FURTHER DEVELOPMENT OF AN OPEN PROBLEM CONCERNING AN INTEGRAL INEQUALITY

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Abstract: In this paper, we generalize an open problem posed by Q. A. Ngô et al. in the paper Notes on an Integral Inequality, *J. Inequal. in Pure and Appl. Math.*, 7(4)(2006), Art. 120 and give an affirmative answer to it without the differentiable restriction on f.

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1. Introduction

Recently, in the paper [6] Ngô et al. studied some very interesting integral inequalities and proved the following result.

Theorem 1.1. *Let $f(x) \geq 0$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f(t)dt \geq \int_x^1 t\,dt, \quad \forall\, x \in [0,1]. \tag{1.1}$$

Then the inequalities

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x^\alpha f(x)dx, \tag{1.2}$$

and

$$\int_0^1 f^{\alpha+1}(x)dx \geq \int_0^1 x f^\alpha(x)dx, \tag{1.3}$$

hold for every positive real number $\alpha > 0$.

Next, they proposed the following open problem:

Problem 1.2. *Let $f(x)$ be a continuous function on $[0,1]$ satisfying*

$$\int_x^1 f(t)dt \geq \int_x^1 t\,dt, \quad \forall\, x \in [0,1]. \tag{1.4}$$

Under what conditions does the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \geq \int_0^1 x^\alpha f^\beta(x)dx, \tag{1.5}$$

holds for α and β?

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We note that, as an open problem, the condition (1.4) may result in an unreasonable restriction on $f(x)$. We remove it herein and propose another more general open problem:

Problem 1.3. *Under what conditions does the inequality*

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b (x-a)^\alpha f^\beta(x)dx, \tag{1.6}$$

hold for $a,\ b,\ \alpha$ *and* β*?*

Shortly after the paper [6] was published, Liu et al. [5] gave an affirmative answer to Problem 1.3 for the case $a = 0$ and obtained the following result:

Theorem 1.4. *Let* $f(x) \geq 0$ *be a continuous function on* $[0, b],\ b \geq 0$ *satisfying*

$$\int_x^b f^\beta(t)dt \geq \int_x^b t^\beta\, dt, \quad \forall\, x \in [0, b]. \tag{1.7}$$

Then the inequality

$$\int_0^b f^{\alpha+\beta}(x)dx \geq \int_0^b x^\alpha f^\beta(x)dx, \tag{1.8}$$

holds for every positive real number $\alpha > 0$ *and* $\beta > 0$.

Almost at the same time, Bougoffa [1] also gave an answer to Problem 1.3 and established the following result (We correct it here according to the presence of the corrigendum in [2]):

Theorem 1.5. *Let* $f(x) \geq 0$ *be a function, continuous on* $[a, b]$ *and differentiable in* (a, b)*. If*

$$\int_x^b f(t)dt \geq \int_x^b (t-a)\, dt, \quad \forall\, x \in [a, b] \tag{1.9}$$

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and

$$f'(x) \le 1, \quad \forall\, x \in (a,b),$$

then the inequality (1.6) holds for every positive real number $\alpha > 0$ *and* $\beta > 0$.

Very recently, Boukerrioua and Guezane-Lakoud [3] obtained the following result:

Theorem 1.6. *Let* $f(x) \ge 0$ *be a continuous function on* $[0,1]$ *satisfying*

$$\int_x^1 f(t)dt \ge \int_x^1 t\,dt, \quad \forall\, x \in [0,1]. \tag{1.10}$$

Then the inequality

$$\int_0^1 f^{\alpha+\beta}(x)dx \ge \int_0^1 x^\alpha f^\beta(x)dx, \tag{1.11}$$

holds for $\alpha > 0$ *and* $\beta \ge 1$.

Comparing the above three results, we note that: the condition (1.7) was required in Theorem 1.4, a differentiability condition was restricted on f in Theorem 1.5 while $\beta \ge 1$ was demanded in Theorem 1.6. In this paper, we will give an affirmative answer to Problem 1.3 without the differentiable restriction on f by improving the methods of [5], [6] and [3]. Our main result is Theorem 2.1 which will be proved in Section 2.

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2. Main Results and Proofs

Theorem 2.1. *Let $f(x) \geq 0$ be a continuous function on $[a,b]$ satisfying*

$$\int_x^b f^{\min\{1,\beta\}}(t)dt \geq \int_x^b (t-a)^{\min\{1,\beta\}}\,dt, \quad \forall\, x \in [a,b]. \tag{2.1}$$

Then the inequality

$$\int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b (x-a)^{\alpha} f^{\beta}(x)dx, \tag{2.2}$$

holds for every positive real number $\alpha > 0$ and $\beta > 0$.

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.2 ([6], General Cauchy inequality). *Let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for all positive real numbers x and y, we always have*

$$\alpha x + \beta y \geq x^{\alpha} y^{\beta}. \tag{2.3}$$

Lemma 2.3. *Under the conditions of Theorem 2.1, we have*

$$\int_a^b (x-a)^{\alpha} f^{\beta}(x)dx \geq \frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1}. \tag{2.4}$$

Proof. We divide the proof into two steps according to the different intervals of β.

Case of $0 < \beta \leq 1$: Integrating by parts, we have

$$\int_a^b (x-a)^{\alpha-1}\left(\int_x^b f^{\beta}(t)dt\right)dx$$

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$$= \frac{1}{\alpha} \int_a^b \left(\int_x^b f^\beta(t)dt \right) d(x-a)^\alpha$$

$$= \frac{1}{\alpha} \left[(x-a)^\alpha \int_x^b f^\beta(t)dt \right]_{x=a}^{x=b} + \frac{1}{\alpha} \int_a^b (x-a)^\alpha f^\beta(x)dx$$

$$= \frac{1}{\alpha} \int_a^b (x-a)^\alpha f^\beta(x)dx.$$

which yields

$$\int_a^b (x-a)^\alpha f^\beta(x)dx = \alpha \int_a^b (x-a)^{\alpha-1} \left(\int_x^b f^\beta(t)dt \right) dx. \tag{2.5}$$

On the other hand, by (2.1), we get

$$\int_a^b (x-a)^{\alpha-1} \left(\int_x^b f^\beta(t)dt \right) dx$$

$$\geq \int_a^b (x-a)^{\alpha-1} \left(\int_x^b (t-a)^\beta \, dt \right) dx$$

$$= \frac{1}{\beta+1} \int_a^b (x-a)^{\alpha-1} \left[(b-a)^{\beta+1} - (x-a)^{\beta+1} \right] dx$$

$$= \frac{(b-a)^{\alpha+\beta+1}}{\alpha(\alpha+\beta+1)}.$$

Therefore, (2.4) holds.

Case of $\beta > 1$: We note that the following result has been proved in the first case

$$\int_a^b (x-a)^\alpha f(x)dx \geq \frac{(b-a)^{\alpha+2}}{\alpha+2}. \tag{2.6}$$

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Using Lemma 2.2, we get

$$\frac{1}{\beta}f^{\beta}(x)+\frac{\beta-1}{\beta}(x-a)^{\beta}\geq f(x)(x-a)^{\beta-1}. \tag{2.7}$$

Multiplying both sides of (2.7) by $(x-a)^{\alpha}$ and integrating the resultant inequality from a to b, we obtain

$$\int_a^b (x-a)^{\alpha}f^{\beta}(x)dx+(\beta-1)\int_a^b (x-a)^{\alpha+\beta}dx\geq\beta\int_a^b (x-a)^{\alpha+\beta-1}f(x)dx, \tag{2.8}$$

which implies

$$\int_a^b (x-a)^{\alpha}f^{\beta}(x)dx+\frac{\beta-1}{\alpha+\beta+1}(b-a)^{\alpha+\beta+1}\geq\beta\int_a^b (x-a)^{\alpha+\beta-1}f(x)dx. \tag{2.9}$$

Moreover, by using (2.6), we get

$$\int_a^b (x-a)^{\alpha}f^{\beta}(x)dx+\frac{\beta-1}{\alpha+\beta+1}(b-a)^{\alpha+\beta+1}\geq\frac{\beta}{\alpha+\beta+1}(b-a)^{\alpha+\beta+1}, \tag{2.10}$$

which implies (2.4). □

We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. Using Lemma 2.2 again, we obtain

$$\frac{\beta}{\alpha+\beta}f^{\alpha+\beta}(x)+\frac{\alpha}{\alpha+\beta}(x-a)^{\alpha+\beta}dx\geq(x-a)^{\alpha}f^{\beta}(x)dx, \tag{2.11}$$

which gives

$$\beta\int_a^b f^{\alpha+\beta}(x)dx+\alpha\int_a^b (x-a)^{\alpha+\beta}dx\geq(\alpha+\beta)\int_a^b (x-a)^{\alpha}f^{\beta}(x)dx. \tag{2.12}$$

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Moreover, by using Lemma 2.3, we get

$$(\alpha+\beta)\int_a^b (x-a)^\alpha f^\beta(x)dx = \alpha\int_a^b (x-a)^\alpha f^\beta(x)dx + \beta\int_a^b (x-a)^\alpha f^\beta(x)dx$$
$$\geq \alpha\frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1} + \beta\int_a^b (x-a)^\alpha f^\beta(x)dx,$$

that is

$$\beta\int_a^b f^{\alpha+\beta}(x)dx + \alpha\frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1} \geq \alpha\frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1} + \beta\int_a^b (x-a)^\alpha f^\beta(x)dx, \tag{2.13}$$

which completes the proof. □

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