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ON NEIGHBORHOODS OF A CERTAIN CLASS OF COMPLEX ORDER DEFINED BY RUSCHEWEYH DERIVATIVE OPERATOR

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Abstract

In this paper, we introduce the subclass $R_b^\lambda(A, B, \alpha, \mu)$ which is defined by concept of subordination. According to this, we obtain a necessary and sufficient condition which is equivalent to this class. Further, we apply to the $\delta-$ neighborhoods for belonging to $R_b^\lambda(A, B, \alpha, \mu)$ to this condition.

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1. Introduction and Definitions

Let $\mathbb{U}=\{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $\mathcal{H}(\mathbb{U})$ be the set of all functions analytic in $\mathbb{U}$, and let

$$\mathcal{A} := \{f \in \mathcal{H}(\mathbb{U}) : f(0) = f'(0) - 1 = 0\}.$$

Given two functions f and g, which are analytic in $\mathbb{U}$. The function f is said to be *subordinate* to g, written

$$f \prec g \quad \text{and} \quad f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω analytic in $\mathbb{U}$, with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \qquad (z \in \mathbb{U}),$$

and such that

$$f(z) = g(\omega(z)) \qquad (z \in \mathbb{U}).$$

In particular, if g is univalent in $\mathbb{U}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ in [7].

Next, for the functions f_j $(j = 1, 2)$ given by

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \qquad (j = 1, 2).$$

Let $f_1 * f_2$ denote the *Hadamard* product *(or convolution)* of f_1 and f_2, defined by

$$(f_1 * f_2)(z) := z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k =: (f_2 * f_1)(z). \tag{1.1}$$

$(a)_v$ denotes the *Pochhammer* symbol *(or the shifted factorial)*, since

$$(1)_n = n! \quad \text{for} \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

defined (for $a, v \in \mathbb{C}$ and in terms of the Gamma function) by

$$(a)_v := \frac{\Gamma(a+v)}{\Gamma(a)} = \begin{cases} 1; & (v = 0, a \in \mathbb{C}\backslash\{0\}), \\ a(a+1)\ldots(a+n-1); & (v = n \in \mathbb{N}; a \in \mathbb{C}). \end{cases}$$

The earlier investigations by Goodman [1] and Ruscheweyh [9], we define the $\delta-$ neighborhood of a function $f \in \mathcal{A}$ by

$$\mathcal{N}_\delta(f) := \Big\{ g \in \mathcal{A} : f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k|a_k - b_k| \leq \delta \Big\}$$

so that, obviously,

$$\mathcal{N}_\delta(e) := \Big\{ g \in \mathcal{A} : \; g(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad \text{and} \quad \sum_{k=2}^{\infty} k|b_k| \leq \delta \Big\},$$

where $e(z) := z$.

Ruscheweyh [8] introduced an linear operator $\mathcal{D}^\lambda : \mathcal{A} \longrightarrow \mathcal{A}$, defined by the Hadamard product as follows:

$$\mathcal{D}^\lambda f(z) := \frac{z}{(1-z)^{\lambda+1}} * f(z) \qquad (\lambda > -1; \; z \in \mathbb{U}),$$

which implies that

$$\mathcal{D}^n f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

Clearly, we have

$$\mathcal{D}^0 f(z) = f(z), \qquad \mathcal{D}^1 f(z) = z f'(z)$$

and

$$\mathcal{D}^n f(z) = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{(1)_k} a_{k+1} z^{k+1} = \left(\sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{(1)_k} z^{k+1} * f\right)(z),$$

where $f \in \mathcal{A}$.

Therefore, we can write the following equality, the easily verified result from the above definitions:

$$\left[(1-\mu)\frac{\mathcal{D}^\lambda f(z)}{z} + \mu\left(\mathcal{D}^\lambda f(z)\right)'\right] * \frac{1}{(1-z)^2} = \left(\mathcal{D}^\lambda f(z)\right)' + \mu z\left(\mathcal{D}^\lambda f(z)\right)'', \tag{1.2}$$

where $f \in \mathcal{A}$, $\lambda\ (\lambda > -1)$, $\mu\ (\mu \geq 0)$ and for all $z \in \mathbb{U}$.

For each A and B such that $-1 \leq B < A \leq 1$ and for all real numbers α such that $0 \leq \alpha < 1$, we define the function

$$h(A, B, \alpha; z) := \frac{1 + \{(1-\alpha)A + \alpha B\} z}{1 + Bz} \qquad (z \in \mathbb{U}).$$

Also, let $h(\alpha)$ denote the extremal function of functions with positive real part of order α $(0 \leq \alpha < 1)$, defined by

$$h(\alpha) := h(1,-1,\alpha;z) = \frac{1+(1-2\alpha)z}{1-z} \qquad (z \in \mathbb{U}).$$

The class $\mathcal{R}^b(A,B)$ is studied by Premabai in [3]. According to this, we introduce the subclass $\mathcal{R}_b^\lambda(A,B,\alpha,\mu)$ which is a generalization of this class, as follows:

$$(1.3) \qquad 1+\frac{1}{b}\left[\left(\mathcal{D}^\lambda f(z)\right)' + \mu z\left(\mathcal{D}^\lambda f(z)\right)'' - 1\right] \prec h(A,B,\alpha;z),$$

where $f \in \mathcal{A}$, $b \in \mathbb{C}/\{0\}$, for some real numbers A, B $(-1 \leq B < A \leq 1)$, λ $(\lambda > -1)$, α $(0 \leq \alpha < 1)$, μ $(\mu \geq 0)$ and for all $z \in \mathbb{U}$ with $\mathcal{R}_b(A,B,\alpha,\mu) := \mathcal{R}_b^0(A,B,\alpha,\mu)$ and $\mathcal{R}^b(A,B) := \mathcal{R}_b(A,B,0,0)$.

We note that the class $\mathcal{R}^b(\mu) := \mathcal{R}_b(1,-1,0,\mu)$ is studied by Altıntaş and Özkan in [4]. Therefore $\mathcal{C}(b) := \mathcal{R}_b(1,-1,0,0)$ is the class of close-to-convex functions of complex order b. $\mathcal{C}(\alpha) := \mathcal{R}_{1-\alpha}(1,-1,0,0)$ is the class of close-to-convex functions of order α $(0 \leq \alpha < 1)$.

Also, let $\mathcal{T}_b(A,B,\alpha)$ denote the class of functions ϕ normalized by

$$(1.4) \qquad \phi(z) := \frac{1+\frac{1}{b}\left\{\frac{1}{(1-z)^2}-1\right\} - \frac{1+\{(1-\alpha)A+\alpha B\}e^{it}}{1+Be^{it}}}{1-\frac{1+\{(1-\alpha)A+\alpha B\}e^{it}}{1+Be^{it}}} \qquad (t \in (0,2\pi)),$$

where $b \in \mathbb{C}/\{0\}$, for some real numbers A, B $(-1 \leq B < A \leq 1)$, for all α $(0 \leq \alpha < 1)$ and for all $z \in \mathbb{U}$ with $\mathcal{T}_b(A,B) := \mathcal{T}_b(A,B,0)$ and $\mathcal{T}(b) := \mathcal{T}_b(1,-1,0)$.

2. The Main Results

A theorem that contains the relationship between the above classes is given as follows:

Theorem 2.1. *$f \in \mathcal{R}_b^{\lambda}(A, B, \alpha, \mu)$ if and only if*

$$\left[(1-\mu)\frac{\mathcal{D}^{\lambda} f(z)}{z} + \mu\left(\mathcal{D}^{\lambda} f(z)\right)'\right] * \phi(z) \neq 0$$

for all $\phi \in \mathcal{T}_b(A, B, \alpha)$ and for all $f \in \mathcal{A}$.

Proof. Firstly, let

$$F^{\lambda}(f, \mu; z) := (1-\mu)\frac{\mathcal{D}^{\lambda} f(z)}{z} + \mu\left(\mathcal{D}^{\lambda} f(z)\right)'$$

and we suppose that

$$F^{\lambda}(f, \mu; z) * \phi(z) \neq 0 \tag{2.1}$$

for all $f \in \mathcal{A}$ and for all $\phi \in \mathcal{T}_b(A, B, \alpha)$. In view of (1.4), we have

$$\begin{aligned} &F^{\lambda}(f, \mu; z) * \phi(z) \\ &= \frac{1 + \frac{1}{b}\left\{F^{\lambda}(f, \mu; z) * \frac{1}{(1-z)^2} - 1\right\} - \frac{1+\{(1-\alpha)A+\alpha B\}e^{it}}{1+Be^{it}}}{1 - \frac{1+\{(1-\alpha)A+\alpha B\}e^{it}}{1+Be^{it}}} \\ &= \frac{1 + \frac{1}{b}\left\{\left(\mathcal{D}^{\lambda} f(z)\right)' + \mu z\left(\mathcal{D}^{\lambda} f(z)\right)'' - 1\right\} - \frac{1+\{(1-\alpha)A+\alpha B\}e^{it}}{1+Be^{it}}}{1 - \frac{1+\{(1-\alpha)A+\alpha B\}e^{it}}{1+Be^{it}}} \\ &\neq 0. \end{aligned}$$

From this inequality we find that

$$1+\frac{1}{b}\left\{\left(\mathcal{D}^{\lambda}f\left(z\right)\right)'+\mu z\left(\mathcal{D}^{\lambda}f\left(z\right)\right)''-1\right\}\neq h\left(A,B,\alpha;e^{it}\right),$$

where $t\in(0,2\pi)$.

This means that $1+\frac{1}{b}\left\{\left(\mathcal{D}^{\lambda}f\left(z\right)\right)'+\mu z\left(\mathcal{D}^{\lambda}f\left(z\right)\right)''-1\right\}$ does not take any value on the image of under $h\left(A,B,\alpha;z\right)$ function of the boundary of $\mathbb{U}$. Therefore we note that $1+\frac{1}{b}\left\{\left(\mathcal{D}^{\lambda}f\left(z\right)\right)'+\mu z\left(\mathcal{D}^{\lambda}f\left(z\right)\right)''-1\right\}$ takes the value 1 for $z=0$. Since $0\leq\alpha<1$ and $B<A$, 1 is contained by the image under $h\left(A,B,\alpha;z\right)$ function of $\mathbb{U}$.Thus, we can write

$$1+\frac{1}{b}\left\{\left(\mathcal{D}^{\lambda}f\left(z\right)\right)'+\mu z\left(\mathcal{D}^{\lambda}f\left(z\right)\right)''-1\right\}\prec h\left(A,B,\alpha;z\right).$$

Hence $f\in\mathcal{R}_{b}^{\lambda}\left(A,B,\alpha,\mu\right)$.

Conversely, assume the function f is in the class $\mathcal{R}_{b}^{\lambda}\left(A,B,\alpha,\mu\right)$. From the definition of subordination, we can write the following inequality:

$$1+\frac{1}{b}\left\{\left(\mathcal{D}^{\lambda}f\left(z\right)\right)'+\mu z\left(\mathcal{D}^{\lambda}f\left(z\right)\right)''-1\right\}\neq h\left(A,B,\alpha;e^{it}\right),$$

where $t\in(0,2\pi)$. From (1.2) we can write

$$1+\frac{1}{b}\left\{F^{\lambda}\left(f,\mu;z\right)*\frac{1}{(1-z)^{2}}-1\right\}\neq h\left(A,B,\alpha;e^{it}\right)$$

or equivalently,

$$F^{\lambda}(f,\mu;z) * \left\{ \frac{1+\frac{1}{b}\left(\frac{1}{(1-z)^2}-1\right)-h\left(A,B,\alpha;e^{it}\right)}{1-h\left(A,B,\alpha;e^{it}\right)} \right\}.$$

Thus, from the definition of the function ϕ, we can write

$$\left[(1-\mu)\frac{\mathcal{D}^{\lambda}f(z)}{z}+\mu\left(\mathcal{D}^{\lambda}f(z)\right)'\right] * \phi(z) \neq 0$$

for all $\phi \in \mathcal{T}_b(A,B,\alpha)$ and for all $f \in \mathcal{A}$. □

Corollary 2.2. *$f \in \mathcal{R}_b(A,B,\alpha,\mu)$ if and only if $\left[(1-\mu)\frac{f(z)}{z}+\mu f'(z)\right] * \phi(z) \neq 0$ for all $\phi \in \mathcal{T}_b(A,B,\alpha)$ and for all $f \in \mathcal{A}$.*

Proof. By putting $\lambda = 0$ in Theorem 2.1. □

Corollary 2.3. *$f \in \mathcal{R}^b(A,B)$ if and only if $\frac{f(z)}{z} * \phi(z) \neq 0$ for all $\phi \in \mathcal{T}_b(A,B)$ and for all $f \in \mathcal{A}$.*

Proof. By putting $\alpha = 0$, $\mu = 0$ in Corollary 2.2. And, we obtain the result of Theorem 1 in [3]. □

Corollary 2.4. *$f \in \mathcal{C}(b)$ if and only if $\frac{f(z)}{z} * \phi(z) \neq 0$ for all $\phi \in \mathcal{T}(b)$ and for all $f \in \mathcal{A}$.*

Proof. By putting $A = 1$, $B = -1$ in Corollary 2.3. □

Theorem 2.5. *Let $F_\epsilon(z) = \frac{f(z)+\epsilon z}{1+\epsilon}$ for $\epsilon \in \mathbb{C}$ and $f \in \mathcal{A}$. If $F_\epsilon \in \mathcal{R}_b^\lambda(A,B,\alpha,\mu)$ for $|\epsilon| < \delta^*$, then*

$$\left|\left[(1-\mu)\frac{\mathcal{D}^\lambda f(z)}{z} + \mu\left(\mathcal{D}^\lambda f(z)\right)'\right] * \phi(z)\right| \geq \delta^* \qquad (z \in \mathbb{U}), \tag{2.2}$$

where $\phi \in \mathcal{T}_b(A,B,\alpha)$ and for all $f \in \mathcal{A}$.

Proof. Let $\phi \in \mathcal{T}_b(A,B,\alpha)$ and $F_\epsilon \in \mathcal{R}_b^\lambda(A,B,\alpha,\mu)$. From Theorem 2.1, we can write

$$\left[(1-\mu)\frac{\mathcal{D}^\lambda F_\epsilon(z)}{z} + \mu\left(\mathcal{D}^\lambda F_\epsilon(z)\right)'\right] * \phi(z) \neq 0.$$

Using $\mathcal{D}^\lambda(\epsilon z) = \epsilon z$, we find that the following inequality

$$\frac{1}{1+\epsilon}\left\{\left[(1-\mu)\frac{\mathcal{D}^\lambda f(z)}{z} + \mu\left(\mathcal{D}^\lambda f(z)\right)'\right] * \phi(z) + \epsilon\right\} \neq 0$$

that is,

$$\left[(1-\mu)\frac{\mathcal{D}^\lambda f(z)}{z} + \mu\left(\mathcal{D}^\lambda f(z)\right)'\right] * \phi(z) \neq -\epsilon$$

or equivalently (2.2). □

Lemma 2.6. *If $\phi(z) = 1 + \sum_{k=1}^\infty c_k z^k \in \mathcal{T}_b(A,B,\alpha)$, then we have*

$$|c_k| \leq \frac{(k+1)(1+|B|)}{(1-\alpha)|b||B-A|} \qquad (k = 1,2,3,\ldots). \tag{2.3}$$

Proof. We suppose that $\phi(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in \mathcal{T}_b(A, B, \alpha)$. From (1.4), we have

$$1 + \sum_{k=1}^{\infty} c_k z^k$$
$$= \frac{1 + \frac{1}{b}\left\{1 + \sum_{k=2}^{\infty} k z^{k-1} - 1\right\} - \frac{1+\{(1-\alpha)A+\alpha B\}e^{it}}{1+Be^{it}}}{1 - \frac{1+\{(1-\alpha)A+\alpha B\}e^{it}}{1+Be^{it}}} \qquad (t \in (0, 2\pi)).$$

We write the following equality result which is easily verified result from the above equality:

$$c_k = \frac{(k+1)}{b(1-\alpha)} \cdot \frac{(1+Be^{it})}{(B-A)e^{it}}.$$

Taking the modulus of both sides, we obtain inequality (2.3). □

Theorem 2.7. *If $F_\epsilon \in \mathcal{R}_b^\lambda(A, B, \alpha, \mu)$ for $|\epsilon| < \delta^*$, then*

$$\mathcal{N}_\delta(f) \subset \mathcal{R}_b^\lambda(A, B, \alpha, \mu),$$

where $\delta := \frac{(1-\alpha)|b||B-A|}{(1+\lambda)(1+\mu)(1+|B|)}\delta^*$.

Proof. Let $g \in \mathcal{N}_\delta(f)$ for $\delta = \frac{(1-\alpha)|b||B-A|}{(1+\lambda)(1+\mu)(1+|B|)}\delta^*$.

If we take

$$F^\lambda(g, \mu; z) := (1-\mu)\frac{\mathcal{D}^\lambda g(z)}{z} + \mu\left(\mathcal{D}^\lambda g(z)\right)',$$

then

$$\begin{aligned}&\left|F^{\lambda}(g,\mu;z)*\phi(z)\right|\\&=\left|\left[(1-\mu)\frac{\mathcal{D}^{\lambda}(f+g-f)(z)}{z}+\mu\left(\mathcal{D}^{\lambda}(f+g-f)(z)\right)'\right]*\phi(z)\right|\\&\geq\left|F^{\lambda}(f,\mu;z)*\phi(z)\right|-\left|F^{\lambda}(g-f,\mu;z)*\phi(z)\right|\end{aligned}$$

and using Theorem 2.5 we can write

$$\geq\delta^{*}-\left|\sum_{k=2}^{\infty}\Psi(k)(b_k-a_k)c_{k-1}z^{k-1}\right|, \tag{2.4}$$

where

$$\Psi(k)=\frac{(\lambda+1)_{(k-1)}}{(1)_k}(\mu k-\mu+1).$$

We know that $\Psi(k)$ is an increasing function of k and

$$0<\Psi(2)=(1+\lambda)(1+\mu)\leq\Psi(k)\qquad\left(\mu\geq0;\ k\in\mathbb{N};\ \lambda\geq\frac{-\mu k}{(1+\mu k)}\right).$$

Since $F_{\epsilon}\in\mathcal{R}_b^{\lambda}(A,B,\alpha,\mu)$ for $|\epsilon|<\delta^{*}$ and using Lemma 2.6 in (2.4) we have

$$\begin{aligned}\left|F^{\lambda}(g,\mu;z)*\phi(z)\right|&>\delta^{*}-\Psi(2)|z|\sum_{k=2}^{\infty}|a_k-b_k|\frac{k(1+|B|)}{(1-\alpha)|b||B-A|}\\&>\delta^{*}-\frac{(1+\lambda)(1+\mu)(1+|B|)}{(1-\alpha)|b||B-A|}\sum_{k=2}^{\infty}k|a_k-b_k|\end{aligned}$$

$$> \delta^* - \delta \frac{(1+\lambda)(1+\mu)(1+|B|)}{(1-\alpha)|b||B-A|} > 0.$$

That is, we can write

$$\left[(1-\mu) \frac{\mathcal{D}^\lambda g(z)}{z} + \mu \left(\mathcal{D}^\lambda g(z) \right)' \right] * \phi(z) \neq 0 \qquad (z \in \mathbb{U}).$$

Thus, from Theorem 2.1 we can find that $g \in \mathcal{R}_b^\lambda (A, B, \alpha, \mu)$. □

Corollary 2.8. *If* $F_\epsilon \in \mathcal{R}_b (A, B, \alpha, \mu)$ *for* $|\epsilon| < \delta^*$, *then*

$$\mathcal{N}_\delta (f) \subset \mathcal{R}_b (A, B, \alpha, \mu),$$

where $\delta := \frac{(1-\alpha)|b||B-A|}{(1+\mu)(1+|B|)} \delta^*$.

Proof. By putting $\lambda = 0$ in Theorem 2.7. □

Corollary 2.9. *If* $F_\epsilon \in \mathcal{R}_b (A, B)$ *for* $|\epsilon| < \delta^*$, *then*

$$\mathcal{N}_\delta (f) \subset \mathcal{R}_b (A, B)$$

where $\delta := \frac{|b||B-A|}{(1+|B|)} \delta^*$.

Proof. By putting $\alpha = 0$, $\mu = 0$ in Corollary 2.8. Thus, we obtain the result of Theorem 2.7 in [3]. □

Corollary 2.10. *If* $F_\epsilon \in \mathcal{C}(b)$ *for* $|\epsilon| < \delta^*$, *then*

$$\mathcal{N}_\delta (f) \subset \mathcal{C}(b),$$

where $\delta := |b| \delta^*$.

Proof. By putting $A = 1$, $B = -1$ in Corollary 2.9. □

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