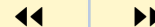
then the function  $h$  has an absolute minimum attained at:

$$s_0 = \begin{cases} \frac{d}{|(\mu_0, \lambda_0)|} - 1 - |(\mu_0, \lambda_0)|; & \text{if } \frac{d}{|(\mu_0, \lambda_0)|} > 1 + |(\mu_0, \lambda_0)|; \\ 0; & \text{if } \frac{d}{|(\mu_0, \lambda_0)|} \leq 1 + |(\mu_0, \lambda_0)|. \end{cases}$$



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Consequently,

$$\int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| d\nu_n(r, x) \leq \frac{1}{h(s_0)} \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{|(r, x)| |(\mu_0, \lambda_0)|}}{(1 + |(r, x)| + |(\mu_0, \lambda_0)|)^d} d\nu_n(r, x) < +\infty.$$

□

**Lemma 3.3.** *Let  $f \in L^2(d\nu_n)$  and let  $d$  be a real number,  $d \geq 0$ . If*

$$\iint_{\Gamma_+} \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| |\mathcal{F}(f)(\mu, \lambda)| e^{|(r, x)| |\theta(\mu, \lambda)|}}{(1 + |(r, x)| + |\theta(\mu, \lambda)|)^d} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda) < +\infty,$$

then there exists  $a > 0$  such that the function  $\tilde{\mathcal{F}}(f)$  is analytic on the set

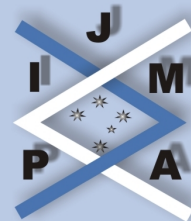
$$\{(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n; |\operatorname{Im} \mu| < a, |\operatorname{Im} \lambda_j| < a; \forall j \in \{1, \dots, n\}\}.$$

*Proof.* From the proof of Lemma 3.2, there exists  $(\mu_0, \lambda_0) \in [0, +\infty[ \times \mathbb{R}^n \setminus \{(0, 0)\}$  such that

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{|(r, x)| |(\mu_0, \lambda_0)|}}{(1 + |(r, x)| + |(\mu_0, \lambda_0)|)^d} d\nu_n(r, x) < +\infty.$$

Let  $a$  be a real number such that  $0 < (n + 1)a < |(\mu_0, \lambda_0)|$ . Then we have

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{|(r, x)| |(\mu_0, \lambda_0)|}}{(1 + |(r, x)| + |(\mu_0, \lambda_0)|)^d} d\nu_n(r, x) \\ &= \int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| e^{(n+1)a|(r, x)|} \frac{e^{|(r, x)| (|(\mu_0, \lambda_0)| - (n+1)a)}}{(1 + |(r, x)| + |(\mu_0, \lambda_0)|)^d} d\nu_n(r, x) < +\infty. \end{aligned}$$



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Let  $g$  be the function defined on  $[0, +\infty[$  by

$$g(s) = \frac{e^{s(|(\mu_0, \lambda_0)| - (n+1)a)}}{(1 + s + |(\mu_0, \lambda_0)|)^d},$$

then  $g$  admits a minimum attained at

$$s_0 = \begin{cases} \frac{d}{|(\mu_0, \lambda_0)| - (n+1)a} - 1 - |(\mu_0, \lambda_0)|; & \text{if } \frac{d}{|(\mu_0, \lambda_0)| - (n+1)a} > 1 + |(\mu_0, \lambda_0)|, \\ 0; & \text{if } \frac{d}{|(\mu_0, \lambda_0)| - (n+1)a} \leq 1 + |(\mu_0, \lambda_0)|. \end{cases}$$

Consequently,

$$(3.1) \quad \int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| e^{(n+1)a|(r, x)|} d\nu_n(r, x) \\ \leq \frac{1}{g(s_0)} \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{|(r, x)||(\mu_0, \lambda_0)|}}{(1 + |(\mu_0, \lambda_0)| + |(r, x)|)^d} d\nu_n(r, x) < +\infty.$$

On the other hand, from the relation (2.2), we deduce that for all  $(r, x) \in [0, +\infty[ \times \mathbb{R}^n$ ; the function

$$(\mu, \lambda) \longmapsto j_{\frac{n-1}{2}}(r\mu) e^{-i\langle \lambda/x \rangle}$$

is analytic on  $\mathbb{C} \times \mathbb{C}^n$  [6], even with respect to the first variable and by the relation (2.3), we deduce that  $\forall (r, x) \in [0, +\infty[ \times \mathbb{R}^n$ ,  $\forall (\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n$ ,

$$(3.2) \quad \left| j_{\frac{n-1}{2}}(r\mu) e^{-i\langle \lambda, x \rangle} \right| \leq e^{r|\operatorname{Im} \mu| + \sum_{j=1}^n |\operatorname{Im} \lambda_j| |x_j|} \\ \leq e^{|(r, x)| [|\operatorname{Im} \mu| + \sum_{j=1}^n |\operatorname{Im} \lambda_j|]}.$$

Then the result follows from the relations (2.9), (3.1), (3.2) and by the analyticity theorem.  $\square$



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**Corollary 3.4.** Let  $f \in L^2(d\nu_n)$ ,  $f \neq 0$  and let  $d$  be a real number,  $d \geq 0$ . If

$$\iint_{\Gamma_+} \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| |\mathcal{F}(f)(\mu, \lambda)|}{(1 + |(r, x)| + |\theta(\mu, \lambda)|)^d} e^{|(r, x)| |\theta(\mu, \lambda)|} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda) < +\infty,$$

then for all real numbers  $a$ ,  $a > 0$ , we have  $m_{n+1}(A_a) > 0$ , where

$$(3.3) \quad A_a = \left\{ (\mu, \lambda) \in \mathbb{R} \times \mathbb{R}^n; \tilde{\mathcal{F}}(f)(\mu, \lambda) \neq 0 \text{ and } |(\mu, \lambda)| > a \right\}.$$

*Proof.* Let  $f$  be a function satisfying the hypothesis. From Lemma 3.2, the function  $f$  belongs to  $L^1(d\nu_n)$  and consequently the function  $\tilde{\mathcal{F}}(f)$  is continuous on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable. Then for all  $a > 0$ , the set  $A_a$  given by the relation (3.3) is an open subset of  $\mathbb{R} \times \mathbb{R}^n$ .

So, if  $m_{n+1}(A_a) = 0$ , then this subset is empty. This means that for every  $(\mu, \lambda) \in \mathbb{R} \times \mathbb{R}^n$ ,  $|(\mu, \lambda)| > a$ , we have  $\tilde{\mathcal{F}}(f)(\mu, \lambda) = 0$ .

From Lemma 3.2, and by analytic continuation, we deduce that  $\tilde{\mathcal{F}}(f) = 0$ , and by the inversion formula (2.11), it follows that  $f = 0$ .  $\square$

*Remark 3.*

- i. Let  $f$  be a function satisfying the hypothesis of Corollary 3.4, then for all real numbers  $a$ ,  $a > 0$ , there exists  $(\mu_0, \lambda_0) \in [0, +\infty[ \times \mathbb{R}^n$  such that  $|(\mu_0, \lambda_0)| > a$  and

$$\int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| \frac{e^{|(r, x)| |(\mu_0, \lambda_0)|}}{(1 + |(r, x)| + |(\mu_0, \lambda_0)|)^d} d\nu_n(r, x) < +\infty.$$

- ii. Let  $d$  and  $\sigma$  be non negative real numbers,  $\sigma + \sigma^2 \geq d$ . Then the function

$$t \longmapsto \frac{e^{\sigma t}}{(1 + t + \sigma)^d}$$

is not decreasing on  $[0, +\infty[$ .

**Lemma 3.5.** *Let  $f$  be a measurable function on  $\mathbb{R} \times \mathbb{R}^n$  even with respect to the first variable, and  $f \in L^2(d\nu_n)$ . Let  $d$  be real number,  $d \geq 0$ . If*

$$\iint_{\Gamma^+} \int_0^\infty \int_{\mathbb{R}^n} |\mathcal{F}(f)(\mu, \lambda)| |f(r, x)| \frac{e^{|\langle r, x \rangle| |\theta(\mu, \lambda)|}}{(1 + |\langle r, x \rangle| + |\theta(\mu, \lambda)|)^d} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda) < +\infty,$$

then the function  $\mathcal{W}_{\frac{n-1}{2}}(f)$  defined by the relation (2.14) belongs to the space  $L^2(dm_{n+1})$ .

*Proof.* From the hypothesis and the relations (2.7) and (2.8), we have

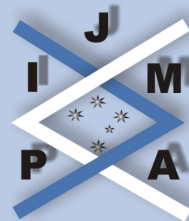
$$\begin{aligned} & \iint_{\Gamma^+} \int_0^\infty \int_{\mathbb{R}^n} |\mathcal{F}(f)(\mu, \lambda)| \frac{|f(r, x)| e^{|\langle r, x \rangle| |\theta(\mu, \lambda)|}}{(1 + |\langle r, x \rangle| + |\theta(\mu, \lambda)|)^d} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda) \\ &= \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \frac{|f(r, x)| e^{|\langle r, x \rangle| |\theta(\mu, \lambda)|}}{(1 + |\langle r, x \rangle| + |\theta(\mu, \lambda)|)^d} d\nu_n(r, x) dm_{n+1}(\mu, \lambda) \\ &< +\infty. \end{aligned}$$

In the same manner as the proof of the inequality (3.1) in Lemma 3.2, there exists  $b \in \mathbb{R}$ ,  $b > 0$  such that

$$\int_0^\infty \int_{\mathbb{R}^n} |\tilde{\mathcal{F}}(f)(\mu, \lambda)| e^{b|\langle \mu, \lambda \rangle|} dm_{n+1}(\mu, \lambda) < +\infty.$$

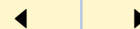
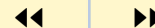
Consequently, the function  $\tilde{\mathcal{F}}(f)$  belongs to the space  $L^1(d\nu_n)$  and by the inversion formula for  $\tilde{\mathcal{F}}$ , we deduce that

$$f(r, x) = \int_0^\infty \int_{\mathbb{R}^n} \tilde{\mathcal{F}}(f)(\mu, \lambda) j_{\frac{n-1}{2}}(r\mu) e^{i\langle \lambda, x \rangle} d\nu_n(\mu, \lambda). \text{ a.e.}$$



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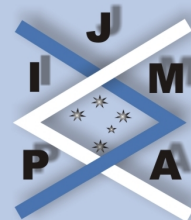


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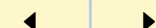
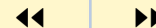
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In particular, the function  $f$  is bounded and

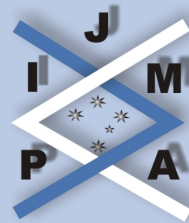
$$(3.4) \quad \|f\|_{\infty, \nu_n} \leq \left\| \tilde{\mathcal{F}}(f) \right\|_{1, \nu_n}.$$

By virtue of the relation (2.14), we get

$$\begin{aligned} \left| \mathcal{W}_{\frac{n-1}{2}}(f)(r, x) \right| &\leq \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \int_t^\infty (r^2 - t^2)^{\frac{n}{2}-1} |f(r, x)| 2r dr \\ &= \frac{r^n}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \int_1^\infty (y^2 - 1)^{\frac{n}{2}-1} |f(ry, x)| 2y dy. \end{aligned}$$

Using Minkowski's inequality for integrals [12], we get:

$$\begin{aligned} (3.5) \quad &\left( \int_0^\infty \int_{\mathbb{R}^n} \left| \mathcal{W}_{\frac{n-1}{2}}(f)(r, x) \right|^2 dm_{n+1}(r, x) \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \left[ \int_0^\infty \int_{\mathbb{R}^n} \left( \int_1^\infty r^n (y^2 - 1)^{\frac{n}{2}-1} |f(ry, x)| 2y dy \right)^2 dm_{n+1}(r, x) \right]^{\frac{1}{2}} \\ &\leq \frac{1}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} \int_1^\infty \left( \int_0^\infty \int_{\mathbb{R}^n} r^{2n} (y^2 - 1)^{n-2} |f(ry, x)|^2 dm_{n+1}(r, x) \right)^{\frac{1}{2}} 2y dy \\ &= \frac{1}{2^{\frac{n}{2}-1} \Gamma(\frac{n}{2})} \left[ \int_1^\infty (y^2 - 1)^{\frac{n}{2}-1} y^{-n+\frac{1}{2}} dy \right] \\ &\quad \times \left[ \int_0^\infty \int_{\mathbb{R}^n} s^{2n} |f(s, x)|^2 dm_{n+1}(s, x) \right]^{\frac{1}{2}} \\ &= \frac{\Gamma(\frac{1}{4})}{2^{\frac{n}{2}} \Gamma(\frac{2n+1}{4})} \left[ \int_0^\infty \int_{\mathbb{R}^n} s^{2n} |f(s, x)|^2 dm_{n+1}(s, x) \right]^{\frac{1}{2}}. \end{aligned}$$



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Using the relations (3.1), (3.4) and (3.5), we deduce that

$$\begin{aligned} \left\| \mathcal{W}_{\frac{n-1}{2}}(f) \right\|_{2, m_{n+1}} &= \left( \int_0^\infty \int_{\mathbb{R}^n} \left| \mathcal{W}_{\frac{n-1}{2}}(f) \right|^2(r, x) dm_{n+1}(r, x) \right)^{\frac{1}{2}} \\ &\leq K_n \int_0^\infty \int_{\mathbb{R}^n} |f(s, x)| e^{(n+1)a|(s,x)|} d\nu_n(s, x) < +\infty, \end{aligned}$$

where

$$K_n = \frac{\Gamma\left(\frac{1}{4}\right)}{2^{\frac{n}{2}} \Gamma\left(\frac{2n+1}{4}\right)} \left( \sqrt{\frac{\pi}{2}} \Gamma\left(\frac{n+1}{2}\right) 2^{\frac{n-1}{2}} \max_{s \geq 0} (s^n e^{-(n+1)as}) \|f\|_{\infty, \nu_n} \right)^{\frac{1}{2}}.$$

□

**Theorem 3.6.** Let  $f \in L^2(d\nu_n)$ ;  $f \neq 0$  and let  $d$  be a real number;  $d \geq 0$ .

If

$$\iint_{\Gamma^+} \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| |\mathcal{F}(f)(\mu, \lambda)| e^{|\theta(\mu, \lambda)|}}{(1 + |(r, x)| + |\theta(\mu, \lambda)|)^d} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda) < +\infty;$$

then

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \left| \mathcal{W}_{\frac{n-1}{2}}(f)(r, x) \right| \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \\ \times \frac{e^{|\theta(\mu, \lambda)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) dm_{n+1}(\mu, \lambda) < +\infty \end{aligned}$$

where  $\mathcal{W}_{\frac{n-1}{2}}$  is the Weyl transform defined by the relation (2.14).





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*Proof.* From the hypothesis, the relations (2.7), (2.8) and Fubini's theorem, we have

$$\begin{aligned}
 (3.6) \quad & \iint_{\Gamma^+} \int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| |\mathcal{F}(f)(\mu, \lambda)| \\
 & \quad \times \frac{e^{|(r,x)||\theta(\mu,\lambda)|}}{(1 + |(r, x)| + |\theta(\mu, \lambda)|)^d} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda) \\
 & = \int_0^\infty \int_{\mathbb{R}^n} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \\
 & \quad \times \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{|(r,x)||(\mu,\lambda)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} d\nu_n(r, x) dm_{n+1}(\mu, \lambda) \\
 & < +\infty.
 \end{aligned}$$

i. If  $d = 0$ , then by the relation (2.13) and Fubini's theorem, we get

$$\begin{aligned}
 (3.7) \quad & \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \left| \mathcal{W}_{\frac{n-1}{2}}(f)(r, x) \right| \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \\
 & \quad \times e^{|(r,x)||(\mu,\lambda)|} dm_{n+1}(r, x) dm_{n+1}(\mu, \lambda) \\
 & \leq \int_0^\infty \int_{\mathbb{R}^n} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \\
 & \quad \times \left( \int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_{\frac{n-1}{2}}(|f|)(r, x) e^{|(r,x)||(\mu,\lambda)|} dm_{n+1}(r, x) \right) dm_{n+1}(\mu, \lambda) \\
 & \leq \int_0^\infty \int_{\mathbb{R}^n} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \\
 & \quad \times \left( \int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| \mathcal{R}_{\frac{n-1}{2}}(e^{|\cdot|\cdot|})(\mu, \lambda)(r, x) d\nu_n(r, x) \right) dm_{n+1}(\mu, \lambda).
 \end{aligned}$$



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However, by (2.12), we deduce that for all  $(r, x) \in [0, +\infty[\times\mathbb{R}^n$ ,

$$(3.8) \quad \mathcal{R}_{\frac{n-1}{2}} \left( e^{|\cdot|} \right) \left( e^{|\cdot|} \right) (r, x) \leq e^{|(r,x)|}.$$

Combining the relations (3.6), (3.7) and (3.8), we deduce that

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \left| \mathcal{W}_{\frac{n-1}{2}}(f)(r, x) \right| \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| e^{|(r,x)|} dm_{n+1}(r, x) dm_{n+1}(\mu, \lambda) \\ & \leq \int_0^\infty \int_{\mathbb{R}^n} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| e^{|(r,x)|} d\nu_n(r, x) dm_{n+1}(\mu, \lambda) < +\infty. \end{aligned}$$

ii. For  $d > 0$ , let  $B_d = \{(r, x) \in [0, +\infty[\times\mathbb{R}^n; |(r, x)| \leq d\}$ . We have

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^n} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \int_0^\infty \int_{\mathbb{R}^n} \frac{|\mathcal{W}_{\frac{n-1}{2}}(f)(r, x)| e^{|(r,x)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) dm_{n+1}(\mu, \lambda) \\ & \leq \iint_{B_d^c} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{\mathcal{W}_{\frac{n-1}{2}}(|f|)(r, x) e^{|(r,x)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) \right) dm_{n+1}(\mu, \lambda) \\ & + \iint_{B_d} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{\mathcal{W}_{\frac{n-1}{2}}(|f|)(r, x) e^{|(r,x)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) \right) dm_{n+1}(\mu, \lambda). \end{aligned}$$

From the relation (2.13), we deduce that

$$(3.9) \quad \iint_{B_d^c} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \times \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{\mathcal{W}_{\frac{n-1}{2}}(|f|)(r, x) e^{|(r,x)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) \right) dm_{n+1}(\mu, \lambda)$$



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$$= \iint_{B_d^c} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| \\ \times \mathcal{R}_{\frac{n-1}{2}} \left( \frac{e^{|\cdot, \cdot|} |(\mu, \lambda)|}{(1 + |(\cdot, \cdot)| + |(\mu, \lambda)|)^d} \right) (r, x) d\nu_n(r, x) dm_{n+1}(\mu, \lambda).$$

However, from the relation (2.12) and ii) of Remark 3, we deduce that for all  $(\mu, \lambda) \in B_d^c$ , we have

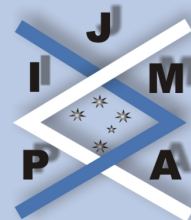
$$(3.10) \quad \mathcal{R}_{\frac{n-1}{2}} \left( \frac{e^{|\cdot, \cdot|} |(\mu, \lambda)|}{(1 + |(\cdot, \cdot)| + |(\mu, \lambda)|)^d} \right) (r, x) \leq \frac{e^{|(r, x)|} |(\mu, \lambda)|}{(1 + |(r, x)| + |(\mu, \lambda)|)^d}.$$

Combining the relations (3.6), (3.9) and (3.10), we get

$$\iint_{B_d^c} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{\mathcal{W}_{\frac{n-1}{2}}(|f|)(r, x) e^{|(r, x)|} |(\mu, \lambda)|}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) \right) dm_{n+1}(\mu, \lambda) \\ \leq \iint_{B_d^c} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{|(r, x)|} |(\mu, \lambda)|}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} d\nu_n(r, x) \right) dm_{n+1}(\mu, \lambda) \\ \leq \int_0^\infty \int_{\mathbb{R}^n} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{|(r, x)|} |(\mu, \lambda)|}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} d\nu_n(r, x) \right) dm_{n+1}(\mu, \lambda) \\ < +\infty.$$

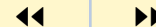
We have

$$\iint_{B_d} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \iint_{B_d} \frac{|\mathcal{W}_{\frac{n-1}{2}}(f)(r, x)| e^{|(r, x)|} |(\mu, \lambda)|}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) dm_{n+1}(\mu, \lambda) \\ \leq e^{d^2} \left( \iint_{B_d} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| dm_{n+1}(\mu, \lambda) \right) \left( \iint_{B_d} \left| \mathcal{W}_{\frac{n-1}{2}}(f)(r, x) \right| dm_{n+1}(r, x) \right)$$



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$$\leq e^{d^2} m_{n+1}(B_d) \|\mathcal{F}(f)\|_{\infty, \gamma_n} \left\| \mathcal{W}_{\frac{n-1}{2}}(f) \right\|_{1, m_{n+1}}.$$

By the relations (2.10) and (2.15), we deduce that

$$\iint_{B_d} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \iint_{B_d} \frac{\left| \mathcal{W}_{\frac{n-1}{2}}(f)(r, x) \right| e^{|(r, x)||(\mu, \lambda)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) dm_{n+1}(\mu, \lambda) \leq e^{d^2} m_{n+1}(B_d) \|f\|_{1, \nu_n}^2 < +\infty.$$

By the relation (2.13), we get

$$\begin{aligned} & \iint_{B_d} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \left( \iint_{B_d^c} \frac{\left| \mathcal{W}_{\frac{n-1}{2}}(f)(r, x) \right| e^{|(r, x)||(\mu, \lambda)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) \right) dm_{n+1}(\mu, \lambda) \\ & \leq \iint_{B_d} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \\ & \quad \times \int_0^\infty \int_{\mathbb{R}^n} \frac{\mathcal{W}_{\frac{n-1}{2}}(|f|)(r, x) e^{|(r, x)||(\mu, \lambda)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} \mathbf{1}_{B_d^c}(r, x) dm_{n+1}(r, x) dm_{n+1}(r, x) \\ & = \iint_{B_d} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \left( \int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| \right. \\ & \quad \times \mathcal{R}_{\frac{n-1}{2}} \left( \frac{e^{|(\cdot, \cdot)||(\mu, \lambda)|}}{(1 + |(\cdot, \cdot)| + |(\mu, \lambda)|)^d} \mathbf{1}_{B_d^c}(\cdot, \cdot) \right) (r, x) d\nu_n(r, x) \left. \right) dm_{n+1}(\mu, \lambda). \end{aligned}$$

However, by ii) of Remark 3 and the relation (2.10), we deduce that for all  $(\mu, \lambda) \in B_d$ :

$$\mathcal{R}_{\frac{n-1}{2}} \left( \frac{e^{|(\cdot, \cdot)||(\mu, \lambda)|}}{(1 + |(\cdot, \cdot)| + |(\mu, \lambda)|)^d} \mathbf{1}_{B_d^c}(\cdot, \cdot) \right) (r, x) \leq \frac{e^{d|(r, x)|}}{(1 + |(r, x)| + d)^d} \mathbf{1}_{B_d^c}(r, x).$$



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Thus,

$$(3.11) \quad \iint_{B_d} \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \\ \times \left( \iint_{B_d^c} \frac{\left| \mathcal{W}_{\frac{n-1}{2}}(f)(r, x) \right| e^{|(r,x)||(\mu,\lambda)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) \right) dm_{n+1}(\mu, \lambda) \\ \leq \|f\|_{1, \nu_n} m_{n+1}(B_d) \iint_{B_d^c} |f(r, x)| \frac{e^{d|(r,x)|}}{(1 + |(r, x)| + d)^d} d\nu_n(r, x).$$

On the other hand, from i) of Remark 3, there exists  $(\mu_0, \lambda_0) \in [0, +\infty[ \times \mathbb{R}^n$ ,  $|(\mu_0, \lambda_0)| > d$  such that

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{e^{|(r,x)||(\mu_0,\lambda_0)|} |f(r, x)|}{(1 + |(r, x)| + |(\mu_0, \lambda_0)|)^d} d\nu_n(r, x) < +\infty.$$

Again, by ii) of Remark 3, we have

$$(3.12) \quad \iint_{B_d^c} |f(r, x)| \frac{e^{d|(r,x)|}}{(1 + |(r, x)| + d)^d} d\nu_n(r, x) \\ \leq \iint_{B_d^c} |f(r, x)| \frac{e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1 + |(r, x)| + |(\mu_0, \lambda_0)|)^d} d\nu_n(r, x) < +\infty.$$

The relations (3.11) and (3.12) imply that

$$\iint_{B_d} \iint_{B_d^c} \left( \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \frac{\left| \mathcal{W}_{\frac{n-1}{2}}(f)(r, x) \right| e^{|(r,x)||(\mu,\lambda)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) \right) dm_{n+1}(\mu, \lambda) < +\infty,$$

and the proof of Theorem 3.1 is complete.  $\square$



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**Theorem 3.7 (Beurling Hörmander for  $\mathcal{R}$ ).** Let  $f$  be a measurable function on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable and such that  $f \in L^2(d\nu_n)$ .

Let  $d$  be a real number,  $d \geq 0$ . If

$$\iint_{\Gamma^+} \int_0^\infty \int_{\mathbb{R}^n} |f(r, x)| \frac{|\mathcal{F}(f)(\mu, \lambda)| e^{|(r,x)||\theta(\mu,\lambda)|}}{(1 + |(r, x)| + |\theta(\mu, \lambda)|)^d} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda) < +\infty,$$

then

- For  $d \leq n + 1$ ,  $f = 0$ .
- For  $d > n + 1$ , there exist a positive constant  $a$  and a polynomial  $P$  on  $\mathbb{R} \times \mathbb{R}^n$  even with respect to the first variable, such that

$$f(r, x) = P(r, x) e^{-a(r^2 + |x|^2)}$$

with  $\text{degree}(P) < \frac{d-(n+1)}{2}$ .

*Proof.* Let  $f$  be a function satisfying the hypothesis. Then, from Theorem 3.1, we have

$$(3.13) \quad \int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \left| \mathcal{W}_{\frac{n-1}{2}}(f)(r, x) \right| \left| \tilde{\mathcal{F}}(f)(\mu, \lambda) \right| \frac{e^{|(r,x)||(\mu,\lambda)|}}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) dm_{n+1}(\mu, \lambda) < +\infty.$$

On the other hand, from Proposition 2.1, Lemma 3.2 and Lemma 3.3, we deduce that the function  $\mathcal{W}_{\frac{n-1}{2}}(f)$  belongs to the space  $L^1(dm_{n+1}) \cap L^2(dm_{n+1})$  and by (2.16), we have

$$\tilde{\mathcal{F}}(f) = \Lambda_{n+1} \left( \mathcal{W}_{\frac{n-1}{2}}(f) \right).$$



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Substituting into (3.13), we get

$$\int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \left| \mathscr{W}_{\frac{n-1}{2}}(f)(r, x) \right| \left| \Lambda_{n+1} \left( \mathscr{W}_{\frac{n-1}{2}}(f) \right) (\mu, \lambda) \right| e^{|(r,x)||(\mu,\lambda)|} \times \frac{1}{(1 + |(r, x)| + |(\mu, \lambda)|)^d} dm_{n+1}(r, x) dm_{n+1}(\mu, \lambda) < +\infty.$$

Applying Theorem 3.1 when  $f$  is replaced by  $\mathscr{W}_{\frac{n-1}{2}}(f)$ , we deduce that

- If  $d \leq n + 1$ ,  $\mathscr{W}_{\frac{n-1}{2}}(f) = 0$  and by Remark 2, we have  $f = 0$ .
- If  $d > n + 1$ , there exist  $a > 0$  and a polynomial  $Q$  on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable such that

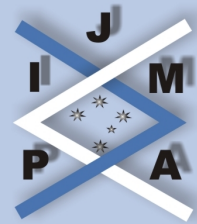
$$\begin{aligned} \mathscr{W}_{\frac{n-1}{2}}(f)(r, x) &= Q(r, x) e^{-a(r^2+|x|^2)} \\ &= \sum_{2k+|\alpha| \leq m} a_{k,\alpha} r^{2k} x^\alpha e^{-a(r^2+|x|^2)}; \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}. \end{aligned}$$

In particular, the function  $\mathscr{W}_{\frac{n-1}{2}}(f)$  lies in  $S_*(\mathbb{R} \times \mathbb{R}^n)$  and by Remark 2, the function  $f$  belongs to  $S_*(\mathbb{R} \times \mathbb{R}^n)$  and we have

$$f = \mathscr{W}_{\frac{n-1}{2}}^{-1} (Q(r, x) e^{-a(r^2+|x|^2)}).$$

Now, using the relation (2.17), we obtain

$$\begin{aligned} (3.14) \quad f(r, x) &= \mathscr{W}_{\frac{n-1}{2}}^{-1} (Q(t, y) e^{-a(t^2+|y|^2)})(r, x) \\ &= (-1)^{\lfloor \frac{n}{2} \rfloor + 1} F_{\lfloor \frac{n}{2} \rfloor - \frac{n}{2} + 1} \left[ \left( \frac{\partial}{\partial t^2} \right)^{\lfloor \frac{n}{2} \rfloor + 1} Q(t, y) e^{-a(t^2+|y|^2)} \right] (r, x) \end{aligned}$$



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$$= (-1)^{\lfloor \frac{n}{2} \rfloor + 1} \sum_{2k + |\alpha| \leq m} a_{k,\alpha} F_{\lfloor \frac{n}{2} \rfloor - \frac{n}{2} + 1} \left[ \left( \frac{\partial}{\partial t^2} \right)^{\lfloor \frac{n}{2} \rfloor + 1} (t^{2k} y^\alpha e^{-a(t^2 + |y|^2)}) \right] (r, x).$$

However, for all  $l \in \mathbb{N}$ ,

$$(3.15) \quad \left( \frac{\partial}{\partial t^2} \right)^l (t^{2k} y^\alpha e^{-a(t^2 + |y|^2)}) \\ = \left( \sum_{j=0}^{\min(l,k)} C_l^j \frac{2^j k!}{(k-j)!} (-2a)^{k-j} t^{2(k-j)} \right) y^\alpha e^{-a(t^2 + |y|^2)}$$

and for all  $b > 0$ ,

$$(3.16) \quad F_b (t^{2k} y^\alpha e^{-a(t^2 + |y|^2)}) (r, x) \\ = \frac{1}{2^b \Gamma(b)} \left( \sum_{j=0}^k C_k^j \frac{\Gamma(b+k-j)}{a^{\mu+k-j} r^{2j}} r^{2j} \right) x^\alpha e^{-a(r^2 + |x|^2)},$$

where the transform  $F_b$  is defined by the relation (2.18).

Combining the relations (3.14), (3.15) and (3.16), we deduce that

$$f(r, x) = P(r, x) e^{-a(r^2 + |x|^2)},$$

where  $P$  is a polynomial, even with respect to the first variable and  $\text{degree}(P) = \text{degree}(Q)$ . □





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## 4. Applications of the Beurling-Hörmander Theorem

This section is devoted to giving some applications of the Beurling-Hörmander theorem for the spherical mean operator. More precisely, we prove a Gelfand-Shilov theorem for the Fourier transform  $\mathcal{F}$  and establish a Cowling Price type theorem for this transform.

**Lemma 4.1.** *Let  $P$  be a polynomial on  $\mathbb{R} \times \mathbb{R}^n$ ;  $P \neq 0$  with  $\text{degree}(P) = m$ . Then there exist two positive constants  $A$  and  $C$  such that*

$$\forall t \geq A, \quad \varphi(t) = \int_{S^n} |P(tw)| d\sigma_n(w) \geq Ct^m,$$

where  $d\sigma_n$  is the surface measure on the unit sphere  $S^n$  of  $\mathbb{R} \times \mathbb{R}^n$ .

*Proof.* Let  $P$  be a polynomial on  $\mathbb{R} \times \mathbb{R}^n$ ,  $P \neq 0$  and  $\text{degree}(P) = m$ . Then we have

$$\varphi(t) = \int_{S^n} \left| \sum_{k=0}^m a_k(w)t^k \right| d\sigma_n(w),$$

where  $a_k$ ,  $0 \leq k \leq m$  are continuous functions on  $S^n$  and  $a_m \neq 0$ .

Then the function  $\varphi$  is continuous on  $[0, +\infty[$  and by the dominated convergence theorem, we have

$$(4.1) \quad \varphi(t) \sim C_m t^m \quad (t \rightarrow +\infty),$$

where

$$C_m = \int_{S^n} |a_m(w)| d\sigma_n(w) > 0.$$

Now, by (4.1), there exists  $A > 0$  such that

$$\forall t \geq A; \quad p(t) \geq \frac{C_m}{2} t^m.$$





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**Theorem 4.2 (Gelfand-Shilov).** Let  $p, q$  be two conjugate exponents,  $p, q \in ]1, +\infty[$ . Let  $\eta, \xi$  be two positive real numbers such that  $\xi\eta \geq 1$ .

Let  $f$  be a measurable function on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable such that  $f \in L^2(d\nu_n)$ .

If

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{\frac{\xi^p |(r, x)|^p}{p}}}{(1 + |(r, x)|)^d} d\nu_n(r, x) < +\infty$$

and

$$\iint_{\Gamma^+} \frac{|\mathcal{F}(f)(\mu, \lambda)| e^{\frac{\xi^q |(\mu, \lambda)|^q}{q}}}{(1 + |\theta(\mu, \lambda)|)^d} d\tilde{\gamma}_n(\mu, \lambda) < +\infty, \quad d \geq 0,$$

then

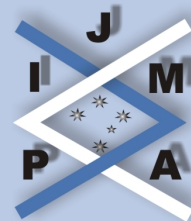
i. For  $d \leq \frac{n+1}{2}$ ,  $f = 0$ .

ii. For  $d > \frac{n+1}{2}$ , we have:

- $f = 0$  for  $\xi\eta > 1$ ;
- $f = 0$  for  $\xi\eta = 1$  and  $p \neq 2$ ;
- $f(r, x) = P(r, x)e^{-a(r^2+|x|^2)}$  for  $\xi\eta = 1$  and  $p = q = 2$ , where  $a > 0$  and  $P$  is a polynomial on  $\mathbb{R} \times \mathbb{R}^n$  even with respect to the first variable, with degree  $(P) < d - \frac{n+1}{2}$ .

*Proof.* Let  $f$  be a function satisfying the hypothesis. Since  $\xi\eta \geq 1$ , by a convexity argument we have

$$(4.2) \quad \iint_{\Gamma^+} \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| |\mathcal{F}(f)(\mu, \lambda)|}{(1 + |(r, x)| + |\theta(\mu, \lambda)|)^{2d}} e^{|\theta(\mu, \lambda)|} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda)$$



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$$\begin{aligned} &\leq \iint_{\Gamma^+} \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| |\mathcal{F}(f)(\mu, \lambda)|}{(1 + |(r, x)|)^d (1 + |\theta(\mu, \lambda)|)^d} \\ &\quad \times e^{\eta \xi |(r, x)| |\theta(\mu, \lambda)|} d\nu_n(r, x) d\tilde{\gamma}_n(\mu, \lambda) \\ &\leq \iint_{\Gamma^+} \frac{|\mathcal{F}(f)(\mu, \lambda)| e^{\frac{\eta^q |\theta(\mu, \lambda)|^q}{q}}}{(1 + |\theta(\mu, \lambda)|)^d} d\tilde{\gamma}_n(\mu, \lambda) \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{\frac{\xi^p |(r, x)|^p}{p}}}{(1 + |(r, x)|)^d} d\nu_n(r, x) \\ &< +\infty. \end{aligned}$$

Then from the Beurling-Hörmander theorem, we deduce that

i. For  $d \leq \frac{n+1}{2}$ ,  $f = 0$ .

ii. For  $d > \frac{n+1}{2}$ , there exist a positive constant  $a$  and a polynomial  $P$  on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable such that

$$(4.3) \quad f(r, x) = P(r, x) e^{-a(r^2 + |x|^2)}$$

with  $\text{degree}(P) < \frac{2d - (n+1)}{2}$ , and using standard calculus, we obtain

$$(4.4) \quad \tilde{\mathcal{F}}(f)(\mu, \lambda) = Q(\mu, \lambda) e^{-\frac{1}{4a}(\mu^2 + |\lambda|^2)},$$

where  $Q$  is a polynomial on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable, with  $\text{degree}(Q) = \text{degree}(P)$ .

On the other hand, from the relations (2.7), (2.8), (4.2), (4.3) and (4.4), we get

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} \frac{|P(r, x)| |Q(\mu, \lambda)| e^{\xi \eta |(r, x)| |\mu, \lambda|}}{(1 + |(r, x)|)^d (1 + |(\mu, \lambda)|)^d} \\ &\quad \times e^{-\frac{(\mu^2 + |\lambda|^2)}{4a}} e^{-a(r^2 + |x|^2)} d\nu_n(r, x) dm_{n+1}(\mu, \lambda) < +\infty. \end{aligned}$$

So,

$$(4.5) \quad \int_0^\infty \int_{\mathbb{R}^n} \frac{\varphi(t)}{(1+t)^d} \frac{\psi(\rho)}{(1+\rho)^d} e^{\xi \eta t \rho} e^{-at^2 - \frac{\rho^2}{4a} t^{2n}} \rho^n dt d\rho < +\infty,$$



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where

$$\varphi(t) = \int_{S^n} |P(tw)| |w_1|^n d\sigma_n(w)$$

and

$$\psi(\rho) = \int_{S^n} |Q(\rho w)| d\sigma_n(w).$$

- Suppose that  $\xi\eta > 1$ . If  $f \neq 0$ , then each of the polynomials  $P$  and  $Q$  is not identically zero. Let  $m = \text{degree}(P) = \text{degree}(Q)$ .

From Lemma 4.1, there exist two positive constants  $A$  and  $C$  such that

$$\forall t \geq A, \quad \varphi(t) \geq Ct^m$$

and

$$\forall \rho \geq A, \quad \psi(\rho) \geq C\rho^m.$$

Then the inequality (4.5) leads to

$$(4.6) \quad \int_A^\infty \int_A^\infty \frac{e^{\xi\eta t\rho}}{(1+t)^d(1+\rho)^d} e^{-at^2} e^{-\frac{\rho^2}{4a}} dt d\rho < +\infty.$$

Let  $\varepsilon > 0$  such that  $c = \eta\xi - \varepsilon > 1$ . The relation (4.6) implies that

$$(4.7) \quad \int_A^\infty \int_A^\infty \frac{e^{\varepsilon\rho t}}{(1+t)^d(1+\rho)^d} e^{c\rho t} e^{-at^2} e^{-\frac{\rho^2}{4a}} dt d\rho < +\infty.$$

However, for all  $t \geq A \geq \frac{d}{\varepsilon}$  and  $\rho \geq A$ , we have

$$\frac{e^{\varepsilon\rho t}}{(1+t)^d(1+\rho)^d} \geq \frac{e^{\varepsilon A^2}}{(1+A)^{2d}}$$



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and by (4.7), it follows that

$$(4.8) \quad \int_A^\infty \int_A^\infty e^{cpt-at^2} e^{-\frac{\rho^2}{4a}} dt d\rho < +\infty.$$

Let  $F(t) = \int_A^\infty e^{cpt-\frac{\rho^2}{4a}} d\rho$ , then the function  $F$  can be written as

$$F(t) = e^{ac^2t^2} \left( \int_A^\infty e^{-\frac{\rho^2}{4a}} d\rho + 2a\gamma e^{-\frac{A^2}{4a}} \int_0^t e^{cAs-ac^2s^2} ds \right).$$

In particular,

$$F(t) \geq e^{ac^2t^2} \int_A^\infty e^{-\frac{\rho^2}{4a}} d\rho.$$

Thus,

$$\int_A^\infty \int_A^\infty e^{cpt-at^2-\frac{\rho^2}{4a}} dt d\rho \geq \int_A^\infty e^{a(c^2-1)t^2} dt \int_A^\infty e^{-\frac{\rho^2}{4a}} d\rho = +\infty$$

because  $c > 1$ . This contradicts the relation (4.8) and shows that  $f = 0$ .

- Suppose that  $\xi\eta = 1$  and  $p \neq 2$ .

In this case, we have  $p > 2$  or  $q > 2$ .

Suppose that  $q > 2$ . Then from the second hypothesis and the relations (2.7), (2.8) and (4.4), we get

$$(4.9) \quad \int_0^\infty \frac{\psi(\rho) e^{-\frac{\rho^2}{4a}} e^{\frac{\eta^q \rho^q}{q}}}{(1+\rho)^d} \rho^n d\rho < +\infty.$$

If  $f \neq 0$ , then the polynomial  $Q$  is not identically zero, and by Lemma 4.1 and the relation (4.9), it follows that there exists  $A > 0$  such that

$$\int_A^\infty \frac{e^{-\frac{\rho^2}{4a}} e^{\frac{\eta^q \rho^q}{q}}}{(1+\rho)^d} d\rho < +\infty,$$

which is impossible because  $q > 2$ .

The proof of Theorem 4.2 is thus complete. □

**Theorem 4.3 (Cowling-Price for spherical mean operator).** *Let  $\eta, \xi, w_1$  and  $w_2$  be non negative real numbers such that  $\eta\xi \geq \frac{1}{4}$ . Let  $p, q$  be two exponents,  $p, q \in [1, +\infty]$  and let  $f$  be a measurable function on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable such that  $f \in L^2(d\nu_n)$ . If*

$$(4.10) \quad \left\| \frac{e^{\xi|\cdot|^2} f}{(1 + |\cdot|^2)^{w_1}} \right\|_{p, \nu_n} < +\infty$$

and

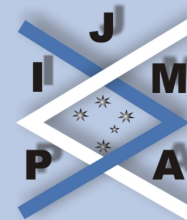
$$(4.11) \quad \left\| \frac{e^{\eta|\theta(\cdot, \cdot)|^2} \mathcal{F}(f)}{(1 + |\theta(\cdot, \cdot)|)^{w_2}} \right\|_{q, \tilde{\gamma}_n} < +\infty,$$

then

- i. For  $\xi\eta > \frac{1}{4}$ ,  $f = 0$ .
- ii. For  $\xi\eta = \frac{1}{4}$ , there exist a positive constant  $a$  and a polynomial  $P$  on  $\mathbb{R} \times \mathbb{R}^n$ , even with respect to the first variable such that

$$f(r, x) = P(r, x)e^{-a(r^2+|x|^2)}.$$

*Proof.* Let  $p'$  and  $q'$  be the conjugate exponents of  $p$  respectively  $q$ .



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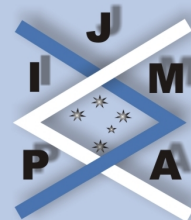


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Let us pick  $d_1, d_2 \in \mathbb{R}$  such that  $d_1 > 2n + 1$  and  $d_2 > n + 1$ . Then from Hölder's inequality and the relations (4.10) and (4.11), we deduce that

$$\begin{aligned}
 (4.12) \quad & \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{\xi |(r, x)|^2}}{(1 + |(r, x)|)^{w_1 + d_1/p'}} d\nu_n(r, x) \\
 & \leq \left\| \frac{e^{\xi |(\cdot, \cdot)|^2} f}{(1 + |(\cdot, \cdot)|)^{w_1}} \right\|_{p, \nu_n} \left\| \frac{1}{(1 + |(\cdot, \cdot)|)^{d_1/p'}} \right\|_{p', \nu_n} \\
 & = \left\| \frac{e^{\xi |(\cdot, \cdot)|^2} f}{(1 + |(\cdot, \cdot)|)^{w_1}} \right\|_{p, \nu_n} \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{d\nu_n(r, x)}{(1 + |(r, x)|)^{d_1}} \right)^{\frac{1}{p'}} < +\infty.
 \end{aligned}$$

and

$$\begin{aligned}
 & \iint_{\Gamma^+} \frac{|\mathcal{F}(f)(\mu, \lambda)| e^{\eta |\theta(\mu, \lambda)|^2}}{(1 + |\theta(\mu, \lambda)|)^{w_2 + d_2/q'}} d\tilde{\gamma}_n(\mu, \lambda) \\
 & \leq \left\| \frac{e^{\eta |\theta(\cdot, \cdot)|^2}}{(1 + |\theta(\cdot, \cdot)|)^{w_2}} \mathcal{F}(f) \right\|_{q, \tilde{\gamma}_n} \left\| \frac{1}{(1 + |\theta(\cdot, \cdot)|)^{d_2/q'}} \right\|_{q', \tilde{\gamma}_n}.
 \end{aligned}$$

By the relation (2.7), we obtain

$$\begin{aligned}
 (4.13) \quad & \iint_{\Gamma^+} \frac{|\mathcal{F}(f)(\mu, \lambda)| e^{\eta |\theta(\mu, \lambda)|^2}}{(1 + |\theta(\mu, \lambda)|)^{w_2 + d_2/q'}} d\tilde{\gamma}_n(\mu, \lambda) \\
 & \leq \left\| \frac{e^{\eta |\theta(\cdot, \cdot)|^2}}{(1 + |\theta(\cdot, \cdot)|)^{w_2}} \mathcal{F}(f) \right\|_{q, \tilde{\gamma}_n} \left( \int_0^\infty \int_{\mathbb{R}^n} \frac{dm_{n+1}(\mu, \lambda)}{(1 + |(\mu, \lambda)|)^{d_2}} \right)^{\frac{1}{q'}} < +\infty.
 \end{aligned}$$

Let  $d > \max \left( w_1 + \frac{d_1}{p'}, w_2 + \frac{d_2}{q'}, \frac{n+1}{2} \right)$ , then from the relations (4.12) and (4.13),

we have

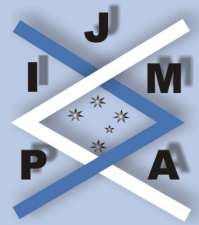
$$\int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r, x)| e^{\xi |(r, x)|^2}}{(1 + |(r, x)|)^d} d\nu_n(r, x) < +\infty$$

and

$$\iint_{\Gamma^+} \frac{|\mathcal{F}(f)(\mu, \lambda)| e^{\eta |\theta(\mu, \lambda)|^2}}{(1 + |\theta(\mu, \lambda)|)^d} d\tilde{\gamma}_n(\mu, \lambda) < +\infty.$$

Then the desired result follows from Theorem 4.2.  $\square$

*Remark 4.* The Hardy theorem is a special case of Theorem 4.2, when  $p = q = +\infty$ .



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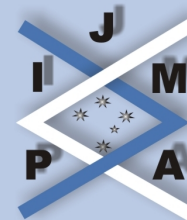
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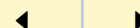
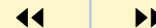
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