ON AN OPEN PROBLEM CONCERNING AN INTEGRAL INEQUALITY

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Abstract: In this note, we generalize an open problem posed by Q. A. Ngô in the paper, Notes on an integral inequality, *J. Inequal. Pure & Appl. Math.*, **7**(4)(2006), Art. 120 and give a positive answer to it using an analytic approach.

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Contents

Open Problem Concerning an Integral Inequality
Wen-jun Liu, Chun-Cheng Li and Jian-wei Dong
vol. 8, iss. 3, art. 74, 2007

journal of inequalities in pure and applied mathematics
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1. Introduction

In the paper [2], Q.A. Ngô studied a very interesting integral inequality and proved the following result.

Theorem 1.1. *Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f(t)dt \geq \int_x^1 t\,dt, \quad \forall\, x \in [0,1]. \tag{1.1}$$

Then the inequalities

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x^\alpha f(x)dx \tag{1.2}$$

and

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x f^\alpha(x)dx \tag{1.3}$$

hold for every positive real number $\alpha > 0$.

Next, they proposed the following open problem.

Problem 1. Let $f(x)$ be a continuous function on $[0,1]$ satisfying

$$\int_x^1 f(t)dt \geq \int_x^1 t\,dt, \quad \forall\, x \in [0,1]. \tag{1.4}$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx, \tag{1.5}$$

hold for α and β?

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We note that, as an open problem, the condition (1.4) maybe result in an unreasonable restriction on $f(x)$. We remove it herein and propose another more general open problem.

Problem 2. Under what conditions does the inequality

$$\int_0^b f^{\alpha+\beta}(x)dx \geq \int_0^b x^\alpha f^\beta(x)dx, \tag{1.6}$$

hold for b, α and β?

In this note, we give an answer to Problem 2 using an analytic approach. Our main results are Theorem 2.1 and Theorem 2.4 which will be proved in Section 2.

vol. 8, iss. 3, art. 74, 2007

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2. Main Results and Proofs

Firstly, we have

Theorem 2.1. *Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f^\beta(t)dt \geq \int_x^1 t^\beta\, dt, \qquad \forall\, x \in [0,1]. \tag{2.1}$$

Then the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx, \tag{2.2}$$

holds for every positive real number $\alpha > 0$ and $\beta > 0$.

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.2 (General Cauchy inequality, [2]). *Let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for all positive real numbers x and y, we have*

$$\alpha x + \beta y \geq x^\alpha y^\beta. \tag{2.3}$$

Lemma 2.3. *Under the conditions of Theorem 2.1, we have*

$$\int_0^1 x^\alpha f^\beta(x)dx \geq \frac{1}{\alpha+\beta+1}. \tag{2.4}$$

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Proof. Integrating by parts, we have

$$\int_0^1 x^{\alpha-1}\left(\int_x^1 f^\beta(t)dt\right)dx = \frac{1}{\alpha}\int_0^1\left(\int_x^1 f^\beta(t)dt\right)d(x^\alpha) \tag{2.5}$$
$$= \frac{1}{\alpha}\left[x^\alpha\int_x^1 f^\beta(t)dt\right]_{x=0}^{x=1} + \frac{1}{\alpha}\int_0^1 x^\alpha f^\beta(x)dx$$
$$= \frac{1}{\alpha}\int_0^1 x^\alpha f^\beta(x)dx,$$

which yields

$$\int_0^1 x^\alpha f^\beta(x)dx = \alpha\int_0^1 x^{\alpha-1}\left(\int_x^1 f^\beta(t)dt\right)dx. \tag{2.6}$$

On the other hand, by (2.1), we get

$$\int_0^1 x^{\alpha-1}\left(\int_x^1 f^\beta(t)dt\right)dx \geq \int_0^1 x^{\alpha-1}\left(\int_x^1 t^\beta\, dt\right)dx \tag{2.7}$$
$$= \frac{1}{\beta+1}\int_0^1 (x^{\alpha-1} - x^{\alpha+\beta})dx$$
$$= \frac{1}{\alpha(\alpha+\beta+1)}.$$

Therefore, (2.4) holds. □

We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. Using Lemma 2.2, we obtain

$$\frac{\beta}{\alpha+\beta}f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta}x^{\alpha+\beta} \geq x^\alpha f^\beta(x), \tag{2.8}$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

which gives

$$\beta\int_0^1 f^{\alpha+\beta}(x)dx+\alpha\int_0^1 x^{\alpha+\beta}dx\geq(\alpha+\beta)\int_0^1 x^{\alpha}f^{\beta}(x)dx. \tag{2.9}$$

Moreover, by using Lemma 2.3, we get

$$\begin{aligned}(\alpha+\beta)\int_0^1 x^{\alpha}f^{\beta}(x)dx&=\alpha\int_0^1 x^{\alpha}f^{\beta}(x)dx+\beta\int_0^1 x^{\alpha}f^{\beta}(x)dx\\&\geq\frac{\alpha}{\alpha+\beta+1}+\beta\int_0^1 x^{\alpha}f^{\beta}(x)dx,\end{aligned} \tag{2.10}$$

that is

$$\beta\int_0^1 f^{\alpha+\beta}(x)dx+\frac{\alpha}{\alpha+\beta+1}\geq\frac{\alpha}{\alpha+\beta+1}+\beta\int_0^1 x^{\alpha}f^{\beta}(x)dx, \tag{2.11}$$

which completes this proof. □

Lastly, we generalize our result.

Theorem 2.4. *Let $f(x)\geq 0$ be a continuous function on $[0,b], b\geq 0$ satisfying*

$$\int_x^b f^{\beta}(t)dt\geq\int_x^b t^{\beta}\,dt,\quad\forall\,x\in[0,b]. \tag{2.12}$$

Then the inequality

$$\int_0^b f^{\alpha+\beta}(x)dx\geq\int_0^b x^{\alpha}f^{\beta}(x)dx \tag{2.13}$$

hold for every positive real number $\alpha>0$ and $\beta>0$.

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To prove Theorem 2.4, we need the following lemma.

Lemma 2.5. *Under the conditions of Theorem 2.4, we have*

$$\int_0^b x^\alpha f^\beta(x)dx \geq \frac{b^{\alpha+\beta+1}}{\alpha+\beta+1}. \tag{2.14}$$

Proof. Integrating by parts, we have

$$\begin{aligned} \int_0^b x^{\alpha-1}\left(\int_x^b f^\beta(t)dt\right)dx &= \frac{1}{\alpha}\int_0^b\left(\int_x^b f^\beta(t)dt\right)d(x^\alpha) \\ &= \frac{1}{\alpha}\left[x^\alpha\int_x^b f^\beta(t)dt\right]_{x=0}^{x=b} + \frac{1}{\alpha}\int_0^b x^\alpha f^\beta(x)dx \\ &= \frac{1}{\alpha}\int_0^b x^\alpha f^\beta(x)dx, \end{aligned} \tag{2.15}$$

which yields

$$\int_0^b x^\alpha f^\beta(x)dx = \alpha\int_0^b x^{\alpha-1}\left(\int_x^b f^\beta(t)dt\right)dx. \tag{2.16}$$

On the other hand, by (2.12), we get

$$\begin{aligned} \int_0^b x^{\alpha-1}\left(\int_x^b f^\beta(t)dt\right)dx &\geq \int_0^b x^{\alpha-1}\left(\int_x^b t^\beta\,dt\right)dx \\ &= \frac{1}{\beta+1}\int_0^b x^{\alpha-1}(b^{\beta+1}-x^{\beta+1})dx \\ &= \frac{b^{\alpha+\beta+1}}{\alpha(\alpha+\beta+1)}. \end{aligned} \tag{2.17}$$

Therefore, (2.14) holds. □

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vol. 8, iss. 3, art. 74, 2007

journal of inequalities in pure and applied mathematics
issn: 1443-5756

We now give the proof of Theorem 2.4.

Proof of Theorem 2.4. Using Lemma 2.2, we obtain

$$\beta \int_0^b f^{\alpha+\beta}(x)dx + \alpha \int_0^b x^{\alpha+\beta}dx \geq (\alpha+\beta) \int_0^b x^\alpha f^\beta(x)dx. \tag{2.18}$$

Moreover, by using Lemma 2.5, we get

$$\begin{aligned}(\alpha+\beta) \int_0^b x^\alpha f^\beta(x)dx &= \alpha \int_0^b x^\alpha f^\beta(x)dx + \beta \int_0^b x^\alpha f^\beta(x)dx \\ &\geq \alpha \frac{b^{\alpha+\beta+1}}{\alpha+\beta+1} + \beta \int_0^b x^\alpha f^\beta(x)dx,\end{aligned} \tag{2.19}$$

that is

$$\beta \int_0^b f^{\alpha+\beta}(x)dx + \alpha \frac{b^{\alpha+\beta+1}}{\alpha+\beta+1} \geq \alpha \frac{b^{\alpha+\beta+1}}{\alpha+\beta+1} + \beta \int_0^b x^\alpha f^\beta(x)dx, \tag{2.20}$$

which completes the proof. □

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