

FURTHER DEVELOPMENT OF AN OPEN PROBLEM CONCERNING AN INTEGRAL INEQUALITY

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Abstract: In this paper, we generalize an open problem posed by Q. A. Ngô et al. in the paper Notes on an Integral Inequality, *J. Inequal. in Pure and Appl. Math.*, 7(4)(2006), Art. 120 and give an affirmative answer to it without the differentiable restriction on f .

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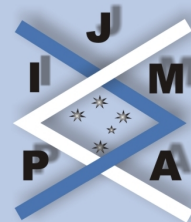
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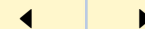
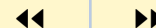
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1. Introduction

Recently, in the paper [6] Ngô et al. studied some very interesting integral inequalities and proved the following result.

Theorem 1.1. Let $f(x) \geq 0$ be a continuous function on $[0, 1]$ satisfying

$$(1.1) \quad \int_x^1 f(t) dt \geq \int_x^1 t dt, \quad \forall x \in [0, 1].$$

Then the inequalities

$$(1.2) \quad \int_0^1 f^{\alpha+1}(x) dx \geq \int_0^1 x^\alpha f(x) dx,$$

and

$$(1.3) \quad \int_0^1 f^{\alpha+1}(x) dx \geq \int_0^1 x f^\alpha(x) dx,$$

hold for every positive real number $\alpha > 0$.

Next, they proposed the following open problem:

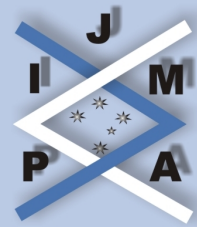
Problem 1.2. Let $f(x)$ be a continuous function on $[0, 1]$ satisfying

$$(1.4) \quad \int_x^1 f(t) dt \geq \int_x^1 t dt, \quad \forall x \in [0, 1].$$

Under what conditions does the inequality

$$(1.5) \quad \int_0^1 f^{\alpha+\beta}(x) dx \geq \int_0^1 x^\alpha f^\beta(x) dx,$$

holds for α and β ?



We note that, as an open problem, the condition (1.4) may result in an unreasonable restriction on $f(x)$. We remove it herein and propose another more general open problem:

Problem 1.3. *Under what conditions does the inequality*

$$(1.6) \quad \int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b (x-a)^\alpha f^\beta(x)dx,$$

hold for a , b , α and β ?

Shortly after the paper [6] was published, Liu et al. [5] gave an affirmative answer to Problem 1.3 for the case $a = 0$ and obtained the following result:

Theorem 1.4. *Let $f(x) \geq 0$ be a continuous function on $[0, b]$, $b \geq 0$ satisfying*

$$(1.7) \quad \int_x^b f^\beta(t)dt \geq \int_x^b t^\beta dt, \quad \forall x \in [0, b].$$

Then the inequality

$$(1.8) \quad \int_0^b f^{\alpha+\beta}(x)dx \geq \int_0^b x^\alpha f^\beta(x)dx,$$

holds for every positive real number $\alpha > 0$ and $\beta > 0$.

Almost at the same time, Bougoffa [1] also gave an answer to Problem 1.3 and established the following result (We correct it here according to the presence of the corrigendum in [2]):

Theorem 1.5. *Let $f(x) \geq 0$ be a function, continuous on $[a, b]$ and differentiable in (a, b) . If*

$$(1.9) \quad \int_x^b f(t)dt \geq \int_x^b (t-a) dt, \quad \forall x \in [a, b]$$

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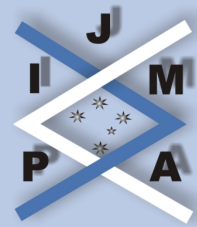
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and

$$f'(x) \leq 1, \quad \forall x \in (a, b),$$

then the inequality (1.6) holds for every positive real number $\alpha > 0$ and $\beta > 0$.

Very recently, Boukerrioua and Guezane-Lakoud [3] obtained the following result:

Theorem 1.6. Let $f(x) \geq 0$ be a continuous function on $[0, 1]$ satisfying

$$(1.10) \quad \int_x^1 f(t) dt \geq \int_x^1 t dt, \quad \forall x \in [0, 1].$$

Then the inequality

$$(1.11) \quad \int_0^1 f^{\alpha+\beta}(x) dx \geq \int_0^1 x^\alpha f^\beta(x) dx,$$

holds for $\alpha > 0$ and $\beta \geq 1$.

Comparing the above three results, we note that: the condition (1.7) was required in Theorem 1.4, a differentiability condition was restricted on f in Theorem 1.5 while $\beta \geq 1$ was demanded in Theorem 1.6. In this paper, we will give an affirmative answer to Problem 1.3 without the differentiable restriction on f by improving the methods of [5], [6] and [3]. Our main result is Theorem 2.1 which will be proved in Section 2.

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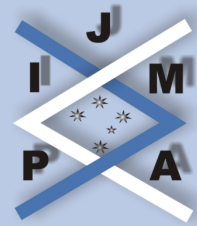
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2. Main Results and Proofs

Theorem 2.1. Let $f(x) \geq 0$ be a continuous function on $[a, b]$ satisfying

$$(2.1) \quad \int_x^b f^{\min\{1,\beta\}}(t)dt \geq \int_x^b (t-a)^{\min\{1,\beta\}} dt, \quad \forall x \in [a, b].$$

Then the inequality

$$(2.2) \quad \int_a^b f^{\alpha+\beta}(x)dx \geq \int_a^b (x-a)^\alpha f^\beta(x)dx,$$

holds for every positive real number $\alpha > 0$ and $\beta > 0$.

To prove Theorem 2.1, we need the following lemmas.

Lemma 2.2 ([6], General Cauchy inequality). Let α and β be positive real numbers satisfying $\alpha + \beta = 1$. Then for all positive real numbers x and y , we always have

$$(2.3) \quad \alpha x + \beta y \geq x^\alpha y^\beta.$$

Lemma 2.3. Under the conditions of Theorem 2.1, we have

$$(2.4) \quad \int_a^b (x-a)^\alpha f^\beta(x)dx \geq \frac{(b-a)^{\alpha+\beta+1}}{\alpha+\beta+1}.$$

Proof. We divide the proof into two steps according to the different intervals of β .

Case of $0 < \beta \leq 1$: Integrating by parts, we have

$$\int_a^b (x-a)^{\alpha-1} \left(\int_x^b f^\beta(t)dt \right) dx$$

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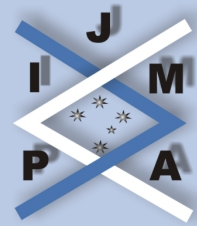
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$$\begin{aligned}
 &= \frac{1}{\alpha} \int_a^b \left(\int_x^b f^\beta(t) dt \right) d(x-a)^\alpha \\
 &= \frac{1}{\alpha} \left[(x-a)^\alpha \int_x^b f^\beta(t) dt \right]_{x=a}^{x=b} + \frac{1}{\alpha} \int_a^b (x-a)^\alpha f^\beta(x) dx \\
 &= \frac{1}{\alpha} \int_a^b (x-a)^\alpha f^\beta(x) dx.
 \end{aligned}$$

which yields

$$(2.5) \quad \int_a^b (x-a)^\alpha f^\beta(x) dx = \alpha \int_a^b (x-a)^{\alpha-1} \left(\int_x^b f^\beta(t) dt \right) dx.$$

On the other hand, by (2.1), we get

$$\begin{aligned}
 &\int_a^b (x-a)^{\alpha-1} \left(\int_x^b f^\beta(t) dt \right) dx \\
 &\geq \int_a^b (x-a)^{\alpha-1} \left(\int_x^b (t-a)^\beta dt \right) dx \\
 &= \frac{1}{\beta+1} \int_a^b (x-a)^{\alpha-1} [(b-a)^{\beta+1} - (x-a)^{\beta+1}] dx \\
 &= \frac{(b-a)^{\alpha+\beta+1}}{\alpha(\alpha+\beta+1)}.
 \end{aligned}$$

Therefore, (2.4) holds.

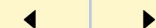
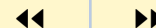
Case of $\beta > 1$: We note that the following result has been proved in the first case

$$(2.6) \quad \int_a^b (x-a)^\alpha f(x) dx \geq \frac{(b-a)^{\alpha+2}}{\alpha+2}.$$



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Using Lemma 2.2, we get

$$(2.7) \quad \frac{1}{\beta} f^\beta(x) + \frac{\beta-1}{\beta} (x-a)^\beta \geq f(x)(x-a)^{\beta-1}.$$

Multiplying both sides of (2.7) by $(x-a)^\alpha$ and integrating the resultant inequality from a to b , we obtain

$$(2.8) \quad \int_a^b (x-a)^\alpha f^\beta(x) dx + (\beta-1) \int_a^b (x-a)^{\alpha+\beta} dx \geq \beta \int_a^b (x-a)^{\alpha+\beta-1} f(x) dx,$$

which implies

$$(2.9) \quad \int_a^b (x-a)^\alpha f^\beta(x) dx + \frac{\beta-1}{\alpha+\beta+1} (b-a)^{\alpha+\beta+1} \geq \beta \int_a^b (x-a)^{\alpha+\beta-1} f(x) dx.$$

Moreover, by using (2.6), we get

$$(2.10) \quad \int_a^b (x-a)^\alpha f^\beta(x) dx + \frac{\beta-1}{\alpha+\beta+1} (b-a)^{\alpha+\beta+1} \geq \frac{\beta}{\alpha+\beta+1} (b-a)^{\alpha+\beta+1},$$

which implies (2.4). □

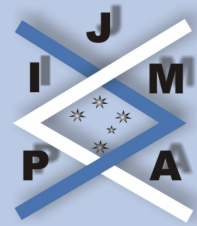
We now give the proof of Theorem 2.1.

Proof of Theorem 2.1. Using Lemma 2.2 again, we obtain

$$(2.11) \quad \frac{\beta}{\alpha+\beta} f^{\alpha+\beta}(x) + \frac{\alpha}{\alpha+\beta} (x-a)^{\alpha+\beta} dx \geq (x-a)^\alpha f^\beta(x) dx,$$

which gives

$$(2.12) \quad \beta \int_a^b f^{\alpha+\beta}(x) dx + \alpha \int_a^b (x-a)^{\alpha+\beta} dx \geq (\alpha+\beta) \int_a^b (x-a)^\alpha f^\beta(x) dx.$$



Moreover, by using Lemma 2.3, we get

$$\begin{aligned}(\alpha + \beta) \int_a^b (x - a)^\alpha f^\beta(x) dx &= \alpha \int_a^b (x - a)^\alpha f^\beta(x) dx + \beta \int_a^b (x - a)^\alpha f^\beta(x) dx \\ &\geq \alpha \frac{(b - a)^{\alpha + \beta + 1}}{\alpha + \beta + 1} + \beta \int_a^b (x - a)^\alpha f^\beta(x) dx,\end{aligned}$$

that is

$$\begin{aligned}(2.13) \quad \beta \int_a^b f^{\alpha + \beta}(x) dx + \alpha \frac{(b - a)^{\alpha + \beta + 1}}{\alpha + \beta + 1} \\ \geq \alpha \frac{(b - a)^{\alpha + \beta + 1}}{\alpha + \beta + 1} + \beta \int_a^b (x - a)^\alpha f^\beta(x) dx,\end{aligned}$$

which completes the proof. \square

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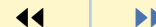
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