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BEURLING-HÖRMANDER UNCERTAINTY PRINCIPLE FOR THE SPHERICAL MEAN OPERATOR

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ABSTRACT. We establish the Beurling-Hörmander theorem for the Fourier transform connected with the spherical mean operator. Applying this result, we prove the Gelfand-Shilov and Cowling-Price type theorems for this transform.

Key words and phrases: Uncertainty principle, Beurling-Hörmander theorem, Gelfand-Shilov theorem, Cowling-Price theorem, Fourier transform, Spherical mean operator.

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1. INTRODUCTION

Uncertainty principles play an important role in harmonic analysis and have been studied by many authors, from many points of view [13, 19]. These principles state that a function f and its Fourier transform $\widehat{f}$ cannot be simultaneously sharply localized. Many aspects of such principles have been studied, for example the Heisenberg-Pauli-Weyl inequality [16] has been established for various Fourier transforms [26, 31, 32] and several generalized forms of this inequality are given in [28, 29, 30]. See also the theorems of Hardy, Morgan, Beurling and Gelfand-Shilov [7, 15, 23, 25, 26]. The most recent Beurling-Hörmander theorem has been proved by Hörmander [20] using an idea of Beurling [3]. This theorem states that if f is an integrable function on $\mathbb{R}$ with respect to the Lebesgue measure and if

$$\iint_{\mathbb{R}^2} |f(x)||\hat{f}(y)|e^{|xy|}dxdy < +\infty,$$

then $f = 0$ almost everywhere.

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A strong multidimensional version of this theorem has been established by Bonami, Demange and Jaming [4] (see also [19]) who have showed that if f is a square integrable function on $\mathbb{R}^n$ with respect to the Lebesgue measure, then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|f(x)||\hat{f}(y)|}{(1+|x|+|y|)^d}e^{|\langle x/y\rangle|}dxdy<+\infty,\qquad d\geq 0;$$

if and only if f can be written as

$$f(x)=P(x)e^{-\langle Ax/x\rangle},$$

where A is a real positive definite symmetric matrix and P is a polynomial with $\mathrm{degree}(P)<\frac{d-n}{2}$.

In particular for $d\leq n;\ f$ is identically zero.

The Beurling-Hörmander uncertainty principle has been studied by many authors for various Fourier transforms. In particular, Trimèche [33] has shown this uncertainty principle for the Dunkl transform, Kamoun and Trimèche [21] have proved an analogue of the Beurling-Hörmander theorem for some singular partial differential operators, Bouattour and Trimèche [5] have shown this theorem for the hypergroup of Chébli-Trimèche. We cite also Yakubovich [37], who has established the same result for the Kontorovich-Lebedev transform.

Many authors are interested in the Beurling-Hörmander uncertainty principle because this principle implies other well known quantitative uncertainty principles such as those of Gelfand-Shilov [14], Cowling Price [7], Morgan [2, 23], and the one of Hardy [15].

On the other hand, the spherical mean operator is defined on $\mathscr{C}_*(\mathbb{R}\times\mathbb{R}^n)$ (the space of continuous functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable) by

$$\mathscr{R}(f)(r,x)\ =\ \int_{S^n}f\big(r\eta,\ x+r\xi\big)d\sigma_n(\eta,\xi),$$

where S^n is the unit sphere $\big\{(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n;\ \eta^2+|\xi|^2=1\big\}$ in $\mathbb{R}\times\mathbb{R}^n$ and σ_n is the surface measure on S^n normalized to have total measure one.

The dual operator ${}^t\mathscr{R}$ of $\mathscr{R}$ is defined by

$${}^t\mathscr{R}(g)(r,x)\ =\ \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}\int_{\mathbb{R}^n}g\left(\sqrt{r^2+|x-y|^2},y\right)dy,$$

where dy is the Lebesgue measure on $\mathbb{R}^n$.

The spherical mean operator $\mathscr{R}$ and its dual ${}^t\mathscr{R}$ play an important role and have many applications, for example; in the image processing of so-called synthetic aperture radar (SAR) data [17, 18], or in the linearized inverse scattering problem in acoustics [11]. These operators have been studied by many authors from many points of view [1, 8, 11, 24, 27].

In [24] (see also [8, 27]); the second author with others, associated to the spherical mean operator $\mathscr{R}$ the Fourier transform $\mathscr{F}$ defined by

$$\mathscr{F}(f)(\mu,\lambda)\ =\ \int_0^{\infty}\int_{\mathbb{R}^n}f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_n(r,x),$$

where

- $\varphi_{\mu,\lambda}(r,x)\ =\ \mathscr{R}\left(\cos(\mu.)e^{-i\langle\lambda/\cdot\rangle}\right)(r,x)$
- $d\nu_n$ is the measure defined on $[0,+\infty[\times\mathbb{R}^n$ by

$$d\nu_n(r,x)=\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}r^n\,dr\otimes\frac{dx}{(2\pi)^{\frac{n}{2}}}.$$

They have constructed the harmonic analysis related to the transform $\mathscr{F}$ (inversion formula, Plancherel formula, Paley-Wiener theorem, Plancherel theorem).

Our purpose in the present work is to study the Beurling-Hörmander uncertainty principle for the Fourier transform $\mathscr{F}$, from which we derive the Gelfand-Shilov and Cowling -Price type theorems for this transform.

More precisely, we collect some basic harmonic analysis results for the Fourier transform $\mathscr{F}$.

In the third section, we establish the main result of this paper, that is, from the Beurling Hörmander theorem:

- Let f be a measurable function on $\mathbb{R}\times\mathbb{R}^n$; even with respect to the first variable and such that $f\in L^2(d\nu_n)$. If

$$\iint_{\Gamma^+}\int_0^\infty\int_{\mathbb{R}^n}|f(r,x)|\frac{|\mathscr{F}(f)(\mu,\lambda)|e^{|(r,x)||\theta(\mu,\lambda)|}}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda)<+\infty;\quad d\geq 0,$$

  then

  i. For $d\leq n+1$; $f=0$;

  ii. For $d>n+1$; there exists a positive constant a and a polynomial P on $\mathbb{R}\times\mathbb{R}^n$ even with respect to the first variable, such that

$$f(r,x)=P(r,x)e^{-a(r^2+|x|^2)}$$

  with $\mathrm{degree}(P)<\frac{d-(n+1)}{2}$;

where

- Γ_+ is the set given by

$$\Gamma_+=[0,+\infty[\times\mathbb{R}^n\cup\left\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n;\ 0\leq\mu\leq|\lambda|\right\}$$

- θ is the bijective function defined on Γ_+ by

$$\theta(\mu,\lambda)=\left(\sqrt{\mu^2+|\lambda|^2},\ \lambda\right)$$

- $d\tilde{\gamma}_n$ is the measure defined on Γ_+ by

$$\iint_{\Gamma^+}g(\mu,\lambda)d\tilde{\gamma}_n(\mu,\lambda)=\sqrt{\frac{2}{\pi}}\frac{1}{(2\pi)^{\frac{n}{2}}}\times\left[\int_0^\infty\int_{\mathbb{R}^n}g(\mu,\lambda)\frac{\mu d\mu d\lambda}{\sqrt{\mu^2+\lambda^2}}+\int_{\mathbb{R}^n}\int_0^{|\lambda|}g(i\mu,\lambda)\frac{\mu d\mu d\lambda}{\sqrt{\lambda^2-\mu^2}}\right].$$

The last section of this paper is devoted to the Gelfand-Shilov and Cowling Price theorems for the transform $\mathscr{F}$.

- Let p,q be two conjugate exponents; $p,\ q\ \in]1,+\infty[$. Let $\eta,\ \xi$ be two positive real numbers such that $\xi\eta\geq 1$. Let f be a measurable function on $\mathbb{R}\times\mathbb{R}^n$; even with respect to the first variable such that $f\in L^2(d\nu_n)$.

  If

$$\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)|e^{\frac{\xi^p|(r,x)|^p}{p}}}{(1+|(r,x)|)^d}d\nu_n(r,x)<+\infty$$

  and

$$\iint_{\Gamma^+}\frac{|\mathscr{F}(f)(\mu,\lambda)|e^{\frac{\xi^q|(r,x)|^q}{q}}}{(1+|\theta(\mu,\lambda)|)^d}d\tilde{\gamma}_n(\mu,\lambda)<+\infty;\quad d\geq 0,$$

  then

  i. For $d\leq\frac{n+1}{2}$; $f=0$.

  ii. For $d>\frac{n+1}{2}$; we have

  - $f = 0$ for $\xi\eta > 1$
  - $f = 0$ for $\xi\eta = 1$ and $p \neq 2$
  - $f(r,x) = P(r,x)e^{-a(r^2+|x|^2)}$ for $\xi\eta = 1$ and $p = q = 2$, where $a > 0$ and P is a polynomial on $\mathbb{R} \times \mathbb{R}^n$ even with respect to the first variable, with $\mathrm{degree}(P) < d - \frac{n+1}{2}$.
- Let η, ξ, w_1 and w_2 be non negative real numbers such that $\eta\xi \geq \frac{1}{4}$. Let p, q be two exponents, $p,\ q \in [1, +\infty]$ and let f be a measurable function on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable such that $f \in L^2(d\nu_n)$.
  If
  $$\left\| \frac{e^{\xi|(\cdot,\cdot)|^2}}{(1 + |(\cdot,\cdot)|)^{w_1}} \right\|_{p,\nu_n} < +\infty$$
  and
  $$\left\| \frac{e^{\eta|\theta(\cdot,\cdot)|^2}}{(1 + |\theta(\cdot,\cdot)|)^{w_2}} \mathscr{F}(f) \right\|_{q,\tilde{\gamma}_n} < +\infty,$$
  then
  i. For $\xi\eta > \frac{1}{4}$; $f = 0$.
  ii. For $\xi\eta = \frac{1}{4}$; there exists a positive constant a and a polynomial P on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable such that
  $$f(r,x) = P(r,x)e^{-a(r^2+|x|^2)}.$$

2. The Spherical Mean Operator

For all $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}^n$; if we denote by $\varphi_{\mu,\lambda}$ the function defined by

$$\varphi_{\mu,\lambda}(r,x) = \mathscr{R}\left(\cos(\mu.)e^{-i\langle\lambda/\cdot\rangle}\right)(r,x),$$

then we have

$$\varphi_{\mu,\lambda}(r,x) = j_{\frac{n-1}{2}}\left(r\sqrt{\mu^2 + \lambda^2}\right) e^{-i\langle\lambda/x\rangle}, \tag{2.1}$$

where

- $\lambda^2 = \lambda_1^2 + \cdots + \lambda_n^2;\ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$;
- $\langle\lambda/x\rangle = \lambda_1 x_1 + \cdots + \lambda_n x_n;\ x = (x_1, \ldots, x_n) \in \mathbb{R}^n$;
- $j_{\frac{n-1}{2}}$ is the modified Bessel function given by

$$\begin{aligned} j_{\frac{n-1}{2}}(s) &= 2^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)\frac{J_{\frac{n-1}{2}}(s)}{s^{\frac{n-1}{2}}} \\ &= \Gamma\left(\frac{n+1}{2}\right)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\frac{n+1}{2})}\left(\frac{s}{2}\right)^{2k}; \end{aligned} \tag{2.2}$$

  and $J_{\frac{n-1}{2}}$ is the usual Bessel function of first kind and order $\frac{n-1}{2}$ [9, 10, 22, 36].

Also, the modified Bessel function $j_{\frac{n-1}{2}}$ has the following integral representation, for all $z \in \mathbb{C}$:

$$j_{\frac{n-1}{2}}(z) = \frac{2\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{n}{2})}\int_0^1 (1-t^2)^{\frac{n}{2}-1}\cos(zt)dt.$$

Thus, for all $z \in \mathbb{C}$; we have

$$\left|j_{\frac{n-1}{2}}(z)\right| \leq e^{|\operatorname{Im} z|}. \tag{2.3}$$

Using the relation (2.1) and the properties of the function $j_{\frac{n-1}{2}}$, we deduce that the function $\varphi_{\mu,\lambda}$ satisfies the following properties [24, 27]:

-
$$\sup_{(r,x)\in\mathbb{R}\times\mathbb{R}^n} \left|\varphi_{\mu,\lambda}(r,x)\right| = 1 \tag{2.4}$$

  if and only if (μ,λ) belongs to the set Γ defined by

$$\Gamma = \mathbb{R}\times\mathbb{R}^n \ \cup\ \left\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n;\ |\mu|\leq\ |\lambda|\right\}. \tag{2.5}$$

- For all $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$; the function $\varphi_{\mu,\lambda}$ is a unique solution of the system

$$\begin{cases} \frac{\partial u}{\partial x_j}(r,x) \ = \ -i\,\lambda_j\, u(r,x);\ 1\leq j\leq n \\ Lu(r,x) \ = \ -\mu^2\, u(r,x) \\ u(0,0)=1;\ \frac{\partial u}{\partial r}\big(\,(0,x_1,\dots,x_n)=0;\ \forall\ (x_1,\dots,x_n)\in\mathbb{R}^n \end{cases}$$

  where

$$L \ = \ \frac{\partial^2}{\partial r^2}+\frac{n}{r}\frac{\partial}{\partial r}-\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}\right)^2.$$

In the following, we denote by

- dm_{n+1} the measure defined on $[0,+\infty[\times\mathbb{R}^n$; by

$$dm_{n+1}(r,x) \ = \ \sqrt{\frac{2}{\pi}}\frac{1}{(2\pi)^{\frac{n}{2}}}dr\otimes dx,$$

  where dx is the Lebesgue measure on $\mathbb{R}^n$.
- $L^p(dm_{n+1});\ p\in[1,+\infty]$, the space of measurable functions f on $[0,+\infty[\times\mathbb{R}^n$ satisfying

$$\|f\|_{p,m_{n+1}} = \begin{cases} \left(\int_0^\infty\int_{\mathbb{R}^n}|f(r,x)|^p\,dm_{n+1}(r,x)\right)^{\frac{1}{p}} < +\infty, & \text{if } 1\leq p<+\infty; \\ \underset{(r,x)\in[0,+\infty[\times\mathbb{R}^n}{\text{ess sup}} \quad |f(r,x)|<+\infty, & \text{if } p=+\infty. \end{cases}$$

- $d\nu_n$ the measure defined on $[0,+\infty[\times\mathbb{R}^n$ by

$$d\nu_n(r,x) \ = \ \frac{r^n\ dr}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}\otimes\frac{dx}{(2\pi)^{\frac{n}{2}}}.$$

- $L^p(d\nu_n),\ p\in[1,+\infty]$, the space of measurable functions f on $[0,+\infty[\times\mathbb{R}^n$ such that $\|f\|_{p,\nu_n}<+\infty$.
- Γ_+ the subset of Γ, given by

$$\Gamma_+ = [0,+\infty[\times\mathbb{R}^n\cup\left\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n;\ 0\leq\mu\leq|\lambda|\right\}.$$

- $\mathscr{B}_{\Gamma_+}$ the $\sigma-$algebra defined on Γ_+ by

$$\mathscr{B}_{\Gamma_+} \ = \ \left\{\theta^{-1}(B);\ B\in\mathscr{B}\text{or}\big([0,+\infty[\times\mathbb{R}^n\big)\right\}, \tag{2.6}$$

  where θ is the bijective function defined on Γ_+ by

$$\theta(\mu,\lambda) = \left(\sqrt{\mu^2+|\lambda|^2},\ \lambda\right).$$

- $d\gamma_n$ the measure defined on $\mathscr{B}_{\Gamma_+}$ by

$$\forall A\in\mathscr{B}_{\Gamma_+};\gamma_n(A) \ = \ \nu_n\big(\theta(A)\big).$$

- $L^p(d\gamma_n),\ p \in [1,+\infty]$, the space of measurable functions g on Γ_+ such that $\|g\|_{p,\gamma_n} < +\infty$.
- $d\tilde{\gamma}_n$ the measure defined on $\mathscr{B}_{\Gamma_+}$ by

$$d\tilde{\gamma}_n(\mu,\lambda) \;=\; \frac{2^{\frac{n}{2}}\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}}\frac{d\gamma_n(\mu,\lambda)}{(\mu^2+|\lambda|^2)^{\frac{n}{2}}}.$$

- $L^p(d\tilde{\gamma}_n),\ p \in [1,+\infty]$, the space of measurable functions g on Γ_+ such that $\|g\|_{p,\tilde{\gamma}_n} < +\infty$.
- $S_*(\mathbb{R}\times\mathbb{R}^n)$ the Schwarz space formed by the infinitely differentiable functions on $\mathbb{R}\times\mathbb{R}^n$, rapidly decreasing together with all their derivatives, and even with respect to the first variable.

Proposition 2.1.

i. *For all non negative measurable functions g on Γ_+ (respectively integrable on Γ_+ with respect to the measure $d\gamma_n$), we have*

$$\iint_{\Gamma_+} g(\mu,\lambda)d\gamma_n(\mu,\lambda) = \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})(2\pi)^{\frac{n}{2}}}\left(\int_0^\infty\int_{\mathbb{R}^n} g(\mu,\lambda)(\mu^2+|\lambda|^2)^{\frac{n-1}{2}}\mu d\mu d\lambda + \int_{\mathbb{R}^n}\int_0^{|\lambda|} g(i\mu,\lambda)(|\lambda|^2-\mu^2)^{\frac{n-1}{2}}\mu d\mu d\lambda\right).$$

ii. *For all non negative measurable functions f on $[0,+\infty[\times\mathbb{R}^n$ (respectively integrable on $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure dm_{n+1}), the function $f\circ\theta$ is measurable on Γ_+ (respectively integrable on Γ_+ with respect to the measure $d\gamma_n$) and we have*

$$\iint_{\Gamma_+} f\circ\theta(\mu,\lambda)d\gamma_n(\mu,\lambda) = \int_0^\infty\int_{\mathbb{R}^n} f(r,x)d\nu_n(r,x).$$

iii. *For all non negative measurable functions f on $[0,+\infty[\times\mathbb{R}^n$ (respectively integrable on $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure dm_{n+1}), we have*

$$\iint_{\Gamma_+} f\circ\theta(\mu,\lambda)d\tilde{\gamma}_n(\mu,\lambda) = \int_0^\infty\int_{\mathbb{R}^n} f(r,x)dm_{n+1}(r,x), \tag{2.7}$$

where θ is the function given by the relation (2.6).

In the sequel, we shall define the Fourier transform associated with the spherical mean operator and give some properties.

Definition 2.1. The Fourier transform $\mathscr{F}$ associated with the spherical mean operator is defined on $L^1(d\nu_n)$ by

$$\forall(\mu,\lambda)\in\Gamma;\quad \mathscr{F}(f)(\mu,\lambda) = \int_0^\infty\int_{\mathbb{R}^n} f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_n(r,x),$$

where $\varphi_{\mu,\lambda}$ is the function given by the relation (2.1) and Γ is the set defined by (2.5).

Remark 1. For all $(\mu,\lambda)\in\Gamma$, we have

$$\mathscr{F}(f)(\mu,\lambda) = \tilde{\mathscr{F}}(f)\circ\theta(\mu,\lambda), \tag{2.8}$$

where

$$\tilde{\mathscr{F}}(f)(\mu,\lambda) = \int_0^{\infty}\int_{\mathbb{R}^n} f(r,x) j_{\frac{n-1}{2}}(r\mu) e^{-i\langle \lambda/x\rangle} d\nu_n(r,x) \tag{2.9}$$

and $j_{\frac{n-1}{2}}$ is the modified Bessel function given by the relation (2.2).

Moreover, by the relation (2.4), the Fourier transform $\mathscr{F}$ is a bounded linear operator from $L^1(d\nu_n)$ into $L^{\infty}(d\gamma_n)$ and for all $f \in L^1(d\nu_n)$:

$$\|\mathscr{F}(f)\|_{\infty,\gamma_n} \leq \|f\|_{1,\nu_n}. \tag{2.10}$$

Theorem 2.2 (Inversion formula)**.** *Let $f \in L^1(d\nu_n)$ such that $\mathscr{F}(f) \in L^1(d\gamma_n)$, then for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}^n$, we have*

$$\begin{aligned} f(r,x) &= \iint_{\Gamma^+} \mathscr{F}(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)} d\gamma_n(\mu,\lambda) \\ &= \int_0^{\infty}\int_{\mathbb{R}^n} \tilde{\mathscr{F}}(f)(\mu,\lambda) j_{\frac{n-1}{2}}(r\mu) e^{i\langle\lambda/x\rangle} d\nu_n(\mu,\lambda). \end{aligned} \tag{2.11}$$

Lemma 2.3. *Let $\mathcal{R}_{\frac{n-1}{2}}$ be the mapping defined for all non negative measurable functions g on $[0,+\infty[\times\mathbb{R}^n$ by*

$$\begin{aligned} \mathcal{R}_{\frac{n-1}{2}}(g)(r,x) &= \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)} r^{1-n}\int_0^r (r^2-t^2)^{\frac{n}{2}-1} g(t,x)dt \\ &= \frac{2\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)} \int_0^1 (1-t^2)^{\frac{n}{2}-1} g(tr,x)dt, \end{aligned} \tag{2.12}$$

then for all non negative measurable functions $f,\ g$ on $[0,+\infty[\times\mathbb{R}^n$, we have

$$\begin{aligned} &\int_0^{\infty}\int_{\mathbb{R}^n} \mathcal{R}_{\frac{n-1}{2}}(g)(r,x) f(r,x) d\nu_n(r,x) \\ &\qquad = \int_0^{\infty}\int_{\mathbb{R}^n} g(t,x)\mathscr{W}_{\frac{n-1}{2}}(f)(t,x) dm_{n+1}(t,x) \end{aligned} \tag{2.13}$$

where $\mathscr{W}_{\frac{n-1}{2}}$ is the classical Weyl transform defined for all non negative measurable functions g on $[0,+\infty[\times\mathbb{R}^n$ by

$$\mathscr{W}_{\frac{n-1}{2}}(f)(t,x) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\int_t^{\infty} (r^2-t^2)^{\frac{n}{2}-1} f(r,x) 2r dr. \tag{2.14}$$

Proposition 2.4. *For all $f \in L^1(d\nu_n)$, the function $\mathscr{W}_{\frac{n-1}{2}}(f)$ given by the relation (2.14) is defined almost every where, belongs to the space $L^1(dm_{n+1})$ and we have*

$$\left\|\mathscr{W}_{\frac{n-1}{2}}(f)\right\|_{1,m_{n+1}} \leq \|f\|_{1,\nu_n}. \tag{2.15}$$

Moreover,

$$\tilde{\mathscr{F}}(f)(\mu,\lambda) = \Lambda_{n+1}\circ\mathscr{W}_{\frac{n-1}{2}}(f)(\mu,\lambda), \tag{2.16}$$

where Λ_{n+1} is the usual Fourier cosine transform defined on $L^1(dm_{n+1})$ by

$$\Lambda_{n+1}(g)(\mu,\lambda) = \int_0^{\infty}\int_{\mathbb{R}^n} g(r,x)\cos(r\mu) e^{-i\langle\lambda,x\rangle} dm_{n+1}(r,x).$$

and $\tilde{\mathscr{F}}$ is the Fourier-Bessel transform defined by the relation (2.9).

Remark 2. It is well known [34, 35] that the Fourier transforms $\tilde{\mathscr{F}}$ and Λ_{n+1} are topological isomorphisms from $S_*(\mathbb{R}\times\mathbb{R}^n)$ onto itself. Then, by the relation (2.16), we deduce that the classical Weyl transform $\mathscr{W}_{\frac{n-1}{2}}$ is also a topological isomorphism from $S_*(\mathbb{R}\times\mathbb{R}^n)$ onto itself, and the inverse isomorphism is given by [24]

$$\mathscr{W}^{-1}_{\frac{n-1}{2}}(f)(r,x) = (-1)^{[\frac{n}{2}]+1} F_{[\frac{n}{2}]-\frac{n}{2}+1}\left(\left(\frac{\partial}{\partial t^2}\right)^{[\frac{n}{2}]+1} f\right)(r,x), \tag{2.17}$$

where $F_a;\ a>0$ is the mapping defined on $S_*(\mathbb{R}\times\mathbb{R}^n)$ by

$$F_a(f)(r,x) = \frac{1}{2^a\Gamma(a)}\int_r^\infty (t^2-r^2)^{a-1} f(t,x)2tdt \tag{2.18}$$

and $\frac{\partial}{\partial r^2}$ is the singular partial differential operator defined by

$$\left(\frac{\partial}{\partial r^2}\right) f(r,x) = \frac{1}{r}\frac{\partial f(r,x)}{\partial r}.$$

3. THE BEURLING-HÖRMANDER THEOREM FOR THE SPHERICAL MEAN OPERATOR

This section contains the main result of this paper, that is the Beurling-Hörmander theorems for the Fourier transform $\mathscr{F}$ associated with the spherical mean operator.

We firstly recall the following result that has been established by Bonami, Demange and Jaming [4].

Theorem 3.1. *Let f be a measurable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable such that $f\in L^2(dm_{n+1})$ and let d be a real number, $d\geq 0$. If*

$$\int_0^\infty\int_{\mathbb{R}^n}\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)||\Lambda_{n+1}(f)(s,y)|}{(1+|(r,x)|+|(s,y)|)^d}e^{|(r,x)||(s,y)|}dm_{n+1}(r,x)dm_{n+1}(s,y) < +\infty,$$

then there exist a positive constant a and a polynomial P on $\mathbb{R}\times\mathbb{R}^n$ even with respect to the first variable, such that

$$f(r,x) = P(r,x)e^{-a(r^2+|x|^2)},$$

with $\mathrm{degree}(P) < \frac{d-(n+1)}{2}$.
In particular, $f=0$ for $d\leq(n+1)$.

Lemma 3.2. *Let $f\in L^2(d\nu_n)$ and let d be a real number, $d\geq 0$. If*

$$\iint_{\Gamma^+}\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)||\mathscr{F}(f)(\mu,\lambda)|e^{|(r,x)||\theta(\mu,\lambda)|}}{\big(1+|(r,x)|+|\theta(\mu,\lambda)|\big)^d}d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda) < +\infty,$$

then the function f belongs to the space $L^1(d\nu_n)$.

Proof. Let $f\in L^2(d\nu_n),\ f\neq 0$. From the relations (2.7) and (2.8), we obtain

$$\begin{aligned}\iint_{\Gamma^+}\int_0^\infty\int_{\mathbb{R}^n}&\frac{|f(r,x)||\mathscr{F}(f)(\mu,\lambda)|}{\big(1+|(r,x)|+|\theta(\mu,\lambda)|\big)^d}e^{|(r,x)||\theta(\mu,\lambda)|}d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda)\\ &=\int_0^\infty\int_{\mathbb{R}^n}\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)||\tilde{\mathscr{F}}(f)(\mu,\lambda)|}{(1+|(r,x)|+|(\mu,\lambda)|)^d}e^{|(r,x)||(\mu,\lambda)|}d\nu_n(r,x)dm_{n+1}(\mu,\lambda) < +\infty.\end{aligned}$$

Then for almost every $(\mu,\lambda)\in[0,+\infty[\times\mathbb{R}^n$,

$$\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}d\nu_n(r,x) < +\infty.$$

In particular, there exists $(\mu_0,\lambda_0)\in[0,+\infty[\times\mathbb{R}^n\setminus\{(0,0)\}$ such that

$$\tilde{\mathscr{F}}(f)(\mu_0,\lambda_0)\neq 0 \text{ and } \int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)|e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}d\nu_n(r,x)\ <\ +\infty.$$

Let h be the function defined on $[0,+\infty[$ by

$$h(s)=\frac{e^{s|(\mu_0,\lambda_0)|}}{(1+s+|(\mu_0,\lambda_0)|)^d},$$

then the function h has an absolute minimum attained at:

$$s_0=\begin{cases}\frac{d}{|(\mu_0,\lambda_0)|}-1-|(\mu_0,\lambda_0)|; & \text{if } \frac{d}{|(\mu_0,\lambda_0)|}>1+|(\mu_0,\lambda_0)|;\\ 0; & \text{if } \frac{d}{|(\mu_0,\lambda_0)|}\leq\ 1+|(\mu_0,\lambda_0)|.\end{cases}$$

Consequently,

$$\int_0^\infty\int_{\mathbb{R}^n}|f(r,x)|d\nu_n(r,x)\leq\frac{1}{h(s_0)}\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)|e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}d\nu_n(r,x)<+\infty.$$

□

Lemma 3.3. *Let $f\in L^2(d\nu_n)$ and let d be a real number, $d\geq 0$. If*

$$\iint_{\Gamma_+}\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)||\mathscr{F}(f)(\mu,\lambda)|e^{|(r,x)||\theta(\mu,\lambda)|}}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda)<+\infty,$$

then there exists $a>0$ such that the function $\tilde{\mathscr{F}}(f)$ is analytic on the set

$$\big\{(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n;\ |\operatorname{Im}\mu|<a,\ |\operatorname{Im}\lambda_j|<a;\ \forall j\in\{1,\dots,n\}\big\}.$$

Proof. From the proof of Lemma 3.2, there exists $(\mu_0,\lambda_0)\in[0,+\infty[\times\mathbb{R}^n\setminus\{(0,0)\}$ such that

$$\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)|e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}d\nu_n(r,x)<+\infty.$$

Let a be a real number such that $0<(n+1)a<|(\mu_0,\lambda_0)|$. Then we have

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)|e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}d\nu_n(r,x)\\&\qquad=\int_0^\infty\int_{\mathbb{R}^n}|f(r,x)|e^{(n+1)a|(r,x)|}\frac{e^{|(r,x)|(|(\mu_0,\lambda_0)|-(n+1)a)}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}d\nu_n(r,x)<+\infty.\end{aligned}$$

Let g be the function defined on $[0,+\infty[$ by

$$g(s)\ =\ \frac{e^{s(|(\mu_0,\lambda_0)|-(n+1)a)}}{(1+s+|(\mu_0,\lambda_0)|)^d},$$

then g admits a minimum attained at

$$s_0=\begin{cases}\frac{d}{|(\mu_0,\lambda_0)|-(n+1)a}-1-|(\mu_0,\lambda_0)|; & \text{if } \frac{d}{|(\mu_0,\lambda_0)|-(n+1)a}>1+|(\mu_0,\lambda_0)|,\\ 0; & \text{if } \frac{d}{|(\mu_0,\lambda_0)|-(n+1)a}\leq\ 1+|(\mu_0,\lambda_0)|.\end{cases}$$

Consequently,

$$\begin{aligned}\text{(3.1)}\qquad&\int_0^\infty\int_{\mathbb{R}^n}|f(r,x)|e^{(n+1)a|(r,x)|}d\nu_n(r,x)\\&\qquad\leq\frac{1}{g(s_0)}\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)|e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(\mu_0,\lambda_0)|+|(r,x)|)^d}d\nu_n(r,x)<+\infty.\end{aligned}$$

On the other hand, from the relation (2.2), we deduce that for all $(r,x) \in [0,+\infty[\times\mathbb{R}^n$; the function

$$(\mu,\lambda) \longmapsto j_{\frac{n-1}{2}}(r\mu)e^{-i\langle\lambda/x\rangle}$$

is analytic on $\mathbb{C}\times\mathbb{C}^n$ [6], even with respect to the first variable and by the relation (2.3), we deduce that $\forall\,(r,x) \in [0,+\infty[\times\mathbb{R}^n,\ \forall(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$,

$$\left|j_{\frac{n-1}{2}}(r\mu)e^{-i\langle\lambda,x\rangle}\right| \leq e^{r|\operatorname{Im}\mu|+\sum_{j=1}^{n}|\operatorname{Im}\lambda_j||x_j|} \leq e^{|(r,x)|\left[|\operatorname{Im}\mu|+\sum_{j=1}^{n}|\operatorname{Im}\lambda_j|\right]}. \tag{3.2}$$

Then the result follows from the relations (2.9), (3.1), (3.2) and by the analyticity theorem. ☐

Corollary 3.4. *Let $f\in L^2(d\nu_n),\ f\neq 0$ and let d be a real number, $d\geq 0$. If*

$$\iint_{\Gamma_+}\int_0^{\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)||\mathscr{F}(f)(\mu,\lambda)|}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}e^{|(r,x)||\theta(\mu,\lambda)|}d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda)<+\infty,$$

then for all real numbers $a,\ a>0$, we have $m_{n+1}(A_a)>0$, where

$$A_a=\left\{(\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n;\ \tilde{\mathscr{F}}(f)(\mu,\lambda)\neq 0 \text{ and } |(\mu,\lambda)|>a\right\}. \tag{3.3}$$

Proof. Let f be a function satisfying the hypothesis. From Lemma 3.2, the function f belongs to $L^1(d\nu_n)$ and consequently the function $\tilde{\mathscr{F}}(f)$ is continuous on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable. Then for all $a>0$, the set A_a given by the relation (3.3) is an open subset of $\mathbb{R}\times\mathbb{R}^n$.

So, if $m_{n+1}(A_a)=0$, then this subset is empty. This means that for every $(\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n,\ |(\mu,\lambda)|>a$, we have $\tilde{\mathscr{F}}(f)(\mu,\lambda)=0$.

From Lemma 3.2, and by analytic continuation, we deduce that $\tilde{\mathscr{F}}(f)=0$, and by the inversion formula (2.11), it follows that $f=0$. ☐

Remark 3.

i. Let f be a function satisfying the hypothesis of Corollary 3.4, then for all real numbers $a,\ a>0$, there exists $(\mu_0,\lambda_0)\in[0,+\infty[\times\mathbb{R}^n$ such that $|(\mu_0,\lambda_0)|>a$ and

$$\int_0^{\infty}\int_{\mathbb{R}^n}|f(r,x)|\frac{e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}d\nu_n(r,x)<+\infty.$$

ii. Let d and σ be non negative real numbers, $\sigma+\sigma^2\geq d$. Then the function

$$t\longmapsto\frac{e^{\sigma t}}{(1+t+\sigma)^d}$$

is not decreasing on $[0,+\infty[$.

Lemma 3.5. *Let f be a measurable function on $\mathbb{R}\times\mathbb{R}^n$ even with respect to the first variable, and $f\in L^2(d\nu_n)$. Let d be real number, $d\geq 0$. If*

$$\iint_{\Gamma^+}\int_0^{\infty}\int_{\mathbb{R}^n}|\mathscr{F}(f)(\mu,\lambda)||f(r,x)|\frac{e^{|(r,x)||\theta(\mu,\lambda)|}}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda)<+\infty,$$

then the function $\mathscr{W}_{\frac{n-1}{2}}(f)$ defined by the relation (2.14) belongs to the space $L^2(dm_{n+1})$.

Proof. From the hypothesis and the relations (2.7) and (2.8), we have

$$\begin{aligned}
&\iint_{\Gamma^+}\int_0^{\infty}\int_{\mathbb{R}^n}|\mathscr{F}(f)(\mu,\lambda)|\,\frac{|f(r,x)|e^{|(r,x)||\theta(\mu,\lambda)|}}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda)\\
&=\int_0^{\infty}\int_{\mathbb{R}^n}\int_0^{\infty}\int_{\mathbb{R}^n}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\frac{|f(r,x)|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}d\nu_n(r,x)dm_{n+1}(\mu,\lambda)\\
&<+\infty.
\end{aligned}$$

In the same manner as the proof of the inequality (3.1) in Lemma 3.2, there exists $b\in\mathbb{R},\ b>0$ such that

$$\int_0^{\infty}\int_{\mathbb{R}^n}|\tilde{\mathscr{F}}(f)(\mu,\lambda)|e^{b|(\mu,\lambda)|}dm_{n+1}(\mu,\lambda)<+\infty.$$

Consequently, the function $\tilde{\mathscr{F}}(f)$ belongs to the space $L^1(d\nu_n)$ and by the inversion formula for $\tilde{\mathscr{F}}$, we deduce that

$$f(r,x)\ =\ \int_0^{\infty}\int_{\mathbb{R}^n}\tilde{\mathscr{F}}(f)(\mu,\lambda)j_{\frac{n-1}{2}}(r\mu)e^{i\langle\lambda/x\rangle}d\nu_n(\mu,\lambda).\ \text{a.e.}$$

In particular, the function f is bounded and

$$\|f\|_{\infty,\nu_n}\le\left\|\tilde{\mathscr{F}}(f)\right\|_{1,\nu_n}. \tag{3.4}$$

By virtue of the relation (2.14), we get

$$\begin{aligned}
\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|&\le\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\int_t^{\infty}(r^2-t^2)^{\frac{n}{2}-1}|f(r,x)|2rdr\\
&=\frac{r^n}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\int_1^{\infty}(y^2-1)^{\frac{n}{2}-1}|f(ry,x)|2ydy.
\end{aligned}$$

Using Minkowski's inequality for integrals [12], we get:

$$\begin{aligned}
&\left(\int_0^{\infty}\int_{\mathbb{R}^n}\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|^2dm_{n+1}(r,x)\right)^{\frac{1}{2}}\\
&\le\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\left[\int_0^{\infty}\int_{\mathbb{R}^n}\left(\int_1^{\infty}r^n(y^2-1)^{\frac{n}{2}-1}|f(ry,x)|2ydy\right)^2dm_{n+1}(r,x)\right]^{\frac{1}{2}}\\
&\le\frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\int_1^{\infty}\left(\int_0^{\infty}\int_{\mathbb{R}^n}r^{2n}(y^2-1)^{n-2}|f(ry,x)|^2dm_{n+1}(r,x)\right)^{\frac{1}{2}}2ydy\\
&=\frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})}\left[\int_1^{\infty}(y^2-1)^{\frac{n}{2}-1}y^{-n+\frac{1}{2}}dy\right]\left[\int_0^{\infty}\int_{\mathbb{R}^n}s^{2n}|f(s,x)|^2dm_{n+1}(s,x)\right]^{\frac{1}{2}}\\
&=\frac{\Gamma(\frac{1}{4})}{2^{\frac{n}{2}}\Gamma(\frac{2n+1}{4})}\left[\int_0^{\infty}\int_{\mathbb{R}^n}s^{2n}|f(s,x)|^2dm_{n+1}(s,x)\right]^{\frac{1}{2}}.
\end{aligned} \tag{3.5}$$

Using the relations (3.1), (3.4) and (3.5), we deduce that

$$\begin{aligned}
\left\|\mathscr{W}_{\frac{n-1}{2}}(f)\right\|_{2,m_{n+1}}&=\left(\int_0^{\infty}\int_{\mathbb{R}^n}\left|\mathscr{W}_{\frac{n-1}{2}}(f)\right|^2(r,x)dm_{n+1}(r,x)\right)^{\frac{1}{2}}\\
&\le K_n\int_0^{\infty}\int_{\mathbb{R}^n}|f(s,x)|e^{(n+1)a|(s,x)|}d\nu_n(s,x)<+\infty,
\end{aligned}$$

where

$$K_n = \frac{\Gamma\left(\frac{1}{4}\right)}{2^{\frac{n}{2}}\Gamma\left(\frac{2n+1}{4}\right)}\left(\sqrt{\frac{\pi}{2}}\Gamma\left(\frac{n+1}{2}\right)2^{\frac{n-1}{2}}\max_{s\geq 0}(s^n e^{-(n+1)as})\|f\|_{\infty,\nu_n}\right)^{\frac{1}{2}}.$$

□

Theorem 3.6. *Let $f \in L^2(d\nu_n)$; $f \neq 0$ and let d be a real number; $d \geq 0$.*
If

$$\iint_{\Gamma^+}\int_0^{\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)||\mathscr{F}(f)(\mu,\lambda)|e^{|(r,x)||\theta(\mu,\lambda)|}}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda) < +\infty;$$

then

$$\int_0^{\infty}\int_{\mathbb{R}^n}\int_0^{\infty}\int_{\mathbb{R}^n}\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right| \times \frac{e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)dm_{n+1}(\mu,\lambda) < +\infty$$

where $\mathscr{W}_{\frac{n-1}{2}}$ is the Weyl transform defined by the relation (2.14).

Proof. From the hypothesis, the relations (2.7), (2.8) and Fubini's theorem, we have

$$\begin{aligned}(3.6)\quad &\iint_{\Gamma^+}\int_0^{\infty}\int_{\mathbb{R}^n}|f(r,x)||\mathscr{F}(f)(\mu,\lambda)|\frac{e^{|(r,x)||\theta(\mu,\lambda)|}}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda)\\ &= \int_0^{\infty}\int_{\mathbb{R}^n}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\int_0^{\infty}\int_{\mathbb{R}^n}\frac{|f(r,x)|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}d\nu_n(r,x)dm_{n+1}(\mu,\lambda)\\ &< +\infty.\end{aligned}$$

i. If $d = 0$, then by the relation (2.13) and Fubini's theorem, we get

$$\begin{aligned}(3.7)\quad &\int_0^{\infty}\int_{\mathbb{R}^n}\int_0^{\infty}\int_{\mathbb{R}^n}\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|e^{|(r,x)||(\mu,\lambda)|}dm_{n+1}(r,x)dm_{n+1}(\mu,\lambda)\\ &\leq\int_0^{\infty}\int_{\mathbb{R}^n}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\left(\int_0^{\infty}\int_{\mathbb{R}^n}\mathscr{W}_{\frac{n-1}{2}}(|f|)(r,x)e^{|(r,x)||(\mu,\lambda)|}dm_{n+1}(r,x)\right)dm_{n+1}(\mu,\lambda)\\ &\leq\int_0^{\infty}\int_{\mathbb{R}^n}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\left(\int_0^{\infty}\int_{\mathbb{R}^n}|f(r,x)|\mathscr{R}_{\frac{n-1}{2}}(e^{|(\cdot,\cdot)||(\mu,\lambda)|})(r,x)d\nu_n(r,x)\right)dm_{n+1}(\mu,\lambda).\end{aligned}$$

However, by (2.12), we deduce that for all $(r,x) \in [0,+\infty[\times\mathbb{R}^n$,

$$(3.8)\qquad \mathcal{R}_{\frac{n-1}{2}}\left(e^{|(\cdot,\cdot)||(\mu,\lambda)|}\right)(r,x) \leq e^{|(r,x)||(\mu,\lambda)|}.$$

Combining the relations (3.6), (3.7) and (3.8), we deduce that

$$\begin{aligned}&\int_0^{\infty}\int_{\mathbb{R}^n}\int_0^{\infty}\int_{\mathbb{R}^n}\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|e^{|(r,x)||(\mu,\lambda)|}dm_{n+1}(r,x)dm_{n+1}(\mu,\lambda)\\ &\leq\int_0^{\infty}\int_{\mathbb{R}^n}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\int_0^{\infty}\int_{\mathbb{R}^n}|f(r,x)|e^{|(r,x)||(\mu,\lambda)|}d\nu_n(r,x)dm_{n+1}(\mu,\lambda) < +\infty.\end{aligned}$$

ii. For $d > 0$, let $B_d = \{(r,x) \in [0,+\infty[\times\mathbb{R}^n;\ |(r,x)| \leq d\}$. We have

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^n}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\int_0^\infty\int_{\mathbb{R}^n}\frac{|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)dm_{n+1}(\mu,\lambda)\\ &\leq \iint_{B_d^c}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\left(\int_0^\infty\int_{\mathbb{R}^n}\frac{\mathscr{W}_{\frac{n-1}{2}}(|f|)(r,x)e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)\right)dm_{n+1}(\mu,\lambda)\\ &\quad+\iint_{B_d}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\left(\int_0^\infty\int_{\mathbb{R}^n}\frac{\mathscr{W}_{\frac{n-1}{2}}(|f|)(r,x)e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)\right)dm_{n+1}(\mu,\lambda).\end{aligned}$$

From the relation (2.13), we deduce that

$$\begin{aligned}\text{(3.9)}\quad &\iint_{B_d^c}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\left(\int_0^\infty\int_{\mathbb{R}^n}\frac{\mathscr{W}_{\frac{n-1}{2}}(|f|)(r,x)e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)\right)dm_{n+1}(\mu,\lambda)\\ &=\iint_{B_d^c}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\int_0^\infty\int_{\mathbb{R}^n}|f(r,x)|\\ &\quad\times\mathscr{R}_{\frac{n-1}{2}}\left(\frac{e^{|(\cdot,\cdot)||(\mu,\lambda)|}}{(1+|(\cdot,\cdot)|+|(\mu,\lambda)|)^d}\right)(r,x)d\nu_n(r,x)dm_{n+1}(\mu,\lambda).\end{aligned}$$

However, from the relation (2.12) and ii) of Remark 3, we deduce that for all $(\mu,\lambda) \in B_d^c$, we have

$$\text{(3.10)}\qquad \mathscr{R}_{\frac{n-1}{2}}\left(\frac{e^{|(\cdot,\cdot)||(\mu,\lambda)|}}{(1+|(\cdot,\cdot)|+|(\mu,\lambda)|)^d}\right)(r,x)\ \leq\ \frac{e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}.$$

Combining the relations (3.6), (3.9) and (3.10), we get

$$\begin{aligned}&\iint_{B_d^c}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\left(\int_0^\infty\int_{\mathbb{R}^n}\frac{\mathscr{W}_{\frac{n-1}{2}}(|f|)(r,x)e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)\right)dm_{n+1}(\mu,\lambda)\\ &\leq\iint_{B_d^c}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\left(\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}d\nu_n(r,x)\right)dm_{n+1}(\mu,\lambda)\\ &\leq\int_0^\infty\int_{\mathbb{R}^n}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\left(\int_0^\infty\int_{\mathbb{R}^n}\frac{|f(r,x)|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}d\nu_n(r,x)\right)dm_{n+1}(\mu,\lambda)\\ &<+\infty.\end{aligned}$$

We have

$$\begin{aligned}&\iint_{B_d}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\iint_{B_d}\frac{|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)dm_{n+1}(\mu,\lambda)\\ &\leq e^{d^2}\left(\iint_{B_d}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|dm_{n+1}(\mu,\lambda)\right)\left(\iint_{B_d}\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|dm_{n+1}(r,x)\right)\\ &\leq e^{d^2}m_{n+1}(B_d)\|\mathscr{F}(f)\|_{\infty,\gamma_n}\left\|\mathscr{W}_{\frac{n-1}{2}}(f)\right\|_{1,m_{n+1}}.\end{aligned}$$

By the relations (2.10) and (2.15), we deduce that

$$\begin{aligned}\iint_{B_d}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\iint_{B_d}\frac{\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)dm_{n+1}(\mu,\lambda)&\\ \leq e^{d^2}m_{n+1}(B_d)\|f\|^2_{1,\nu_n}<+\infty.&\end{aligned}$$

By the relation (2.13), we get

$$\iint_{B_d}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\left(\iint_{B_d^c}\frac{\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)\right)dm_{n+1}(\mu,\lambda)$$

$$\leq \iint_{B_d}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\int_0^\infty\int_{\mathbb{R}^n}\frac{\mathscr{W}_{\frac{n-1}{2}}(|f|)(r,x)e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}\mathbf{1}_{B_d^c}(r,x)dm_{n+1}(r,x)dm_{n+1}(r,x)$$

$$= \iint_{B_d}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\left(\int_0^\infty\int_{\mathbb{R}^n}|f(r,x)|\right.$$

$$\left.\times\,\mathscr{R}_{\frac{n-1}{2}}\left(\frac{e^{|(\cdot,\cdot)||(\mu,\lambda)|}}{(1+|(\cdot,\cdot)|+|(\mu,\lambda)|)^d}\mathbf{1}_{B_d^c}(\cdot,\cdot)\right)(r,x)d\nu_n(r,x)\right)dm_{n+1}(\mu,\lambda).$$

However, by ii) of Remark 3 and the relation (2.10), we deduce that for all $(\mu,\lambda)\in B_d$:

$$\mathscr{R}_{\frac{n-1}{2}}\left(\frac{e^{|(\cdot,\cdot)||(\mu,\lambda)|}}{(1+|(\cdot,\cdot)|+|(\mu,\lambda)|)^d}\mathbf{1}_{B_d^c}(\cdot,\cdot)\right)(r,x)\leq\frac{e^{d|(r,x)|}}{(1+|(r,x)|+d)^d}\mathbf{1}_{B_d^c}(r,x).$$

Thus,

$$\iint_{B_d}\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\left(\iint_{B_d^c}\frac{\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)\right)dm_{n+1}(\mu,\lambda)$$
$$\leq\|f\|_{1,\nu_n}m_{n+1}(B_d)\iint_{B_d^c}|f(r,x)|\frac{e^{d|(r,x)|}}{(1+|(r,x)|+d)^d}d\nu_n(r,x). \tag{3.11}$$

On the other hand, from i) of Remark 3, there exists $(\mu_0,\lambda_0)\in[0,+\infty[\times\mathbb{R}^n$, $|(\mu_0,\lambda_0)|>d$ such that

$$\int_0^\infty\int_{\mathbb{R}^n}\frac{e^{|(r,x)||(\mu_0,\lambda_0)|}|f(r,x)|}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}d\nu_n(r,x)<+\infty.$$

Again, by ii) of Remark 3, we have

$$\iint_{B_d^c}|f(r,x)|\frac{e^{d|(r,x)|}}{(1+|(r,x)|+d)^d}d\nu_n(r,x)$$
$$\leq\iint_{B_d^c}|f(r,x)|\frac{e^{|(r,x)||(\mu_0,\lambda_0)|}}{(1+|(r,x)|+|(\mu_0,\lambda_0)|)^d}d\nu_n(r,x)<+\infty. \tag{3.12}$$

The relations (3.11) and (3.12) imply that

$$\iint_{B_d}\iint_{B_d^c}\left(\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right|\frac{\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)\right)dm_{n+1}(\mu,\lambda)<+\infty,$$

and the proof of Theorem 3.1 is complete. □

Theorem 3.7 (Beurling Hörmander for $\mathscr{R}$)**.** *Let f be a measurable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable and such that $f\in L^2(d\nu_n)$.*

Let d be a real number, $d\geq 0$. If

$$\iint_{\Gamma^+}\int_0^\infty\int_{\mathbb{R}^n}|f(r,x)|\frac{|\mathscr{F}(f)(\mu,\lambda)|e^{|(r,x)||\theta(\mu,\lambda)|}}{(1+|(r,x)|+|\theta(\mu,\lambda)|)^d}d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda)<+\infty,$$

then

- *For* $d \leq n+1,\ f = 0$.
- *For* $d > n+1$, *there exist a positive constant* a *and a polynomial* P *on* $\mathbb{R} \times \mathbb{R}^n$ *even with respect to the first variable, such that*

$$f(r,x) = P(r,x)e^{-a(r^2+|x|^2)}$$

with $\operatorname{degree}(P) < \frac{d-(n+1)}{2}$.

Proof. Let f be a function satisfying the hypothesis. Then, from Theorem 3.1, we have

$$\int_0^{\infty}\int_{\mathbb{R}^n}\int_0^{\infty}\int_{\mathbb{R}^n}\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|\left|\tilde{\mathscr{F}}(f)(\mu,\lambda)\right| \times \frac{e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)dm_{n+1}(\mu,\lambda) < +\infty. \tag{3.13}$$

On the other hand, from Proposition 2.1, Lemma 3.2 and Lemma 3.3, we deduce that the function $\mathscr{W}_{\frac{n-1}{2}}(f)$ belongs to the space $L^1(dm_{n+1}) \cap L^2(dm_{n+1})$ and by (2.16), we have

$$\tilde{\mathscr{F}}(f) = \Lambda_{n+1}\left(\mathscr{W}_{\frac{n-1}{2}}(f)\right).$$

Substituting into (3.13), we get

$$\int_0^{\infty}\int_{\mathbb{R}^n}\int_0^{\infty}\int_{\mathbb{R}^n}\left|\mathscr{W}_{\frac{n-1}{2}}(f)(r,x)\right|\left|\Lambda_{n+1}\left(\mathscr{W}_{\frac{n-1}{2}}(f)\right)(\mu,\lambda)\right| \times \frac{e^{|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|+|(\mu,\lambda)|)^d}dm_{n+1}(r,x)dm_{n+1}(\mu,\lambda) < +\infty.$$

Applying Theorem 3.1 when f is replaced by $\mathscr{W}_{\frac{n-1}{2}}(f)$, we deduce that

- If $d \leq n+1,\ \mathscr{W}_{\frac{n-1}{2}}(f) = 0$ and by Remark 2, we have $f = 0$.
- If $d > n+1$, there exist $a > 0$ and a polynomial Q on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable such that

$$\begin{aligned}\mathscr{W}_{\frac{n-1}{2}}(f)(r,x) &= Q(r,x)e^{-a(r^2+|x|^2)} \\ &= \sum_{2k+|\alpha|\leq m} a_{k,\alpha}r^{2k}x^{\alpha}e^{-a(r^2+|x|^2)}; \quad x^{\alpha} = x_1^{\alpha_1}\cdots x_n^{\alpha_n}.\end{aligned}$$

In particular, the function $\mathscr{W}_{\frac{n-1}{2}}(f)$ lies in $S_*(\mathbb{R}\times\mathbb{R}^n)$ and by Remark 2, the function f belongs to $S_*(\mathbb{R}\times\mathbb{R}^n)$ and we have

$$f = \mathscr{W}_{\frac{n-1}{2}}^{-1}\left(Q(r,x)e^{-a(r^2+|x|^2)}\right).$$

Now, using the relation (2.17), we obtain

$$\begin{aligned}f(r,x) =& \mathscr{W}_{\frac{n-1}{2}}^{-1}(Q(t,y)e^{-a(t^2+|y|^2)})(r,x) \\ =&(-1)^{[\frac{n}{2}]+1}F_{[\frac{n}{2}]-\frac{n}{2}+1}\left[\left(\frac{\partial}{\partial t^2}\right)^{[\frac{n}{2}]+1}Q(t,y)e^{-a(t^2+|y|^2)}\right](r,x) \\ =&(-1)^{[\frac{n}{2}]+1}\sum_{2k+|\alpha|\leq m}a_{k,\alpha}F_{[\frac{n}{2}]-\frac{n}{2}+1}\left[\left(\frac{\partial}{\partial t^2}\right)^{[\frac{n}{2}]+1}(t^{2k}y^{\alpha}e^{-a(t^2+|y|^2)})\right](r,x).\end{aligned} \tag{3.14}$$

However, for all $l \in \mathbb{N}$,

$$\left(\frac{\partial}{\partial t^2}\right)^l \left(t^{2k} y^\alpha e^{-a(t^2+|y|^2)}\right) = \left(\sum_{j=0}^{\min(l,k)} C_l^j \frac{2^j k!}{(k-j)!}(-2a)^{k-j} t^{2(k-j)}\right) y^\alpha e^{-a(t^2+|y|^2)} \tag{3.15}$$

and for all $b > 0$,

$$F_b\big(t^{2k} y^\alpha e^{-a(t^2+|y|^2)}\big)(r,x) = \frac{1}{2^b\Gamma(b)}\left(\sum_{j=0}^{k} C_k^j \frac{\Gamma(b+k-j)}{a^{\mu+k-j} r^{2j}} r^{2j}\right) x^\alpha e^{-a(r^2+|x|^2)}, \tag{3.16}$$

where the transform F_b is defined by the relation (2.18).

Combining the relations (3.14), (3.15) and (3.16), we deduce that

$$f(r,x) \;=\; P(r,x) e^{-a(r^2+|x|^2)},$$

where P is a polynomial, even with respect to the first variable and $\mathrm{degree}(P) = \mathrm{degree}(Q)$. □

4. APPLICATIONS OF THE BEURLING-HÖRMANDER THEOREM

This section is devoted to giving some applications of the Beurling-Hörmander theorem for the spherical mean operator. More precisely, we prove a Gelfand-Shilov theorem for the Fourier transform $\mathscr{F}$ and establish a Cowling Price type theorem for this transform.

Lemma 4.1. *Let P be a polynomial on $\mathbb{R} \times \mathbb{R}^n$; $P \neq 0$ with $\mathrm{degree}(P) = m$. Then there exist two positive constants A and C such that*

$$\forall\, t \geq A, \quad \varphi(t) = \int_{S^n} |P(tw)| d\sigma_n(w) \geq C t^m,$$

where $d\sigma_n$ is the surface measure on the unit sphere S^n of $\mathbb{R} \times \mathbb{R}^n$.

Proof. Let P be a polynomial on $\mathbb{R} \times \mathbb{R}^n$, $P \neq 0$ and $\mathrm{degree}\ (P) = m$. Then we have

$$\varphi(t) = \int_{S^n} \left| \sum_{k=0}^{m} a_k(w) t^k \right| d\sigma_n(w),$$

where $a_k,\ 0 \leq k \leq m$ are continuous functions on S^n and $a_m \neq 0$.

Then the function φ is continuous on $[0, +\infty[$ and by the dominated convergence theorem, we have

$$\varphi(t) \sim C_m t^m \quad (t \longrightarrow +\infty), \tag{4.1}$$

where

$$C_m = \int_{S^n} \big|a_m(w)\big| d\sigma_n(w) > 0.$$

Now, by (4.1), there exists $A > 0$ such that

$$\forall\, t \geq A; \quad p(t) \geq \frac{C_m}{2} t^m.$$

□

Theorem 4.2 (Gelfand-Shilov)**.** *Let p, q be two conjugate exponents, $p,\ q \in]1, +\infty[$. Let $\eta,\ \xi$ be two positive real numbers such that $\xi\eta \geq 1$.*

Let f be a measurable function on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable such that $f \in L^2(d\nu_n)$.

If

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r,x)|e^{\frac{\xi^p |(r,x)|^p}{p}}}{(1+|(r,x)|)^d} d\nu_n(r,x) < +\infty$$

and

$$\iint_{\Gamma^+} \frac{|\mathscr{F}(f)(\mu,\lambda)|e^{\frac{\xi^q |(r,x)|^q}{q}}}{(1+|\theta(\mu,\lambda)|)^d} d\tilde{\gamma}_n(\mu,\lambda) < +\infty, \quad d \geq 0,$$

then

i. *For* $d \leq \frac{n+1}{2},\ f = 0$.

ii. *For* $d > \frac{n+1}{2}$*, we have:*
- $f = 0$ *for* $\xi\eta > 1$;
- $f = 0$ *for* $\xi\eta = 1$ *and* $p \neq 2$;
- $f(r,x) = P(r,x)e^{-a(r^2+|x|^2)}$ *for* $\xi\eta = 1$ *and* $p = q = 2$*, where* $a > 0$ *and* P *is a polynomial on* $\mathbb{R} \times \mathbb{R}^n$ *even with respect to the first variable, with* $\mathrm{degree}\,(P) < d - \frac{n+1}{2}$.

Proof. Let f be a function satisfying the hypothesis. Since $\xi\eta \geq 1$, by a convexity argument we have

(4.2) $$\begin{aligned}&\iint_{\Gamma^+}\int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r,x)||\mathscr{F}(f)(\mu,\lambda)|}{\big(1+|(r,x)|+|\theta(\mu,\lambda)|\big)^{2d}} e^{|(r,x)||\theta(\mu,\lambda)|} d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda)\\ &\leq \iint_{\Gamma^+}\int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r,x)||\mathscr{F}(f)(\mu,\lambda)|}{(1+|(r,x)|)^d(1+|\theta(\mu,\lambda)|)^d} e^{\eta\xi|(r,x)||\theta(\mu,\lambda)|} d\nu_n(r,x)d\tilde{\gamma}_n(\mu,\lambda)\\ &\leq \iint_{\Gamma^+} \frac{|\mathscr{F}(f)(\mu,\lambda)|e^{\frac{\eta^q|\theta(\mu,\lambda)|^q}{q}}}{(1+|\theta(\mu,\lambda)|)^d} d\tilde{\gamma}_n(\mu,\lambda) \int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r,x)|e^{\frac{\xi^p|(r,x)|^p}{p}}}{(1+|(r,x)|)^d} d\nu_n(r,x)\\ &< +\infty.\end{aligned}$$

Then from the Beurling-Hörmander theorem, we deduce that
i. For $d \leq \frac{n+1}{2},\ f = 0$.
ii. For $d > \frac{n+1}{2}$, there exist a positive constant a and a polynomial P on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable such that

(4.3) $$f(r,x) = P(r,x)e^{-a(r^2+|x|^2)}$$

with $\mathrm{degree}(P) < \frac{2d-(n+1)}{2}$, and using standard calculus, we obtain

(4.4) $$\tilde{\mathscr{F}}(f)(\mu,\lambda) = Q(\mu,\lambda)e^{-\frac{1}{4a}(\mu^2+|\lambda|^2)},$$

where Q is a polynomial on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, with $\mathrm{degree}\,(Q) = \mathrm{degree}(P)$.

On the other hand, from the relations (2.7), (2.8), (4.2), (4.3) and (4.4), we get

$$\int_0^\infty \int_{\mathbb{R}^n}\int_0^\infty \int_{\mathbb{R}^n} \frac{|P(r,x)||Q(\mu,\lambda)|e^{\xi\eta|(r,x)||(\mu,\lambda)|}}{(1+|(r,x)|)^d(1+|(\mu,\lambda)|)^d} \times e^{\frac{-\left(\mu^2+|\lambda|^2\right)}{4a}} e^{-a\left(r^2+|x|^2\right)} d\nu_n(r,x)dm_{n+1}(\mu,\lambda) < +\infty.$$

So,

$$\int_0^{\infty}\int_{\mathbb{R}^n}\frac{\varphi(t)}{(1+t)^d}\frac{\psi(\rho)}{(1+\rho)^d}e^{\xi\eta t\rho}e^{-at^2-\frac{\rho^2}{4a}}t^{2n}\rho^n dtd\rho<+\infty, \tag{4.5}$$

where

$$\varphi(t)=\int_{S^n}|P(tw)||w_1|^n d\sigma_n(w)$$

and

$$\psi(\rho)=\int_{S^n}|Q(\rho w)|d\sigma_n(w).$$

- Suppose that $\xi\eta>1$. If $f\neq 0$, then each of the polynomials P and Q is not identically zero. Let $m=\mathrm{degree}(P)=\mathrm{degree}(Q)$.
  From Lemma 4.1, there exist two positive constants A and C such that

  $$\forall t\geq A,\qquad \varphi(t)\geq Ct^m$$

  and

  $$\forall \rho\geq A,\qquad \psi(\rho)\geq C\rho^m.$$

  Then the inequality (4.5) leads to

  $$\int_A^{\infty}\int_A^{\infty}\frac{e^{\xi\eta t\rho}}{(1+t)^d(1+\rho)^d}e^{-at^2}e^{-\frac{\rho^2}{4a}}dtd\rho<+\infty. \tag{4.6}$$

  Let $\varepsilon>0$ such that $c=\eta\xi-\varepsilon>1$. The relation (4.6) implies that

  $$\int_A^{\infty}\int_A^{\infty}\frac{e^{\varepsilon\rho t}}{(1+t)^d(1+\rho)^d}e^{c\rho t}e^{-at^2}e^{-\frac{\rho^2}{4a}}dtd\rho<+\infty. \tag{4.7}$$

  However, for all $t\geq A\geq\frac{d}{\varepsilon}$ and $\rho\geq A$, we have

  $$\frac{e^{\varepsilon\rho t}}{(1+t)^d(1+\rho)^d}\geq\frac{e^{\varepsilon A^2}}{(1+A)^{2d}}$$

  and by (4.7), it follows that

  $$\int_A^{\infty}\int_A^{\infty}e^{c\rho t-at^2}e^{-\frac{\rho^2}{4a}}dtd\rho<+\infty. \tag{4.8}$$

  Let $F(t)=\int_A^{\infty}e^{c\rho t-\frac{\rho^2}{4a}}d\rho$, then the function F can be written as

  $$F(t)=e^{ac^2t^2}\left(\int_A^{\infty}e^{-\frac{\rho^2}{4a}}d\rho+2a\gamma e^{-\frac{A^2}{4a}}\int_0^t e^{cAs-ac^2s^2}ds\right).$$

  In particular,

  $$F(t)\geq e^{ac^2t^2}\int_A^{\infty}e^{-\frac{\rho^2}{4a}}d\rho.$$

  Thus,

  $$\int_A^{\infty}\int_A^{\infty}e^{c\rho t-at^2-\frac{\rho^2}{4a}}dtd\rho\geq\int_A^{\infty}e^{a(c^2-1)t^2}dt\int_A^{\infty}e^{-\frac{\rho^2}{4a}}d\rho=+\infty$$

  because $c>1$. This contradicts the relation (4.8) and shows that $f=0$.

- Suppose that $\xi\eta = 1$ and $p \neq 2$.
  In this case, we have $p > 2$ or $q > 2$.
  Suppose that $q > 2$. Then from the second hypothesis and the relations (2.7), (2.8) and (4.4), we get

$$\int_0^\infty \frac{\psi(\rho)e^{-\frac{\rho^2}{4a}}e^{\frac{\eta^q\rho^q}{q}}}{(1+\rho)^d}\rho^n d\rho < +\infty. \tag{4.9}$$

  If $f \neq 0$, then the polynomial Q is not identically zero, and by Lemma 4.1 and the relation (4.9), it follows that there exists $A > 0$ such that

$$\int_A^\infty \frac{e^{-\frac{\rho^2}{4a}}e^{\frac{\eta^q\rho^q}{q}}}{(1+\rho)^d}d\rho < +\infty,$$

  which is impossible because $q > 2$.

The proof of Theorem 4.2 is thus complete. □

Theorem 4.3 (Cowling-Price for spherical mean operator)**.** *Let η, ξ, w_1 and w_2 be non negative real numbers such that $\eta\xi \geq \frac{1}{4}$. Let p, q be two exponents, $p,\ q \in [1, +\infty]$ and let f be a measurable function on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable such that $f \in L^2(d\nu_n)$. If*

$$\left\| \frac{e^{\xi|(\cdot,\cdot)|^2} f}{(1+|(\cdot,\cdot)|)^{w_1}} \right\|_{p,\nu_n} < +\infty \tag{4.10}$$

and

$$\left\| \frac{e^{\eta|\theta(\cdot,\cdot)|^2}}{(1+|\theta(\cdot,\cdot)|)^{w_2}} \mathscr{F}(f) \right\|_{q,\tilde{\gamma}_n} < +\infty, \tag{4.11}$$

then

i. *For $\xi\eta > \frac{1}{4}$, $f = 0$.*
ii. *For $\xi\eta = \frac{1}{4}$, there exist a positive constant a and a polynomial P on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable such that*

$$f(r,x) = P(r,x)e^{-a(r^2+|x|^2)}.$$

Proof. Let p' and q' be the conjugate exponents of p respectively q.

Let us pick $d_1,\ d_2 \in \mathbb{R}$ such that $d_1 > 2n+1$ and $d_2 > n+1$. Then from Hölder's inequality and the relations (4.10) and (4.11), we deduce that

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r,x)|e^{\xi|(r,x)|^2}}{(1+|(r,x)|)^{w_1+d_1/p'}} d\nu_n(r,x) \\ &\leq \left\| \frac{e^{\xi|(\cdot,\cdot)|^2} f}{(1+|(\cdot,\cdot)|)^{w_1}} \right\|_{p,\nu_n} \left\| \frac{1}{(1+|(\cdot,\cdot)|)^{d_1/p'}} \right\|_{p',\nu_n} \\ &= \left\| \frac{e^{\xi|(\cdot,\cdot)|^2} f}{(1+|(\cdot,\cdot)|)^{w_1}} \right\|_{p,\nu_n} \left(\int_0^\infty \int_{\mathbb{R}^n} \frac{d\nu_n(r,x)}{(1+|(r,x)|)^{d_1}} \right)^{\frac{1}{p'}} < +\infty. \end{aligned} \tag{4.12}$$

and

$$\iint_{\Gamma^+} \frac{|\mathscr{F}(f)(\mu,\lambda)|e^{\eta|\theta(\mu,\lambda)|^2}}{(1+|\theta(\mu,\lambda)|)^{w_2+d_2/q'}} d\tilde{\gamma}_n(\mu,\lambda) \leq \left\| \frac{e^{\eta|\theta(\cdot,\cdot)|^2}}{(1+|\theta(\cdot,\cdot)|)^{w_2}} \mathscr{F}(f) \right\|_{q,\tilde{\gamma}_n} \left\| \frac{1}{(1+|\theta(\cdot,\cdot)|)^{d_2/q'}} \right\|_{q',\tilde{\gamma}_n}.$$

By the relation (2.7), we obtain

$$\iint_{\Gamma^+} \frac{|\mathscr{F}(f)(\mu,\lambda)|e^{\eta|\theta(\mu,\lambda)|^2}}{(1+|\theta(\mu,\lambda)|)^{w_2+d_2/q'}} d\tilde{\gamma}_n(\mu,\lambda) \leq \left\| \frac{e^{\eta|\theta(\cdot,\cdot)|^2}}{(1+|\theta(\cdot,\cdot)|)^{w_2}} \mathscr{F}(f) \right\|_{q,\tilde{\gamma}_n} \left(\int_0^\infty \int_{\mathbb{R}^n} \frac{dm_{n+1}(\mu,\lambda)}{(1+|(\mu,\lambda)|)^{d_2}} \right)^{\frac{1}{q'}} < +\infty. \tag{4.13}$$

Let $d > \max\left(w_1 + \frac{d_1}{p'},\ w_2 + \frac{d_2}{q'},\ \frac{n+1}{2}\right)$, then from the relations (4.12) and (4.13), we have

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{|f(r,x)|e^{\xi|(r,x)|^2}}{(1+|(r,x)|)^d} d\nu_n(r,x) < +\infty$$

and

$$\iint_{\Gamma^+} \frac{|\mathscr{F}(f)(\mu,\lambda)|e^{\eta|\theta(\mu,\lambda)|^2}}{(1+|\theta(\mu,\lambda)|)^d} d\tilde{\gamma}_n(\mu,\lambda) < +\infty.$$

Then the desired result follows from Theorem 4.2. □

Remark 4. The Hardy theorem is a special case of Theorem 4.2, when $p = q = +\infty$.

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