

# ON AN OPEN PROBLEM CONCERNING AN INTEGRAL INEQUALITY

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*Abstract:* In this note, we generalize an open problem posed by Q. A. Ngô in the paper, Notes on an integral inequality, *J. Inequal. Pure & Appl. Math.*, 7(4)(2006), Art. 120 and give a positive answer to it using an analytic approach.

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Open Problem Concerning  
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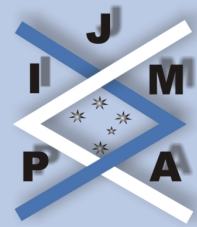
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## 1. Introduction

In the paper [2], Q.A. Ngô studied a very interesting integral inequality and proved the following result.

**Theorem 1.1.** Let  $f(x) \geq 0$  be a continuous function on  $[0, 1]$  satisfying

$$(1.1) \quad \int_x^1 f(t)dt \geq \int_x^1 t dt, \quad \forall x \in [0, 1].$$

Then the inequalities

$$(1.2) \quad \int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x^\alpha f(x)dx$$

and

$$(1.3) \quad \int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x f^\alpha(x)dx$$

hold for every positive real number  $\alpha > 0$ .

Next, they proposed the following open problem.

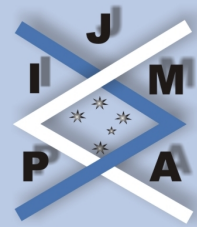
*Problem 1.* Let  $f(x)$  be a continuous function on  $[0, 1]$  satisfying

$$(1.4) \quad \int_x^1 f(t)dt \geq \int_x^1 t dt, \quad \forall x \in [0, 1].$$

Under what conditions does the inequality

$$(1.5) \quad \int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx,$$

hold for  $\alpha$  and  $\beta$ ?



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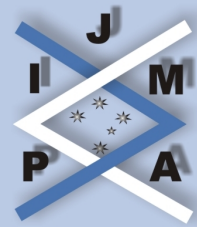
We note that, as an open problem, the condition (1.4) may result in an unreasonable restriction on  $f(x)$ . We remove it herein and propose another more general open problem.

*Problem 2.* Under what conditions does the inequality

$$(1.6) \quad \int_0^b f^{\alpha+\beta}(x)dx \geq \int_0^b x^\alpha f^\beta(x)dx,$$

hold for  $b$ ,  $\alpha$  and  $\beta$ ?

In this note, we give an answer to Problem 2 using an analytic approach. Our main results are Theorem 2.1 and Theorem 2.4 which will be proved in Section 2.



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## 2. Main Results and Proofs

Firstly, we have

**Theorem 2.1.** *Let  $f(x) \geq 0$  be a continuous function on  $[0, 1]$  satisfying*

$$(2.1) \quad \int_x^1 f^\beta(t) dt \geq \int_x^1 t^\beta dt, \quad \forall x \in [0, 1].$$

*Then the inequality*

$$(2.2) \quad \int_0^1 f^{\alpha+\beta}(x) dx \geq \int_0^1 x^\alpha f^\beta(x) dx,$$

*holds for every positive real number  $\alpha > 0$  and  $\beta > 0$ .*

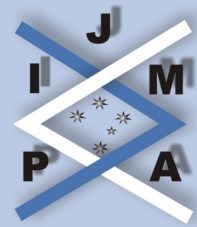
To prove Theorem 2.1, we need the following lemmas.

**Lemma 2.2 (General Cauchy inequality, [2]).** *Let  $\alpha$  and  $\beta$  be positive real numbers satisfying  $\alpha + \beta = 1$ . Then for all positive real numbers  $x$  and  $y$ , we have*

$$(2.3) \quad \alpha x + \beta y \geq x^\alpha y^\beta.$$

**Lemma 2.3.** *Under the conditions of Theorem 2.1, we have*

$$(2.4) \quad \int_0^1 x^\alpha f^\beta(x) dx \geq \frac{1}{\alpha + \beta + 1}.$$



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*Proof.* Integrating by parts, we have

$$\begin{aligned}(2.5) \quad \int_0^1 x^{\alpha-1} \left( \int_x^1 f^\beta(t) dt \right) dx &= \frac{1}{\alpha} \int_0^1 \left( \int_x^1 f^\beta(t) dt \right) d(x^\alpha) \\ &= \frac{1}{\alpha} \left[ x^\alpha \int_x^1 f^\beta(t) dt \right]_{x=0}^{x=1} + \frac{1}{\alpha} \int_0^1 x^\alpha f^\beta(x) dx \\ &= \frac{1}{\alpha} \int_0^1 x^\alpha f^\beta(x) dx,\end{aligned}$$

which yields

$$(2.6) \quad \int_0^1 x^\alpha f^\beta(x) dx = \alpha \int_0^1 x^{\alpha-1} \left( \int_x^1 f^\beta(t) dt \right) dx.$$

On the other hand, by (2.1), we get

$$\begin{aligned}(2.7) \quad \int_0^1 x^{\alpha-1} \left( \int_x^1 f^\beta(t) dt \right) dx &\geq \int_0^1 x^{\alpha-1} \left( \int_x^1 t^\beta dt \right) dx \\ &= \frac{1}{\beta+1} \int_0^1 (x^{\alpha-1} - x^{\alpha+\beta}) dx \\ &= \frac{1}{\alpha(\alpha+\beta+1)}.\end{aligned}$$

Therefore, (2.4) holds. □

We now give the proof of Theorem 2.1.

*Proof of Theorem 2.1.* Using Lemma 2.2, we obtain

$$(2.8) \quad \frac{\beta}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta} x^{\alpha+\beta} \geq x^\alpha f^\beta(x),$$



which gives

$$(2.9) \quad \beta \int_0^1 f^{\alpha+\beta}(x)dx + \alpha \int_0^1 x^{\alpha+\beta}dx \geq (\alpha + \beta) \int_0^1 x^\alpha f^\beta(x)dx.$$

Moreover, by using Lemma 2.3, we get

$$(2.10) \quad \begin{aligned} (\alpha + \beta) \int_0^1 x^\alpha f^\beta(x)dx &= \alpha \int_0^1 x^\alpha f^\beta(x)dx + \beta \int_0^1 x^\alpha f^\beta(x)dx \\ &\geq \frac{\alpha}{\alpha + \beta + 1} + \beta \int_0^1 x^\alpha f^\beta(x)dx, \end{aligned}$$

that is

$$(2.11) \quad \beta \int_0^1 f^{\alpha+\beta}(x)dx + \frac{\alpha}{\alpha + \beta + 1} \geq \frac{\alpha}{\alpha + \beta + 1} + \beta \int_0^1 x^\alpha f^\beta(x)dx,$$

which completes this proof. □

Lastly, we generalize our result.

**Theorem 2.4.** *Let  $f(x) \geq 0$  be a continuous function on  $[0, b]$ ,  $b \geq 0$  satisfying*

$$(2.12) \quad \int_x^b f^\beta(t)dt \geq \int_x^b t^\beta dt, \quad \forall x \in [0, b].$$

*Then the inequality*

$$(2.13) \quad \int_0^b f^{\alpha+\beta}(x)dx \geq \int_0^b x^\alpha f^\beta(x)dx$$

*hold for every positive real number  $\alpha > 0$  and  $\beta > 0$ .*

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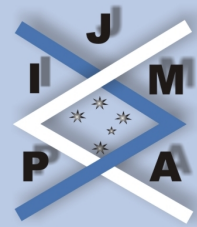
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To prove Theorem 2.4, we need the following lemma.

**Lemma 2.5.** *Under the conditions of Theorem 2.4, we have*

$$(2.14) \quad \int_0^b x^\alpha f^\beta(x) dx \geq \frac{b^{\alpha+\beta+1}}{\alpha + \beta + 1}.$$

*Proof.* Integrating by parts, we have

$$(2.15) \quad \begin{aligned} \int_0^b x^{\alpha-1} \left( \int_x^b f^\beta(t) dt \right) dx &= \frac{1}{\alpha} \int_0^b \left( \int_x^b f^\beta(t) dt \right) d(x^\alpha) \\ &= \frac{1}{\alpha} \left[ x^\alpha \int_x^b f^\beta(t) dt \right]_{x=0}^{x=b} + \frac{1}{\alpha} \int_0^b x^\alpha f^\beta(x) dx \\ &= \frac{1}{\alpha} \int_0^b x^\alpha f^\beta(x) dx, \end{aligned}$$

which yields

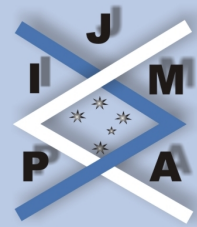
$$(2.16) \quad \int_0^b x^\alpha f^\beta(x) dx = \alpha \int_0^b x^{\alpha-1} \left( \int_x^b f^\beta(t) dt \right) dx.$$

On the other hand, by (2.12), we get

$$(2.17) \quad \begin{aligned} \int_0^b x^{\alpha-1} \left( \int_x^b f^\beta(t) dt \right) dx &\geq \int_0^b x^{\alpha-1} \left( \int_x^b t^\beta dt \right) dx \\ &= \frac{1}{\beta + 1} \int_0^b x^{\alpha-1} (b^{\beta+1} - x^{\beta+1}) dx \\ &= \frac{b^{\alpha+\beta+1}}{\alpha(\alpha + \beta + 1)}. \end{aligned}$$

Therefore, (2.14) holds. □





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We now give the proof of Theorem 2.4.

*Proof of Theorem 2.4.* Using Lemma 2.2, we obtain

$$(2.18) \quad \beta \int_0^b f^{\alpha+\beta}(x)dx + \alpha \int_0^b x^{\alpha+\beta}dx \geq (\alpha + \beta) \int_0^b x^\alpha f^\beta(x)dx.$$

Moreover, by using Lemma 2.5, we get

$$(2.19) \quad \begin{aligned} (\alpha + \beta) \int_0^b x^\alpha f^\beta(x)dx &= \alpha \int_0^b x^\alpha f^\beta(x)dx + \beta \int_0^b x^\alpha f^\beta(x)dx \\ &\geq \alpha \frac{b^{\alpha+\beta+1}}{\alpha + \beta + 1} + \beta \int_0^b x^\alpha f^\beta(x)dx, \end{aligned}$$

that is

$$(2.20) \quad \beta \int_0^b f^{\alpha+\beta}(x)dx + \alpha \frac{b^{\alpha+\beta+1}}{\alpha + \beta + 1} \geq \alpha \frac{b^{\alpha+\beta+1}}{\alpha + \beta + 1} + \beta \int_0^b x^\alpha f^\beta(x)dx,$$

which completes the proof. □

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