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A LOCAL MINIMUM ENERGY CONDITION OF HEXAGONAL CIRCLE PACKING

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ABSTRACT. A sufficient condition for the energy of a point such that a local minimum of the energy exists at every triangular lattice point is obtained. The condition is expressed as a certain type of strong convexity condition of the function which defines the energy. New results related to Riemann sum of a function with such the convexity and new inequalities related to sums on triangular lattice points are also presented.

Key words and phrases: Packing, Energy, Convex function.

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1. INTRODUCTION

In some scientific or engineering fields, we are sometimes required to give or measure well-distributed objects in a space. From a purely mathematical point of view, these requirements are satisfied by solving a question which asks whether the well-distributed points are given by the minimization of a total energy of arbitrarily distributed points. In [7], assuming the well-distributed points to be arranged as in a periodic sphere packing [10, pp.25], we have obtained the minimum energy condition in a one-dimensional case; this condition is given as a certain strong convexity condition of the function which defines the energy. A natural question arising in this context is whether the one-dimensional condition can be theoretically extended to higher dimensional spaces.

In this study, we consider the two-dimensional case by imposing two strong restrictions. The first constraint restricts the packing structure to a hexagonal circle packing. Although general circle packing structures are unknown [5, D1], the densest (ideal) circle packing is achieved by the hexagonal circle packing [10] [11, Theorem 1.3 (Lagrange (1773), Thue (1910), L. Fejes Tóth (1940), Segre and Mahler (1944))], which is equivalent to the structure with the center of each circle placed on the triangular lattice points. The second constraint restricts the minimum energy analyses to the point-based local minimum analysis, which addresses whether a small

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perturbation of a point increases the energy of the point. These restrictions are motivated by a suggestion about the study of local minima for optimal configurations [5, F17].

Hence, we investigate the condition for the energy of a point such that each triangular lattice point has a locally optimal configuration with respect to the energy.

2. DEFINITION

Definition 2.1. For a point set $X \subset \mathbb{R}^2$ and $f : (0, \infty) \to \mathbb{R}$, let *the energy of a point* $x \in X$ be

$$J(X, x, f) = \sum_{y \in X \setminus \{x\}} f(\|x - y\|),$$

where $\| \cdot \|$ is the Euclidean norm.

For ease of analysis, we use the above-mentioned definition for defining the energy that is different from the energy in the one-dimensional case [7]. However, the obtained results in this study are also valid for energies having the same form as that of the energy in the one-dimensional case when X is a finite set and $f(0)$ is defined.

Definition 2.2. Let $d > 0$, $\mathbf{v}_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, and $\mathbf{v}_2 = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. Let *one-sixth of the triangular lattice points* be given by

$$\Lambda_d = \left\{ d(i\mathbf{v}_1 + j\mathbf{v}_2) : i \in \mathbb{N}, j = 0, \ldots, i-1 \right\}. \tag{2.1}$$

Let *one-sixth of equally spaced points on equally spaced concentric circles* be given by

$$\Omega_d = \left\{ (id \cos \tau_{ij}, id \sin \tau_{ij}) : \tau_{ij} = \pi/3 \cdot (1 - j/i), i \in \mathbb{N}, j = 0, \ldots, i-1 \right\}. \tag{2.2}$$

In addition, let *the triangular lattice points* Λ_d^* and *equally spaced points on equally spaced concentric circles* Ω_d^* be the unions of the rotations of Λ_d and Ω_d, respectively, around the origin by angles $\frac{\pi}{3}j$ for $j = 0, \ldots, 5$.

Figure 2.1(a) and (b) illustrates Λ_d and Ω_d, respectively. From the definition of Λ_d^*, it can be easily checked that $\Lambda_d^* = \{d(i\mathbf{v}_1 + j\mathbf{v}_2) : i, j \in \mathbb{Z}\} \setminus \{\mathbf{0}\}$.

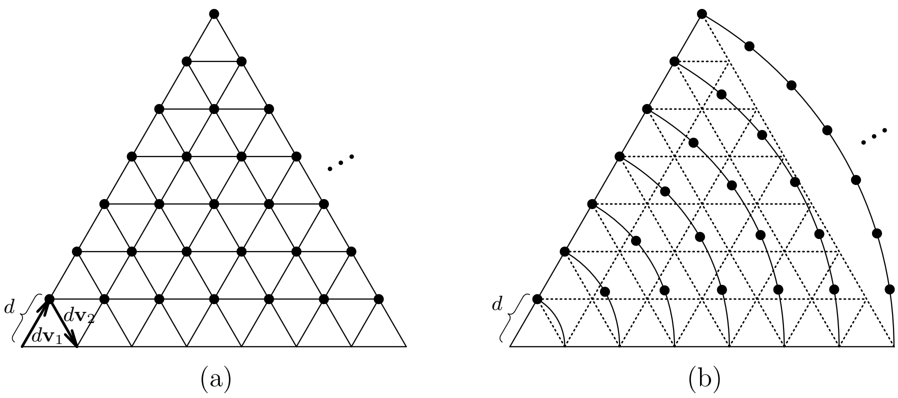

Figure 2.1: Illustration of two point sets along with related parameters: (a) Λ_d and (b) Ω_d.

3. ANALYTICAL CONDITION FOR LOCAL MINIMUM ENERGY

In this section, we derive the analytical condition for the energy J such that it has a local minimum at $\mathbf{v}$ when X consists of equally spaced points on each of the concentric circles with arbitrary radii centered at $\mathbf{v}$.

Proposition 3.1. *Let* $n \in \mathbb{N}$. *For* $i = 1, \ldots, n$, *let* $k_i \in \mathbb{N}$ *with* $k_i \geq 3$, $\theta_i \in [0, 2\pi)$, *and* $0 < r_i \leq 1$. *For* $i = 1, \ldots, n$ *and* $j = 0, \ldots, k_i - 1$, *let* $\tau_{ij} = 2\pi j/k_i + \theta_i$ *and vectors* $\mathbf{u}_{ij} = (r_i \cos\tau_{ij}, r_i \sin\tau_{ij})$. *Let* $\mathbf{v} \in \mathbb{R}^2$ *and a point set* $X \subset \mathbb{R}^2$ *satisfying*

$$X \cap \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{y}| \leq 1\} = \bigcup_{i=1}^{n} \{\mathbf{u}_{ij} : j = 0, \ldots, k_i - 1\}. \tag{3.1}$$

Let $f : (0, \infty) \to \mathbb{R}$ *belong to the class* C^2 *and* $f(x) = 0$ *on* $x \geq 1$. *If*

$$\sum_{\mathbf{y} \in X} \left[f''(|\mathbf{y}|) + \frac{f'(|\mathbf{y}|)}{|\mathbf{y}|} \right] = \sum_{i=1}^{n} k_i \left[f''(r_i) + \frac{f'(r_i)}{r_i} \right] > 0, \tag{3.2}$$

then the energy $J((\mathbf{v} + X) \cup \{\mathbf{x}\}, \mathbf{x}, f)$ *has a local minimum at* $\mathbf{x} = \mathbf{v}$.

Proof. We analyze the derivative of J and the Hessian matrix of the derivative. Without loss of generality, we may assume $\mathbf{v} = \mathbf{0}$ and $\theta_i = 0$ for each i because these restrictions do not influence the value of J. Then, the energy of a point $\mathbf{x}$ is given by

$$J(X \cup \{\mathbf{x}\}, \mathbf{x}, f) = \sum_{\mathbf{y} \in X} f(|\mathbf{x} - \mathbf{y}|) = \sum_{i=1}^{n} \sum_{j=0}^{k_i - 1} f(|\mathbf{x} - \mathbf{u}_{ij}|).$$

From the assumption, $f = f' = f'' = 0$ on $x \geq 1$. Thus, J is certainly twice differentiable with respect to $\mathbf{x}$.

First, we consider ∇J. Since the derivative of $|\mathbf{x}|$ with respect to $\mathbf{x}$ is $\frac{\mathbf{x}}{|\mathbf{x}|}$, we get

$$\nabla J = \sum_{i=1}^{n} \sum_{j=0}^{k_i - 1} f'(|\mathbf{x} - \mathbf{u}_{ij}|) \cdot \frac{\mathbf{x} - \mathbf{u}_{ij}}{|\mathbf{x} - \mathbf{u}_{ij}|}.$$

Note that at the point $\mathbf{x} = \mathbf{0}$, we have $|\mathbf{x} - \mathbf{u}_{ij}| = |\mathbf{u}_{ij}| = r_i$. Here, for $m, p \in \mathbb{N}$ with $m < p$ and for $\eta \in \mathbb{R}$ with $\cos\eta \neq 1$, a general exponential sum formula holds in $\mathbb{C}$:

$$\begin{aligned} \sum_{m=0}^{p-1} (\cos m\eta + i \sin m\eta) &= \sum_{m=0}^{p-1} e^{im\eta} \\ &= \frac{1 - e^{ip\eta}}{1 - e^{i\eta}} = \frac{1 - (\cos p\eta + i \sin p\eta)}{1 - (\cos\eta + i \sin\eta)}. \end{aligned} \tag{3.3}$$

(In (3.3), i denotes the imaginary unit.) Substituting $m = j$, $p = k_i$, and $\eta = 2\pi/k_i$ in (3.3), we obtain $\sum_{j=0}^{k_i - 1} \mathbf{u}_{ij} = \mathbf{0}$ for each i. Hence, $\nabla J = \mathbf{0}$ holds at $\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{0}$ is a stationary point.

Next, we analyze the Hessian matrix of ∇J to determine whether J has a local minimum at $\mathbf{x} = \mathbf{0}$. Using the notations $\mathbf{x} = (x_1, x_2)$ and $\mathbf{u}_{ij} = (u_{ij1}, u_{ij2})$, we get

$$\frac{\partial^2 J}{\partial {x_m}^2} = \sum_{i=1}^{n} \sum_{j=0}^{k_i - 1} \left[f''(|\mathbf{x} - \mathbf{u}_{ij}|) - \frac{f'(|\mathbf{x} - \mathbf{u}_{ij}|)}{|\mathbf{x} - \mathbf{u}_{ij}|} \right] \cdot \frac{(x_m - u_{ijm})^2}{|\mathbf{x} - \mathbf{u}_{ij}|^2} + \sum_{i=1}^{n} \sum_{j=0}^{k_i - 1} \frac{f'(|\mathbf{x} - \mathbf{u}_{ij}|)}{|\mathbf{x} - \mathbf{u}_{ij}|},$$

where $m = 1, 2$ and

$$\frac{\partial^2 J}{\partial x_2 \partial x_1} = \frac{\partial^2 J}{\partial x_1 \partial x_2} = \sum_{i=1}^{n} \sum_{j=0}^{k_i - 1} \left[f''(|\mathbf{x} - \mathbf{u}_{ij}|) - \frac{f'(|\mathbf{x} - \mathbf{u}_{ij}|)}{|\mathbf{x} - \mathbf{u}_{ij}|} \right] \cdot \frac{(x_1 - u_{ij1})(x_2 - u_{ij2})}{|\mathbf{x} - \mathbf{u}_{ij}|^2}.$$

Note that $\cos\eta \neq 1$ from the assumption $k_i \geq 3$. Hence, by substituting $m = j$, $p = k_i$, and $\eta = 4\pi/k_i$ in (3.3), we obtain

$$\sum_{j=0}^{k_i-1} \cos 2\tau_{ij} = \sum_{j=0}^{k_i-1} \sin 2\tau_{ij} = 0$$

for each i. Hence, using double-angle formulas, for $i = 1, \dots, n$, we have

$$\sum_{j=0}^{k_i-1} {u_{ij1}}^2 = \sum_{j=0}^{k_i-1} {u_{ij2}}^2 = \frac{k_i {r_i}^2}{2}, \qquad \sum_{j=0}^{k_i-1} u_{ij1} u_{ij2} = 0.$$

From these equalities, at $\mathbf{x} = \mathbf{0}$, we have $\frac{\partial^2 J}{\partial {x_1}^2} = \frac{\partial^2 J}{\partial {x_2}^2}$, $\frac{\partial^2 J}{\partial x_1 \partial x_2} = \frac{\partial^2 J}{\partial x_2 \partial x_1} = 0$, and

$$\begin{aligned} \frac{\partial^2 J}{\partial {x_1}^2} &= \sum_{i=1}^{n} \sum_{j=0}^{k_i-1} \left[f''(r_i) - \frac{f'(r_i)}{r_i} \right] \cdot \frac{{u_{ij1}}^2}{{r_i}^2} + \sum_{i=1}^{n} \sum_{j=0}^{k_i-1} \frac{f'(r_i)}{r_i} \\ &= \sum_{i=1}^{n} \frac{k_i}{2} \left[f''(r_i) + \frac{f'(r_i)}{r_i} \right]. \end{aligned}$$

Hence, at $\mathbf{x} = \mathbf{0}$, both the discriminant and the term $\frac{\partial^2 J}{\partial {x_1}^2}$ are positive from the assumption. Thus, J has a local minimum at $\mathbf{x} = \mathbf{0}$. □

We can apply Proposition 3.1 to Λ_d^* and Ω_d^* because each set can satisfy the form (3.1) for fixed $k_i = 6$. Furthermore, we can use Λ_d and Ω_d for the estimations of (3.2) because the values of (3.2) for $X = \Lambda_d^*$ and $X = \Omega_d^*$ are 6 times those obtained for $X = \Lambda_d$ and $X = \Omega_d$, respectively. In particular, substituting $r = d^{-1}$ in (2.2), on Ω_d, we obtain

$$\begin{aligned} \sum_{\mathbf{y} \in \Omega_d} \left[f''(|\mathbf{y}|) + \frac{f'(|\mathbf{y}|)}{|\mathbf{y}|} \right] &= \sum_{i=1}^{\lfloor r \rfloor} i \left[f''\left(\frac{i}{r}\right) + \frac{r}{i} f'\left(\frac{i}{r}\right) \right] \\ &= r \sum_{i=1}^{\lfloor r \rfloor} \left[f'\left(\frac{i}{r}\right) + \frac{i}{r} f''\left(\frac{i}{r}\right) \right]. \end{aligned} \tag{3.4}$$

Thus, the local minimum energy condition on Ω_d is simplified into the positivity of the sum of a single-variable function. Since it might be difficult to directly analyze (3.2) with respect to Λ_d, we would first analyze the right-hand side of (3.4) for Ω_d.

4. RIEMANN SUM OF A FUNCTION WITH A CERTAIN STRONG CONVEXITY

In [7], for the minimum energy analysis in a one-dimensional case, we have proved a variant of a result obtained by Bennett and Jameson [1]. Here, in order to investigate a sufficient condition such that the expression (3.4) may be greater than 0, we again adopt the same approach.

For a function f on $(0, 1]$, let $S_n(f)$ be the upper Riemann sum for the integral $\int_0^1 f$ resulting from division of $[0, 1]$ into n equal subintervals:

$$S_n(f) = \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right).$$

Theorem 3A in [1] states that if f is increasing and either convex or concave, then $S_n(f)$ decreases with n. The same theorem has been independently proved by Kuang [9]. Further related results have been presented in [1, 3]. Here, we show that $S_n(f)$ also decreases if f is increasing, $\left(f'\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}\right)^2$ is convex, and $\lim_{x\to 1} f'(x) = 0$.

Before presenting the result, we prove the following lemma.

Lemma 4.1. *Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$. Let $p \geq 1$ be a real number. If $f \geq 0$, f is decreasing, and $f(x)^p$ is convex, then*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{p}{p+1}f(a) + \frac{1}{p+1}f(b). \tag{4.1}$$

Equality holds iff any one of the following conditions is satisfied:

(a) $p = 1$ *and f is linear on* $[a, b]$*;*
(b) *f is constant on $[a, b]$; and*
(c) *$f(x)^p$ is linear on $[a, b]$ and $f(b) = 0$.*

Proof. We show that in fact a stronger inequality

$$f(xa + (1-x)b) \leq x^{\frac{1}{p}} f(a) + \left(1 - x^{\frac{1}{p}}\right) f(b) \tag{4.2}$$

holds, where $0 \leq x \leq 1$. By integrating (4.2) over $x \in [0, 1]$, we can obtain (4.1).

If $p = 1$, then the result follows from the convexity of f.

We assume that $p > 1$. By using the substitution $g(x) = f(xa + (1-x)b)$, it is sufficient to show

$$g(x) \leq \left(1 - x^{\frac{1}{p}}\right) g(0) + x^{\frac{1}{p}} g(1) \tag{4.3}$$

for $g : [0, 1] \to \mathbb{R}$, where $g \geq 0$ is increasing and $g(x)^p$ is convex.

First, consider the case when $g(0) = 0$. Since $g(x)^p$ is convex, $g(x)^p \leq xg(1)^p$, thus,

$$g(x) \leq x^{\frac{1}{p}} g(1) \tag{4.4}$$

on $[0, 1]$. Equality holds iff $g(x)^p$ is linear; this case corresponds to case (c).

Next, suppose that $g(0) = c > 0$. If we can show that $[g(x)-c]^p$ is convex, then (4.3) follows from substituting $g(x) - c$ for $g(x)$ in (4.4). Let $h(x) = g(x)^p$ and $k(x) = [g(x) - c]^p$. Since a convex function is differentiable at all but at most countably many points, we may rely on the differentiability of h, and therefore g and k. Then, $h'(x) = pg(x)^{p-1}g'(x)$ and

$$k'(x) = p[g(x) - c]^{p-1}g'(x) = h'(x)\left(1 - \frac{c}{g(x)}\right)^{p-1}.$$

Both $h'(x)$ and $(1 - c/g(x))^{p-1}$ are positive and increasing. Hence, $k'(x)$ is increasing, as required. Equality in (4.3) holds iff

$$g(x)^p = [(1 - x^{1/p})g(0) + x^{1/p}g(1)]^p,$$

which gives

$$[g(x)^p]' = (g(1) - g(0))\left[g(1) - g(0) + x^{-1/p}g(0)\right]^{p-1}.$$

Here, it follows that $g(0) = g(1)$ because $[g(x)^p]'$ cannot be increasing for $p > 1$ if $g(0) < g(1)$. This equality condition corresponds to the condition in case (b). □

Theorem 4.2. *Let $f : (0, 1] \to \mathbb{R}$ be differentiable. If f is increasing, $\left(f'\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}\right)^2$ is convex, and $\lim_{x\to 1} f'(x) = 0$, then $S_n(f)$ decreases with n.*

Proof. From the assumption, $f' \geq 0$ holds and $f'\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}$ is decreasing. Without loss of generality, we may assume that $f(1) = 0$ and extend $f = f' = 0$ on $x \geq 1$. For a real number $r \geq 1$, let

$$S_r(f) = \frac{1}{r}\sum_{i=1}^{\lfloor r \rfloor} f\left(\frac{i}{r}\right).$$

We show that ${S_r}'(f) \le 0$ for $r \ge 1$, where ${S_r}'(f)$ is the differential coefficient of $S_r(f)$ with respect to r. The existence of ${S_r}'(f)$ is confirmed by the differentiability of f on $(0, \infty)$ and $f = f' = 0$ on $x \ge 1$. In fact, we have

$$ {S_r}'(f) = -\frac{1}{r^2}\sum_{i=1}^{\lfloor r \rfloor}\left[f\left(\frac{i}{r}\right) + \frac{i}{r} f'\left(\frac{i}{r}\right)\right]. $$

The substitution $x = t^{\frac{1}{2}}$ gives

$$ \int_{a^{\frac{1}{2}}}^{b^{\frac{1}{2}}} f'(x)dx = \int_a^b \frac{f'\big(t^{\frac{1}{2}}\big)}{2t^{\frac{1}{2}}}dt. $$

Thus, by applying $f'\big(x^{\frac{1}{2}}\big)/x^{\frac{1}{2}}$ to f in Lemma 4.1 with $p = 2$, for $0 < a < b$, we obtain

$$ \frac{1}{b-a}\int_{a^{\frac{1}{2}}}^{b^{\frac{1}{2}}} f'(x)dx \le \frac{2}{6}\cdot\frac{f'\big(a^{\frac{1}{2}}\big)}{a^{\frac{1}{2}}} + \frac{1}{6}\cdot\frac{f'\big(b^{\frac{1}{2}}\big)}{b^{\frac{1}{2}}}. \tag{4.5} $$

Substituting $a = \left(\frac{j}{r}\right)^2$ and $b = \left(\frac{j+1}{r}\right)^2$ in (4.5), we get

$$ f\left(\frac{j+1}{r}\right) - f\left(\frac{j}{r}\right) \le \frac{2}{6r}\cdot\frac{2j+1}{j} f'\left(\frac{j}{r}\right) + \frac{1}{6r}\cdot\frac{2(j+1)-1}{j+1} f'\left(\frac{j+1}{r}\right). $$

Summing over $j = i, \dots, \lfloor r \rfloor$ and using $f(1) = 0$, we obtain

$$ -f\left(\frac{i}{r}\right) \le \frac{1}{r}\sum_{j=i}^{\lfloor r \rfloor} f'\left(\frac{j}{r}\right) + \frac{1}{6r}\sum_{j=i}^{\lfloor r \rfloor}\frac{1}{j} f'\left(\frac{j}{r}\right) - \frac{1}{6r}\cdot\frac{2i-1}{i} f'\left(\frac{i}{r}\right). \tag{4.6} $$

Thus, from (4.6) and $f' \ge 0$, we obtain

$$ \begin{aligned} \sum_{i=1}^{\lfloor r \rfloor}\left[f\left(\frac{i}{r}\right) + \frac{i}{r} f'\left(\frac{i}{r}\right)\right] &= \sum_{i=1}^{\lfloor r \rfloor}\left[f\left(\frac{i}{r}\right) + \frac{1}{r}\sum_{j=i}^{\lfloor r \rfloor} f'\left(\frac{j}{r}\right)\right] \\ &\ge \frac{1}{6r}\sum_{i=1}^{\lfloor r \rfloor}\frac{2i-1}{i} f'\left(\frac{i}{r}\right) - \frac{1}{6r}\sum_{i=1}^{\lfloor r \rfloor}\sum_{j=i}^{\lfloor r \rfloor}\frac{1}{j} f'\left(\frac{j}{r}\right) \\ &= \frac{1}{6r}\sum_{i=1}^{\lfloor r \rfloor}\frac{i-1}{i} f'\left(\frac{i}{r}\right) \\ &\ge 0. \end{aligned} \tag{4.7} $$

Hence, ${S_r}'(f) \le 0$ holds. Thus, $S_r(f)$ decreases with $r \ge 1$.

From Lemma 4.1, equality in (4.5) holds iff either $f' = 0$ on $[a, b]$ or $\big(f'\big(x^{\frac{1}{2}}\big)/x^{\frac{1}{2}}\big)^2$ is linear on $[a, b]$ with $f'\big(b^{\frac{1}{2}}\big) = 0$. Thus, from (4.6) and (4.7), ${S_r}'(f) = 0$ holds iff either $f' = 0$ on $[\frac{1}{r}, 1]$ or $\big(f'\big(x^{\frac{1}{2}}\big)/x^{\frac{1}{2}}\big)^2$ is linear on $[\frac{1}{r}, \frac{2}{r}]$ with $f'(\frac{2}{r}) = 0$, indicating that $f' = 0$ on $[\frac{2}{r}, 1]$. Here, the latter condition with $f(\frac{1}{r}) \ne 0$ can hold only for one fixed r. Hence, for any $r_1 \ge 1$, $S_r(f)$ strictly decreases with $r \ge r_1$ iff $f' > 0$ on $\big(0, \frac{1}{r_1}\big)$. □

Therefore, $S_n(f)$ decreases with n if f is increasing and either

(i) f is convex or concave (from [1, Theorem 3A]), or
(ii) $\big(f'\big(x^{\frac{1}{2}}\big)/x^{\frac{1}{2}}\big)^2$ is convex and $\lim_{x\to 1} f'(x) = 0$ (from Theorem 4.2).

Here, conditions (i) and (ii) are independent of each other, which can be observed in the following examples. Let $p > 0$.

Example 4.1. Let $f(x) = -(1-x)^p$ and $f'(x) = p(1-x)^{p-1}$. Then, f is convex if $0 < p \le 1$ and concave if $p \ge 1$. Further, $\left(f'\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}\right)^2$ is convex and $\lim_{x\to 1} f'(x) = 0$ if $p \ge 1.5$. Further, $f'\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}$ is convex if $p \ge 2$.

Example 4.2. Let $f(x) = -(1-x^2)^p$ and $f'(x) = 2px(1-x^2)^{p-1}$. Then, f is convex if $0 < p \le 1$ and neither convex nor concave if $p > 1$. Further, $\left(f'\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}\right)^2$ is convex and $\lim_{x\to 1} f'(x) = 0$ if $p \ge 1.5$. Further, $f'\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}$ is convex if $p \ge 2$.

Theorem 4.3. *Let* $f : (0,1] \to \mathbb{R}$. *If* $f\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}$ *is decreasing and convex and* $f(1) = 0$*, then*

$$\int_{\frac{1}{n}}^{1} f(x)dx \le \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) \le \int_0^1 f(x)dx. \tag{4.8}$$

Proof. Let $a < b$. The Hermite-Hadamard inequality for a convex function f gives

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

By applying the convexity of $f\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}$ to this inequality, we obtain

$$\begin{aligned} (b-a)f\left(\left[\frac{a+b}{2}\right]^{\frac{1}{2}}\right)\cdot\left(\frac{a+b}{2}\right)^{-\frac{1}{2}} &\le \int_a^b \frac{f\left(x^{\frac{1}{2}}\right)}{x^{\frac{1}{2}}}dx \\ &\le \frac{b-a}{2}\left[\frac{f\left(a^{\frac{1}{2}}\right)}{a^{\frac{1}{2}}} + \frac{f\left(b^{\frac{1}{2}}\right)}{b^{\frac{1}{2}}}\right]. \end{aligned} \tag{4.9}$$

Substituting $a = (\frac{i}{n})^2$ and $b = (\frac{i+1}{n})^2$ on the right-hand inequality in (4.9), we get

$$\int_{\frac{i}{n}}^{\frac{i+1}{n}} f(x)dx \le \frac{1}{4n}\cdot\frac{2i+1}{i}f\left(\frac{i}{n}\right) + \frac{1}{4n}\cdot\frac{2(i+1)-1}{i+1}f\left(\frac{i+1}{n}\right).$$

Summing over $i = j, \dots, n-1$ and using $f \ge 0$, for each $j = 1, \dots, n-1$, we obtain

$$\begin{aligned} \int_{\frac{j}{n}}^{1} f(x)dx &\le \frac{1}{n}\sum_{i=j}^{n} f\left(\frac{i}{n}\right) - \frac{1}{4n}\cdot\frac{2j-1}{j}f\left(\frac{j}{n}\right) - \frac{1}{4n}\cdot\frac{2n-1}{n}f(1) \\ &\le \frac{1}{n}\sum_{i=j}^{n} f\left(\frac{i}{n}\right). \end{aligned} \tag{4.10}$$

Hence, the left-hand inequality in (4.8) follows from (4.10) when $j = 1$. Next, we extend $f = 0$ on $x \ge 1$, which yields the convexity of $f\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}$ on $(0,\infty)$. Then, substituting $a = \frac{i^2-i}{n^2}$ and $b = \frac{i^2+i}{n^2}$ on the left-hand inequality in (4.9), we get

$$\frac{2i}{n^2}f\left(\frac{i}{n}\right)\cdot\left(\frac{i}{n}\right)^{-1} \le \int_{\frac{i^2-i}{n^2}}^{\frac{i^2+i}{n^2}} \frac{f\left(x^{\frac{1}{2}}\right)}{x^{\frac{1}{2}}}dx.$$

Summing over $i = 1, \dots, n$, we obtain the right-hand inequality in (4.8).

In (4.10), equalities hold iff $f'\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}$ is linear on $\left[\frac{j}{n}, 1\right]$ and $f\left(\frac{j}{n}\right) = 0$, that is, $f = 0$ on $\left[\frac{j}{n}, 1\right]$. Hence, equality on the left-hand inequality in (4.8) holds iff $f = 0$ on $\left[\frac{1}{n}, 1\right]$. Similarly, equality on the right-hand inequality in (4.8) holds iff $f = 0$ on $(0, 1]$. □

If f is decreasing on $[0,1]$ and $f(1) = 0$, then (4.8) and $\int_0^1 f \le \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n}\right)$ are trivial. However, when we use a function f on $[0,1]$ in Theorem 4.3, such an additional upper estimation no longer holds. From (4.10), if a stronger condition that $f'\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}$ is convex is assumed in Theorem 4.2, then in (4.7),

$$f\left(\frac{i}{r}\right) + \frac{1}{r}\sum_{j=i}^{\lfloor r\rfloor} f'\left(\frac{j}{r}\right) \ge 0$$

holds for each i.

5. LOCAL MINIMUM ENERGY CONDITION

We can obtain the local minimum energy condition for Ω_d^* from the result obtained in the previous section.

Proposition 5.1. *Let $0 < d \le 1$ and Ω_d^* be defined by Definition 2.2. Let $\mathbf{v} \in \mathbb{R}^2$ and a point set $X \subset \mathbb{R}^2$ satisfying*

$$\{\mathbf{x} \in X : |\mathbf{x}| \le 1\} = \{\mathbf{x} \in \Omega_d^* : |\mathbf{x}| \le 1\}.$$

Let $f : (0,\infty) \to \mathbb{R}$ belong to the class C^2 with $f = 0$ on $x \ge 1$ and $f'' \not\equiv 0$ on $[d,1]$. If f is convex and either

(i) *f' is concave or*

(ii) *$\left(f''\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}\right)^2$ is convex and either is strictly convex on $[d,2d]$ or $f(2d) \ne 0$,*

then the energy $J((\mathbf{v}+X)\cup\{\mathbf{x}\},\mathbf{x},f)$ has a local minimum at $\mathbf{x} = \mathbf{v}$.

Proof. Let $r = d^{-1}$. As stated after Proposition 3.1, it is sufficient to show that (3.4) is greater than 0. In case (i), f'' is decreasing. Moreover, there is an interval contained in $[d,1]$ in which f'' is strictly decreasing because $f'' \not\equiv 0$ on $[d,1]$ and $f''(1) = 0$. Thus,

$$\sum_{i=1}^{\lfloor r\rfloor}\left[f'\left(\frac{i}{r}\right) + \frac{i}{r}f''\left(\frac{i}{r}\right)\right] = \sum_{i=1}^{\lfloor r\rfloor}\left[-\int_{\frac{i}{r}}^1 f''(x)dx + \frac{1}{r}\sum_{j=i}^{\lfloor r\rfloor} f''\left(\frac{j}{r}\right)\right] > 0.$$

In case (ii), the result follows from (4.7) and related arguments presented after that. □

It is expected that a result similar to that of Proposition 5.1 can be obtained for the triangular lattice points Λ_d^* being similar in structure to Ω_d^*, thereby leading to the following theorem. In the proof, two inequalities related to the triangular lattice points are required. The proofs of these inequalities are given in Section 7. In the statement of Theorem 5.2, a specific value of p is given. The meaning of the value p is explained in the proof of the theorem.

Theorem 5.2. *Let $0 < d \le 1$ and Λ_d^* be defined by Definition 2.2. Let $\mathbf{v} \in \mathbb{R}^2$ and a point set $X \subset \mathbb{R}^2$ satisfying $\{\mathbf{x} \in X : |\mathbf{x}| \le 1\} = \{\mathbf{x} \in \Lambda_d^* : |\mathbf{x}| \le 1\}$. Let $f : (0,\infty) \to \mathbb{R}$ belong to the class C^2 with $f = 0$ on $x \ge 1$ and $f'' \not\equiv 0$ on $[d,1]$. If f is convex and either*

(i) *f' is concave or*

(ii) *$\left(f''\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}\right)^p$ is convex for $p = \frac{2}{11}(47 + 27\sqrt{3}) = 17.048\ldots$,*

then the energy $J((\mathbf{v}+X)\cup\{\mathbf{x}\},\mathbf{x},f)$ has a local minimum at $\mathbf{x} = \mathbf{v}$.

Proof. As stated in Section 3, we can use the one-sixth version set Λ_d instead of the triangular lattice points set Λ_d^*. Thus, from Proposition 3.1, it is sufficient to show that

$$\sum_{i=1}^{\infty}\left[f''(a_i) + \frac{f'(a_i)}{a_i}\right] > 0, \tag{5.1}$$

where the sequence $\{a_i\}$ is obtained by sorting the value $|\mathbf{y}|$ for all $\mathbf{y} \in \Lambda_d$ in increasing order. More precisely, each a_i is defined by

$$a_i = \max\big\{|\mathbf{y}| : \mathbf{y} \in \Lambda_d,\ \#\{\mathbf{z} \in \Lambda_d : |\mathbf{z}| < |\mathbf{y}|\} < i\big\}.$$

The first 10 values of a_i are $d, \sqrt{3}d, 2d, \sqrt{7}d, \sqrt{7}d, 3d, \sqrt{12}d, \sqrt{13}d, \sqrt{13}d$ and $4d$; these values are illustrated in Figure 7.1 in Section 7.

First, we summarize the inequalities that are required for the proof. From Theorem 7.1, which will be stated later in Section 7, for all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \frac{a_{n+1}}{a_i} < 2n. \tag{5.2}$$

In addition, from Theorem 7.3, for $n \in \mathbb{N}$ with $n \geq 4$,

$$\sum_{i=1}^{n} \frac{{a_{n+1}}^2}{a_i} < \sum_{i=1}^{n} 3a_i. \tag{5.3}$$

Since a_n increases with n, for all $n \in \mathbb{N}$ and any real number $p \geq 0$, we have

$$\sum_{i=1}^{n} \left[\frac{p}{p+1} \cdot \frac{{a_{n+1}}^2}{a_i} + \frac{1}{p+1} \cdot \frac{{a_n}^2}{a_i}\right] \leq \sum_{i=1}^{n} \frac{{a_{n+1}}^2}{a_i}.$$

Thus, by substituting $p = \frac{2}{11}(47 + 27\sqrt{3})$ and from (5.3), for all $n \in \mathbb{N}$, we obtain

$$\sum_{i=1}^{n} \left[\frac{p}{p+1} \cdot \frac{{a_{n+1}}^2}{a_i} + \frac{1}{p+1} \cdot \frac{{a_n}^2}{a_i}\right] \leq \sum_{i=1}^{n} 3a_i, \tag{5.4}$$

where equality holds iff $n = 3$. Note that (5.4) strictly holds for any (large) $p \geq 0$ when $n \neq 3$. The specific value of p is the upper bound of p for satisfying (5.4) when $n = 3$.

Next, we prove (5.1) for cases of (i) and (ii) by using (5.2) and (5.4), respectively. From the assumption, suppose that $f = f' = f'' = 0$ on $x \geq 1$ and $f'' \not\equiv 0$ on $[a_1, 1]$.

Case (i): Let f' be concave. Then, f'' is decreasing. Thus, for $0 < a < b$,

$$\int_a^b f''(x)dx \leq (b - a)f''(a).$$

Substituting $a = a_j$ and $b = a_{j+1}$ and summing over $j = i, i + 1, \ldots$, we obtain

$$f'(a_i) \geq -\sum_{j=i}^{\infty} (a_{j+1} - a_j) f''(a_j). \tag{5.5}$$

Thus, from (5.2) and (5.5) and considering that $f'' = 0$ on $x \geq 1$, $f'' \not\equiv 0$ on $[a_1, 1]$, and f'' is decreasing, we obtain

$$\begin{aligned}
\sum_{i=1}^{\infty} \left[f''(a_i) + \frac{f'(a_i)}{a_i}\right] &\geq \sum_{i=1}^{\infty} \left[f''(a_i) - \sum_{j=i}^{\infty} \frac{a_{j+1} - a_j}{a_i} f''(a_j)\right] \\
&= \sum_{i=1}^{\infty} \left[2f''(a_i) - \sum_{j=1}^{i} \frac{a_{i+1}}{a_j} \left(f''(a_i) - f''(a_{i+1})\right)\right] \\
&> \sum_{i=1}^{\infty} \Big[2f''(a_i) - 2i \left(f''(a_i) - f''(a_{i+1})\right)\Big] = 0.
\end{aligned}$$

Case (ii): Let $p = \frac{2}{11}(47 + 27\sqrt{3})$ and $\left(f''\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}\right)^p$ be convex. Then, $f''\left(x^{\frac{1}{2}}\right)/x^{\frac{1}{2}}$ is decreasing since $f'' \geq 0$ and $f''(1) = 0$. Thus, in the same way as in the derivation of (4.5), from Lemma 4.1, for $0 < a < b$, we obtain

$$\frac{1}{b-a}\int_{a^{\frac{1}{2}}}^{b^{\frac{1}{2}}} f''(x)dx \leq \frac{p}{2(p+1)} \cdot \frac{f''\left(a^{\frac{1}{2}}\right)}{a^{\frac{1}{2}}} + \frac{1}{2(p+1)} \cdot \frac{f''\left(b^{\frac{1}{2}}\right)}{b^{\frac{1}{2}}}.$$

Substituting $a = {a_j}^2$ and $b = {a_{j+1}}^2$ and summing over $j = i, i+1, \ldots$, we have

$$\text{(5.6)} \qquad f'(a_i) \geq -\frac{p}{2(p+1)}\sum_{j=i}^{\infty}\frac{{a_{j+1}}^2 - {a_j}^2}{a_j}f''(a_j) - \frac{1}{2(p+1)}\sum_{j=i}^{\infty}\frac{{a_{j+1}}^2 - {a_j}^2}{a_{j+1}}f''(a_{j+1}).$$

Thus, from (5.4) and (5.6) and considering that $f'' = 0$ on $x \geq 1$, $f'' \not\equiv 0$ on $[a_1, 1]$, and $f''(x)/x$ is decreasing, we obtain

$$\begin{aligned}
&2(p+1)\sum_{i=1}^{\infty}\left[f''(a_i) + \frac{f'(a_i)}{a_i}\right] \\
&\geq \sum_{i=1}^{\infty}\left[2(p+1)f''(a_i) - p\sum_{j=i}^{\infty}\frac{{a_{j+1}}^2 - {a_j}^2}{a_i} \cdot \frac{f''(a_j)}{a_j}\right. \\
&\qquad\qquad \left. - \sum_{j=i}^{\infty}\frac{{a_{j+1}}^2 - {a_j}^2}{a_i} \cdot \frac{f''(a_{j+1})}{a_{j+1}}\right] \\
&= \sum_{i=1}^{\infty}\left[3(p+1)f''(a_i)\right. \\
&\qquad\qquad \left. - \left(p\sum_{j=1}^{i}\frac{{a_{i+1}}^2}{a_j} + \sum_{j=1}^{i}\frac{{a_i}^2}{a_j}\right) \cdot \left(\frac{f''(a_i)}{a_i} - \frac{f''(a_{i+1})}{a_{i+1}}\right)\right] \\
&> 3(p+1)\sum_{i=1}^{\infty}\left[f''(a_i) - \left(\sum_{j=1}^{i} a_j\right) \cdot \left(\frac{f''(a_i)}{a_i} - \frac{f''(a_{i+1})}{a_{i+1}}\right)\right] = 0.
\end{aligned}$$

Here, the second inequality certainly holds strictly because in (5.4), strict inequality holds for $n \neq 3$, and $f''(a_1)/a_1 - f''(a_2)/a_2 > 0$ holds from $f \not\equiv 0$ on $[a_1, 1]$. □

In cases (i) and (ii), the assumption that f is convex can be omitted because the other conditions yield $f'' \geq 0$. Nevertheless, it is natural to assume this condition in case (ii).

Now, we address the (second) question presented in the introduction.

Remark 1. Let us consider the relation between Theorem 5.2 and the one-dimensional result [7]. The one-dimensional result was as follows. Consider a finite point set $X \subset \mathbb{R}/\mathbb{Z}$ with the Euclidean distance $\|\cdot\|$ defined by

$$\|x - y\| = \min\{|x - y + e| : e = -1, 0, 1\}$$

and the energy of X defined by the average value of $f(\|x-y\|)$ for $x, y \in X$, where $f : [0, 1/2] \to \mathbb{R}$. If f is convex, then among all m-point sets for fixed $m \geq 1$, the energy is (globally) minimized by an equally spaced m-point set. Additionally, if f is convex, $f'\left(x^{\frac{1}{2}}\right)$ is concave, and $\lim_{x \to \frac{1}{2}} f'(x) = 0$, then among all m-point sets for $1 \leq m \leq n$, the energy is minimized by an equally spaced n-point set.

It is easy to verify that the condition in Theorem 5.2 is stronger than these one-dimensional conditions. Thus, in the two-dimensional case, even for the existence of a local minimum, the function f should have a stronger convexity than the convexity which is defined by these one-dimensional conditions.

As stated in Section 2, in the two-dimensional case, by defining an affine transformation $g : \mathbb{R}^2 \to \mathbb{R}^2$ and the periodic space $g(\mathbb{R}^2/\mathbb{Z}^2)$ with the Euclidean distance $\|\cdot\|$ defined by

$$\|\mathbf{x}-\mathbf{y}\| = \min\{|\mathbf{x}-\mathbf{y}+e_1\cdot g(1,0)+e_2\cdot g(0,1)| : e_1, e_2 = -1, 0, 1\},$$

we can also define the energy of a point as

$$I(X,x,f) = \frac{1}{|X|}\sum_{y\in X} f\big(\|x-y\|\big),$$

where X is a point set in $g(\mathbb{R}^2/\mathbb{Z}^2)$ and $|X|$ is the cardinality of X. Then, Theorem 5.2 is also valid for the energy I only if $|X|$ is finite and $f(0)$ is defined.

6. EXAMPLES

Remark 2. If $p > 0$, $f(x) \geq 0$, and $f(x)^p$ is convex, then $f(x)^q$ is convex for all $q \geq p$. This is because for $g(x) = x^{q/p}$, g is increasing and convex on $[0,\infty)$. Thus,

$$g(f(ax+by)^p) \leq g(af(x)^p + bf(y)^p) \leq ag(f(x)^p) + bg(f(y)^p)$$

holds for $a, b \in [0,1]$ with $a+b=1$.

Example 6.1. For $n \in \mathbb{N}$, let $\omega_n(r) = \pi^{\frac{n}{2}} r^n \big/ \Gamma(\frac{n}{2}+1)$ denote the volume of an n-dimensional ball of radius r. Let $V_n(x)$ be the volume of the cross region of two identical n-dimensional balls of unit diameter with their centers at distance r from each other.

$$V_n(r) = 2\int_{\frac{r}{2}}^{\frac{1}{2}} \omega_{n-1}\left(\sqrt{\frac{1}{4}-x^2}\right)dx = \frac{\pi^{\frac{n-1}{2}}}{2^{n-1}\Gamma\left(\frac{n+1}{2}\right)}\int_r^1 (1-x^2)^{\frac{n-1}{2}}dx.$$

By omitting the constant coefficient of $V_n(r)$, we define $g_n(r) = \int_r^1 (1-x^2)^{(n-1)/2}dx$ for $0 \leq r \leq 1$ and further extend $g_n(r) = 0$ for $r > 1$. Then, each $g_n(x)$ on $[0,1]$ for $n = 1,\dots,5$ is given by

$$\begin{aligned}
g_1(x) &= 1-x,\\
g_2(x) &= \frac{1}{2}\cos^{-1}x - \frac{1}{4}\sin(2\cos^{-1}x) = \frac{1}{2}\cos^{-1}x - \frac{1}{2}x\sqrt{1-x^2},\\
g_3(x) &= \frac{2}{3} - x + \frac{1}{3}x^3,\\
g_4(x) &= \frac{3}{8}\cos^{-1}x - \frac{1}{4}\sin(2\cos^{-1}x) + \frac{1}{32}\sin(4\cos^{-1}x)\\
&= \frac{3}{8}\cos^{-1}x - \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{8}x(2x^2-1)\sqrt{1-x^2},\\
g_5(x) &= \frac{8}{15} - x + \frac{2}{3}x^3 - \frac{1}{5}x^5.
\end{aligned}$$

For $p > 0$, let $f_{np}(x) = g_n(x)^p$. Then, $f_{np}(x)$ is convex for all $n \geq 1$ and $p > 0$.

Table 6.1 shows the conditions required for p to satisfy the convexities in Proposition 5.1 and Theorem 5.2 with respect to $f_{np}{}'$ and $f_{np}{}''$ under the restriction $f_{np} = f_{np}{}' = f_{np}{}'' = 0$ on $x = 1$ for $n = 1,\dots,5$. In the table, the values indicated with an asterisk are approximation values obtained from the numerical analysis, while the others are exact values. In this example, among cases (i) and (ii) in Proposition 5.1 or Theorem 5.2, we may confirm that case (ii) is more valid than the case (i) when $n \geq 2$. In particular, case (ii) is valid for all $p \geq 1$ if $n \geq 4$. In the case

Table 6.1: Convexities with respect to ${f_{np}}'$ and ${f_{np}}''$ (results obtained from numerical analysis).*

	${f_{np}}'(x)$ is concave	${f_{np}}'(x^{\frac{1}{2}})$ is concave	$({f_{np}}''(x^{\frac{1}{2}})/x^{\frac{1}{2}})^{17.048}$ is convex	$({f_{np}}''(x^{\frac{1}{2}})/x^{\frac{1}{2}})^{2}$ is convex	${f_{np}}''(x^{\frac{1}{2}})/x^{\frac{1}{2}}$ is convex
$n=1$	$p>2$	$p>2$	$p \geq 2.06^*$	$p \geq 2.5$	$p \geq 3$
$n=2$	$p \geq 2.44^*$	$p > \frac{4}{3}$	$p \geq 1.38^*$	$p \geq \frac{5}{3}$	$p \geq 2$
$n=3$	$p \geq 2.57^*$	$p>1$	$p \geq 1.03^*$	$p \geq 1.25$	$p \geq 1.5$
$n=4$	$p \geq 2.64^*$	$p \geq 1$	$p \geq 1$	$p \geq 1$	$p \geq 1.2$
$n=5$	$p \geq 2.68^*$	$p \geq 1$	$p \geq 1$	$p \geq 1$	$p \geq 1$

of $n = 3$ and $p = 1$, the two-dimensional condition of Theorem 5.2 is not satisfied, while the one-dimensional condition mentioned in Remark 1 is satisfied.

7. INEQUALITIES RELATED TO SUMS ON TRIANGULAR LATTICE POINTS

In the rest of the paper, we focus on a variation of lattice point problems to prove (5.2) and (5.3). In lattice point theory, the well-known Gauss' (lattice point or circle) problem is the problem of counting up the number of square lattice points which are inside a circle of radius r centered at the origin [6, F1] [8]. Meanwhile, the lattice sum is the problem of determining the sums of a variety of quantities on lattice points [2, Chap. 9]. Although it is not clearly defined, the lattice sum usually targets infinite sums. Our problem may occupy an intermediate position between the two problems because we will investigate a relation between certain lattice sums of finite type and the number of triangular lattice points which are inside a circle.

Hereafter, the interval of the lattice is fixed at $d = 1$ because the inequalities (5.2) and (5.3) are not influenced by d. These inequalities can be analyzed by an appropriate approximation of a_i on Λ_1 as follows.

Remark 3. Let $\{a_i\}$ be a sequence of the values of $|\mathbf{v}|$ for $\mathbf{v} \in \Lambda_1$ sorted in increasing order. To obtain an approximation for $\{a_i\}$, let us consider the case that there are k triangular lattice points in a circle of radius $r > 1$ centered at the origin. Then, the area of the circle, πr^2, can be approximated by the total area of k identical equilateral triangles of the area $\sqrt{3}/2$. Here, if $r = a_i$, we have $k = 6i$. Thus, we have $a_i \approx b_i$, where

$$b_i = 3^{\frac{3}{4}} \cdot \pi^{-\frac{1}{2}} \cdot i^{\frac{1}{2}}.$$

Next, we consider $\{b_i\}$. Since $x^{-\frac{1}{2}}$ is decreasing,

$$\frac{1}{(i+1)^{\frac{1}{2}}} < \int_i^{i+1} \frac{1}{x^{\frac{1}{2}}} dx < \frac{1}{i^{\frac{1}{2}}}. \tag{7.1}$$

Considering that $x^{-\frac{1}{2}}$ is decreasing, and from the left-hand inequality in (7.1), we have

$$\begin{aligned} \frac{1}{i^{\frac{1}{2}}} &< -\frac{1}{(i+1)^{\frac{1}{2}}} + \frac{2}{i^{\frac{1}{2}}} \\ &< 2(i+1)^{\frac{1}{2}} - 2i^{\frac{1}{2}} - \frac{2}{(i+1)^{\frac{1}{2}}} + \frac{2}{i^{\frac{1}{2}}} \\ &= \frac{2i}{(i+1)^{\frac{1}{2}}} - \frac{2(i-1)}{i^{\frac{1}{2}}}. \end{aligned} \tag{7.2}$$

Likewise, from the right-hand inequality in (7.1), we have

$$2(i+1)^{\frac{1}{2}} - \frac{2(i-1)}{i^{\frac{1}{2}}} < \frac{3}{i^{\frac{1}{2}}}. \tag{7.3}$$

Thus, summing each of (7.2) and (7.3) multiplied by i over $i = 1, \dots, n$, we obtain

$$\sum_{i=1}^{n} \frac{b_{n+1}}{b_i} < 2n < \sum_{i=1}^{n} \frac{3b_i}{b_{n+1}}.$$

Hence, if we use the sequence $\{b_i\}$ instead of the sequence $\{a_i\}$, then (5.2) and (5.3) holds on the basis of the local inequalities (nearby $i^{\frac{1}{2}}$) obtained from the concavity of $x^{\frac{1}{2}}$.

For convenience, we also prepare the representation of the triangular lattice points by means of number theory [4, pp.110]. Let $N(n)$ denote the number of triangular lattice points placed at distance $\sqrt{n}$ from the origin. Let $N'(n) = N(n)/6$. Then, $N'(n)$ is specified by the following values:

$$\begin{cases} N'(3^a) = 1 & \text{for} \quad a \geq 0, \\ N'(p^a) = a+1 & \text{for} \quad p \equiv 1 \pmod 3, \\ N'(p^a) = 0 & \text{for} \quad p \equiv 2 \pmod 3, \quad a \text{ odd}, \\ N'(p^a) = 1 & \text{for} \quad p \equiv 2 \pmod 3, \quad a \text{ even}, \end{cases}$$

where $p \neq 3$ is prime. That is, by factorizing the natural number n into prime factors by

$$n = 3^a \cdot {p_1}^{b_1} \cdots {p_k}^{b_k} \cdot {q_1}^{c_1} \cdots {q_l}^{c_l},$$

where $p_1, \dots, p_k \equiv 1 \pmod 3$ and $q_1, \dots, q_l \equiv 2 \pmod 3$, we have

$$N'(n) = N'(3^a) \cdot N'({p_1}^{b_1}) \cdots N'({p_k}^{b_k}) \cdot N'({q_1}^{c_1}) \cdots N'({q_l}^{c_l}).$$

For example, $N'(27) = N'(3^3) = 1$, $N'(39) = N'(3^1) \cdot N'(13^1) = 2$, and $N'(49) = N'(7^2) = 3$. Figure 7.1 shows the distances of points in $\Lambda_1 \cup \{\mathbf{0}\}$ from the origin for $i \leq 9$.

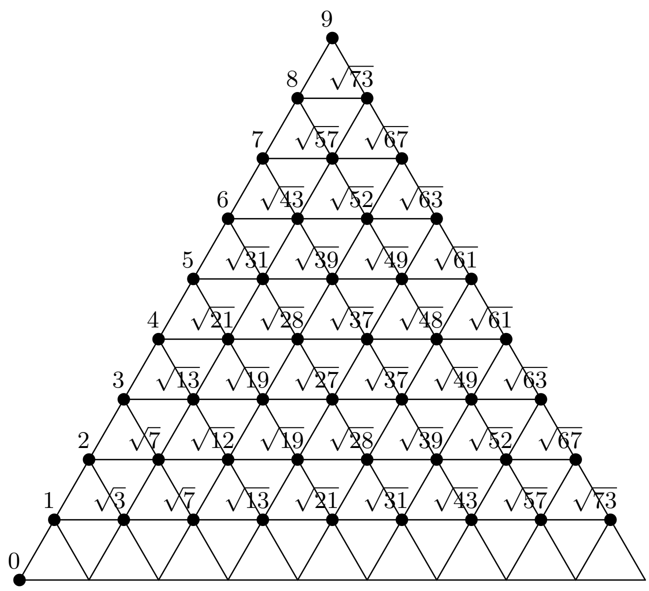

Figure 7.1: Triangular lattice points $\Lambda_1 \cup \{\mathbf{0}\}$ along with their distances from the origin ($i \leq 9$).

Theorem 7.1. *Let $r > 1$. Then, for the triangular lattice points Λ_1 defined by (2.1),*

$$\sum_{\mathbf{x}\in\Lambda_1\cap B_r}\left[\frac{1}{|\mathbf{x}|}-\frac{2}{r}\right]<0 \tag{7.4}$$

holds, where $B_r = \{\mathbf{x} : |\mathbf{x}| < r\}$. Moreover, (7.4) is equivalent to (5.2) and

$$\sum_{i=1}^{n-1}\left[\frac{1}{\sqrt{i}}-\frac{2}{\sqrt{n}}\right]N'(i)<0 \tag{7.5}$$

for $n > 1$ with $N'(n) \neq 0$.

Before presenting the proof of Theorem 7.1, we prove the following lemma.

Lemma 7.2. *Let $k \in \mathbb{N}$, $r \in \mathbb{R}$ with $k < r$, and $f \leq 0$ be convex on $\left[k-\frac{1}{2}, r\right]$. Then,*

$$\sum_{i=k}^{\lceil r\rceil-1} f(i) \leq \int_{k-\frac{1}{2}}^{r-\frac{1}{2}} f(x)dx.$$

Proof. Since $k \leq \lceil r\rceil - 1 < r$ and f is convex on $\left[k-\frac{1}{2}, r\right]$, we have

$$\sum_{i=k}^{\lceil r\rceil-1} f(i) \leq f(\lceil r\rceil-1)+\int_{k-\frac{1}{2}}^{\lceil r\rceil-\frac{3}{2}} f(x)dx \tag{7.6}$$

and

$$(r-\lceil r\rceil+1)f\left(\frac{r+\lceil r\rceil-2}{2}\right) \leq \int_{\lceil r\rceil-\frac{3}{2}}^{r-\frac{1}{2}} f(x)dx. \tag{7.7}$$

Next, again from the convexity of f and $0 \leq \frac{\lceil r\rceil-r}{r-\lceil r\rceil+2} < 1$, we have

$$f(\lceil r\rceil-1) \leq \frac{2(r-\lceil r\rceil+1)}{r-\lceil r\rceil+2} f\left(\frac{r+\lceil r\rceil-2}{2}\right)+\frac{\lceil r\rceil-r}{r-\lceil r\rceil+2} f(r).$$

Thus, considering $0 < \frac{r-\lceil r\rceil+2}{2} \leq 1$ and $f \leq 0$, we have

$$0 \leq -\frac{\lceil r\rceil-r}{2} f(r) \leq (r-\lceil r\rceil+1)f\left(\frac{r+\lceil r\rceil-2}{2}\right)-f(\lceil r\rceil-1). \tag{7.8}$$

Then, the required inequality follows by summing up (7.6) and (7.7) side by side and using (7.8). When $f \not\equiv 0$, equality holds iff $r \in \mathbb{N}$ and f is linear. □

The proof of Theorem 7.1 comprises 9 steps. As illustrated in Figure 7.2(a), dividing a circular sector at distance r from the origin into two regions, A (an equilateral triangle) and B (a circular segment), we shall prove (7.4) on $A \cup B$. By referring to the observations in Remark 3, our approach to the proof is based on simple convexity and monotonicity. The point is to use a mutual elimination between the two terms in (7.4) on B. Figure 7.2(b) illustrates points related to B, which will be explained in step 2 of the proof.

Proof of Theorem 7.1. Step 1 (equivalence of (7.4),(5.2), and (7.5)). Suppose that (5.2) is satisfied. For $r > 1$, choose n such that $a_{n+1} \geq r > a_n$. Then, considering that $\#\{\Lambda_1 \cap B_r\} = n$, we obtain

$$\sum_{\mathbf{x}\in\Lambda_1\cap B_r}\frac{2}{r} \geq \frac{2}{a_{n+1}}\#\{\Lambda_1\cap B_r\} = \frac{2n}{a_{n+1}} > \sum_{i=1}^{n}\frac{1}{a_i} = \sum_{\mathbf{x}\in\Lambda_1\cap B_r}\frac{1}{|\mathbf{x}|},$$

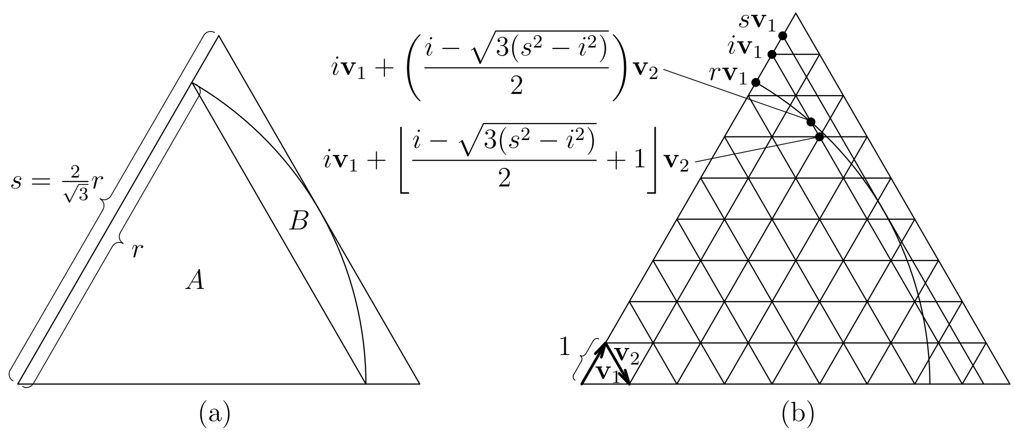

Figure 7.2: Illustration of (a) regions A and B and constants r and s, and (b) points related to B.

which gives (7.4). When (7.4) holds, clearly (7.5) holds. Suppose that (7.5) is satisfied. For each $n \in \mathbb{N}$, let $m = {a_{n+1}}^2$ and select the maximum $k \le n$ such that $a_k < a_{n+1}$. Then, from the definition of a_i and considering that $\sum_{i=1}^{m-1} N'(i) = k$, we have $N'(m) \neq 0$ and

$$\sum_{i=1}^{n} \frac{2}{a_{n+1}} - \frac{n-k}{a_{n+1}} = \frac{n+k}{\sqrt{m}} \ge \frac{2k}{\sqrt{m}} = \sum_{i=1}^{m-1} \frac{2N'(i)}{\sqrt{m}}$$
$$> \sum_{i=1}^{m-1} \frac{N'(i)}{\sqrt{i}} = \sum_{i=1}^{k} \frac{1}{a_i} = \sum_{i=1}^{n} \frac{1}{a_i} - \frac{n-k}{a_{n+1}},$$

which gives (5.2). Consequently, (5.2), (7.4), and (7.5) are all equivalent to each other.

In the following steps, we concentrate on the proof of inequality (7.4).

Step 2 (division into A and B). In (2.1), note that each $\mathbf{x} \in \Lambda_1$ is given by $\mathbf{x} = i\mathbf{v}_1 + j\mathbf{v}_2$ for some $i \in \mathbb{N}$ and $j \in \{0, \dots, i-1\}$, and $|\mathbf{x}| = [i^2 - ij + j^2]^{\frac{1}{2}}$. Let

(7.9) $$s = \frac{2}{\sqrt{3}}r.$$

Henceforth, for convenience, we will often use s as well as r. For $i \in \mathbb{N} \cap [r, s]$, let

(7.10) $$k_i = \frac{i - \sqrt{3(s^2 - i^2)}}{2} + 1.$$

Let

$$A = \{(i,j) : i = 1, \dots, \lceil r \rceil - 1, j = 0, \dots, i-1\},$$
$$B = \{(i,j) : i = \lceil r \rceil, \dots, \lceil s \rceil - 1, j = \lfloor k_i \rfloor, \dots, i - \lfloor k_i \rfloor\}.$$

Then, we have

(7.11) $$\{(i,j) : i \in \mathbb{N}, j = 0, \dots, i-1, [i^2 - ij + j^2]^{\frac{1}{2}} < r\} = A \cup B.$$

The proof of (7.11) is given as follows. If $i \in \{1, \dots, \lceil r \rceil - 1\}$, then $[i^2 - ij + j^2]^{\frac{1}{2}} < r$ holds for all $j \in \{0, \dots, i-1\}$. If $i \in \{\lceil r \rceil, \dots, \lceil s \rceil - 1\}$, then $[i^2 - ij + j^2]^{\frac{1}{2}} < r$ is equivalent to

$$k_i - 1 = \frac{i - \sqrt{3(s^2 - i^2)}}{2} < j < \frac{i + \sqrt{3(s^2 - i^2)}}{2} = i - k_i + 1$$

and $k_i - 1 < \lfloor k_i \rfloor \le j \le i - \lfloor k_i \rfloor = i - k_i + 1$. Thus, (7.11) holds. Figure 7.2(b) illustrates the relationship between k_i and the curved boundary of B.

Hence, from (7.11), it follows that

$$\sum_{\mathbf{x}\in\Lambda_1\cap B_r}\left[\frac{1}{|\mathbf{x}|}-\frac{2}{r}\right] = \sum_{i=1}^{\lceil r\rceil-1}\sum_{j=0}^{i-1}\left[\frac{1}{[i^2-ij+j^2]^{\frac{1}{2}}}-\frac{2}{r}\right] + \sum_{i=\lceil r\rceil}^{\lceil s\rceil-1}\sum_{j=\lfloor k_i\rfloor}^{i-\lfloor k_i\rfloor}\left[\frac{1}{[i^2-ij+j^2]^{\frac{1}{2}}}-\frac{2}{r}\right]. \tag{7.12}$$

Step 3 (proof for $s \le 7$). In the case of $s \le 7$, B is equal to an empty set. Thus, we treat this case independently. From the argument in step 1, considering that $r = \frac{\sqrt{3}}{2}s \le \frac{\sqrt{3}}{2}7 = \sqrt{36.75}$, it is sufficient to verify (7.5) for cases when n has the following values:

$$3, 4, 7, 9, 12, 13, 16, 19, 21, 25, 27, 28, 31, 36, \text{ and } 37. \tag{7.13}$$

For example, when $n = 21$, we have

$$\begin{aligned}&\sum_{i=1}^{20}\left[\frac{1}{\sqrt{i}}-\frac{2}{\sqrt{21}}\right]N'(i)\\ &= 1+\frac{1}{2}+\frac{1}{\sqrt{3}}+\frac{1}{3}+\frac{2}{\sqrt{7}}+\frac{1}{4}+\frac{2}{\sqrt{13}}+\frac{1}{\sqrt{12}}+\frac{2}{\sqrt{19}}-\frac{2}{\sqrt{21}}\cdot 12\\ &= 4.7188\ldots - 5.2372\ldots = -0.52\ldots < 0.\end{aligned}$$

Similarly, omitting detailed calculations, by substituting the values in (7.13) in the variable n on the left-hand side of (7.5), we obtain

$$\begin{aligned}&-0.15, -0.42, -0.19, -0.50, -0.29, -0.42, -0.49, -0.32, -0.51,\\ &-0.44, -0.41, -0.49, -0.54, -0.38, \text{ and } -0.45, \text{ respectively}.\end{aligned}$$

Step 4 (estimation of (7.12) related to A). Henceforth, assume that $s > 7$. For $0 \le x \le i$, let

$$h_i(x) = \ln\left|\frac{2x-i}{2}+[i^2-ix+x^2]^{\frac{1}{2}}\right|. \tag{7.14}$$

Then,

$$h_i{}'(x) = \frac{1}{[i^2-ix+x^2]^{\frac{1}{2}}}.$$

Here, $h_i{}'(x)$ is strictly concave on $[0, i]$, and $h_i{}'(x) = h_i{}'(i-x)$ holds. Hence, we have

$$\begin{aligned}\sum_{j=0}^{i-1}h_i{}'(j) &= \frac{1}{2}\sum_{j=0}^{i-1}\left[h_i{}'(j)+h_i{}'(j+1)\right]\\ &< \int_0^i h_i{}'(x)dx = h_i(i)-h_i(0) = \ln 3.\end{aligned}$$

Define a negative variable $\varepsilon(n)$ as

$$\varepsilon(n) = \sum_{i=1}^{n}\sum_{j=0}^{i-1}h_i{}'(j) - n\ln 3. \tag{7.15}$$

Then, as an estimation of (7.12) related to A, we obtain

$$\begin{aligned}(7.16)\qquad \sum_{i=1}^{\lceil r\rceil-1}\sum_{j=0}^{i-1}\left[h_i{}'(j)-\frac{2}{r}\right] &= \sum_{i=1}^{n}\sum_{j=0}^{i-1}h_i{}'(j)+\sum_{i=n+1}^{\lceil r\rceil-1}\sum_{j=0}^{i-1}h_i{}'(j)-\sum_{i=1}^{\lceil r\rceil-1}\sum_{j=0}^{i-1}\frac{2}{r}\\ &< \varepsilon(n)+n\ln 3+(\lceil r\rceil-n-1)\ln 3-\frac{(\lceil r\rceil-1)\lceil r\rceil}{r}\\ &= \varepsilon(n)+(\lceil r\rceil-1)\ln 3-\frac{(\lceil r\rceil-1)\lceil r\rceil}{r},\end{aligned}$$

where n is an arbitrary natural number with $1\le n<\lceil r\rceil-1$. Thus, in (7.16), the negative value $\varepsilon(n)$ can be regarded as an adjustment value. Since $\varepsilon(n)$ decreases with n, a larger n gives a better upper estimation in (7.16). However, from [1, Theorem 3A], we can find that $\sum_{j=0}^{i-1}h_i{}'(j)$ increases to $\ln 3$ with i. Thus, $\varepsilon(n)-\varepsilon(n+1)$ decreases to 0 with n. Thus, even a small n may be rather effective. In the final estimation in step 9, we shall use the fixed value $\varepsilon(5)$ as the largest allowed value for $s>7$ obtained from

$$\lceil r\rceil-1=\left\lceil\frac{\sqrt{3}}{2}s\right\rceil-1\ge\left\lceil\frac{\sqrt{3}}{2}7\right\rceil-1=6.$$

Step 5 (estimation of (7.12) related to B for $j=\lfloor k_i\rfloor,\dots,i-\lfloor k_i\rfloor$ when $\lfloor k_i\rfloor\le\frac{i}{2}$). This is the key part of the proof. Suppose that $\lfloor k_i\rfloor\le\frac{i}{2}$. Since $h_i{}'(x)$ is strictly concave on $[0,i]$, from $h_i{}'(x)=h_i{}'(i-x)$ and $\lfloor k_i\rfloor\le i-\lfloor k_i\rfloor$, we get

$$(7.17)\qquad \sum_{j=\lfloor k_i\rfloor}^{i-\lfloor k_i\rfloor}\left[h_i{}'(j)-\frac{2}{r}\right]<h_i{}'(\lfloor k_i\rfloor)+h_i(i-\lfloor k_i\rfloor)-h_i(\lfloor k_i\rfloor)-\frac{2}{r}(i-2\lfloor k_i\rfloor+1).$$

On replacing $\lfloor k_i\rfloor$ with x, the value of the terms on the right-hand side of (7.17) increases with x because by using $r>1$, $[i^2-ix+x^2]^{\frac12}\le\frac{\sqrt{3}}{2}i\le\frac{\sqrt{3}}{2}r$, and $2x-i<x\le s$, its derivative satisfies

$$\begin{aligned}h_i{}''(x)-2h_i{}'(x)+\frac{4}{r} &= -\frac{2x-i}{2[i^2-ix+x^2]^{\frac32}}-\frac{2}{[i^2-ix+x^2]^{\frac12}}+\frac{4}{r}\\ &> -\frac{s}{2}\left(\frac{2}{\sqrt{3}r}\right)^3-\frac{4}{\sqrt{3}r}+\frac{4}{r}\\ &= \frac{4}{9r^2}(-2-3\sqrt{3}r+9r)>0.\end{aligned}$$

Since $h_i{}'$ is strictly concave, for any $t\in\mathbb{R}$ with $0\le t\le i-1$, we have

$$\begin{aligned}(7.18)\qquad & h_i(i-t+1)-h_i(i-t)+h_i(t)-h_i(t-1)-[h_i{}'(t-1)+h_i{}'(t)]\\ &= 2\int_{t-1}^{t}h_i{}'(x)dx-[h_i{}'(t-1)+h_i{}'(t)]>0.\end{aligned}$$

From (7.17), (7.18), and the increase in the value of the terms on the right-hand side of (7.17), we have

$$\begin{aligned}\sum_{j=\lfloor k_i\rfloor}^{i-\lfloor k_i\rfloor}\left[h_i{}'(j)-\frac{2}{r}\right] &< h_i{}'(k_i)+h_i(i-k_i)-h_i(k_i)-\frac{2}{r}(i-2k_i+1)\\ &< -h_i{}'(k_i-1)+h_i(i-k_i+1)-h_i(k_i-1)-\frac{2}{r}(i-2k_i+1).\end{aligned}\tag{7.19}$$

From (7.10), we have

$$\begin{aligned}[i^2-i(k_i-1)+(k_i-1)^2]^{\frac{1}{2}} &= [i^2-i(i-k_i+1)+(i-k_i+1)^2]^{\frac{1}{2}}\\ &= r=\frac{\sqrt{3}}{2}s\end{aligned}\tag{7.20}$$

and

$$\frac{2(k_i-1)-i}{2}=-\frac{2(i-k_i+1)-i}{2}=-\frac{\sqrt{3(s^2-i^2)}}{2}.\tag{7.21}$$

Thus, we have $h_i{}'(k_i-1)=r^{-1}$ and

$$\begin{aligned}-h_i{}'(k_i-1)-\frac{2}{r}(i-2k_i+1) &= -\frac{1}{r}-\frac{2}{r}\left(\sqrt{3(s^2-i^2)}-1\right)\\ &= \frac{1}{r}-\frac{4\sqrt{s^2-i^2}}{s}.\end{aligned}\tag{7.22}$$

In (7.22), the calculation $-1/r+2/r=1/r$ corresponds to the mutual elimination stated before the proof. Thus, by substituting (7.22) and (7.14) in the right-hand side of (7.19), and then by using (7.20) and (7.21), the inequality (7.19) is rewritten as

$$\sum_{j=\lfloor k_i\rfloor}^{i-\lfloor k_i\rfloor}\left[h_i{}'(j)-\frac{2}{r}\right]<\frac{1}{r}+2\ln\left|\frac{i}{s-\sqrt{s^2-i^2}}\right|-\frac{4\sqrt{s^2-i^2}}{s}.\tag{7.23}$$

In fact, (7.23) also holds in the case of $\lfloor k_i\rfloor>\frac{i}{2}$. This will be proved in step 7.

Step 6 (a function for further estimations of (7.12) related to B for $i=\lceil r\rceil,\dots,\lceil s\rceil-1$). In this step, we present some properties related to the variable term of the right-hand side of (7.23). For $0\le x\le s$, let

$$f_1(x)=2x\ln\left|\frac{x}{s-\sqrt{s^2-x^2}}\right|-\frac{2x\sqrt{s^2-x^2}}{s}.\tag{7.24}$$

Then,

$$f_1{}'(x)=2\ln\left|\frac{x}{s-\sqrt{s^2-x^2}}\right|-\frac{4\sqrt{s^2-x^2}}{s}$$

is the variable term on the right-hand side of (7.23). Here, for $\frac{s}{\sqrt{2}}<x<s$,

$$f_1{}''(x)=\frac{2}{s}\left[\frac{x}{\sqrt{s^2-x^2}}-\frac{\sqrt{s^2-x^2}}{x}\right]=\frac{2(2x^2-s^2)}{sx\sqrt{s^2-x^2}}>0.$$

Further, for all $0<x<s$,

$$f_1{}'''(x)=\frac{2}{s}\left[\frac{x}{\sqrt{s^2-x^2}}+\frac{\sqrt{s^2-x^2}}{x^2}+\frac{1}{\sqrt{s^2-x^2}}+\frac{x^2}{(s^2-x^2)^{\frac{3}{2}}}\right]>0.$$

Hence, f_1 is strictly convex on $\left[\frac{s}{\sqrt{2}}, s\right]$, and $f_1{}'$ is strictly convex on $[0, s]$ and increasing on $\left[\frac{s}{\sqrt{2}}, s\right]$. Since $f_1{}'(s) = 0$, $f_1{}'(x) \le 0$ also holds on $\left[\frac{s}{\sqrt{2}}, s\right]$.

Step 7 (proof of (7.23) when $\lfloor k_i \rfloor > \frac{i}{2}$). Suppose that $\lfloor k_i \rfloor > \frac{i}{2}$. Since both $\lfloor k_i \rfloor$ and i are natural numbers, $\lfloor k_i \rfloor \ge \frac{i+1}{2}$ holds, hence,

$$\frac{i - \sqrt{3(s^2 - i^2)}}{2} + 1 = k_i \ge \lfloor k_i \rfloor \ge \frac{i+1}{2}.$$

Thus, $i^2 \ge s^2 - \frac{1}{3}$. Since $f_1{}'$ is increasing on $\left[\frac{s}{\sqrt{2}}, s\right]$, by substituting $s = \frac{x+1}{\sqrt{3(x-1)}}$, we get

$$\begin{aligned} f_1{}'(i) + \frac{2}{\sqrt{3}s} &\ge f_1{}'\left(\sqrt{s^2 - \frac{1}{3}}\right) + \frac{2}{\sqrt{3}s} \\ &= \ln\left|\frac{s + \frac{1}{\sqrt{3}}}{s - \frac{1}{\sqrt{3}}}\right| - \frac{2}{\sqrt{3}s} \\ &= \frac{(x+1)\ln x - 2(x-1)}{x+1} > 0, \end{aligned}$$

where the last inequality holds by the convexity of $(x+1)\ln x$. The left-hand side of (7.23) can be naturally defined to be equal to 0 when $\lfloor k_i \rfloor > \frac{i}{2}$. Hence, (7.23) holds for all $i \in [r, s] \cap \mathbb{N}$.

Step 8 (estimation of two functional values defined in step 6). We derive two estimations for $f_1(x)$. The first estimation is made at $x = \lceil r \rceil - \frac{1}{2}$. Since f_1 is strictly convex on $\left[\frac{s}{\sqrt{2}}, s\right]$ and the interval $\left[\min\{\lceil r \rceil - \frac{1}{2}, r\}, \max\{\lceil r \rceil - \frac{1}{2}, r\}\right]$ is contained in $\left[\frac{s}{\sqrt{2}}, s\right]$, we have

$$\begin{aligned} f_1\left(\lceil r \rceil - \frac{1}{2}\right) &> f_1(r) + f_1{}'(r)\left(\lceil r \rceil - r - \frac{1}{2}\right) \\ &= r\ln 3 - r + (\ln 3 - 2)\left(\lceil r \rceil - r - \frac{1}{2}\right). \end{aligned} \tag{7.25}$$

The second estimation is made at $x = s - \frac{1}{2}$ as follows:

$$f_1\left(s - \frac{1}{2}\right) < \frac{2\left(s - \frac{1}{2}\right)^{\frac{1}{2}}}{3s}. \tag{7.26}$$

The proof of (7.26) is given as follows. Let

$$x^2 = \frac{s - \frac{1}{2}}{s - \sqrt{s^2 - \left(s - \frac{1}{2}\right)^2}},$$

where $x > 1$. Then, we get

$$4(x^2-1)^2 s^2 - 4(x^4 - x^2 + 1)s + (x^4 + 1) = 0,$$

and thus,

$$s = \frac{x^4 - x^2 + 1 + \sqrt{(x^4 - x^2 + 1)^2 - (x^2-1)^2(x^4+1)}}{2(x^2-1)^2} = \frac{x^4+1}{2(x^2-1)^2}.$$

This equality can be rewritten as follows:

$$\left(s - \frac{1}{4}\right)^{\frac{1}{2}} = \frac{x^2+1}{2(x^2-1)}, \qquad \left(s - \frac{1}{2}\right)^{\frac{1}{2}} = \frac{x}{x^2-1}.$$

Hence, for $x > 1$, we can write

$$f_1\left(s-\frac{1}{2}\right) - \frac{2\left(s-\frac{1}{2}\right)^{\frac{1}{2}}}{3s} = 2\left(s-\frac{1}{2}\right)g(x),$$

where

$$g(x) = 2\ln x - \frac{x^4-1}{x^4+1} - \frac{2(x^2-1)^3}{3x(x^4+1)}.$$

Here, we obtain $g'(x) < 0$ for $x > 1$ from the following straightforward calculation.

$$\begin{aligned} g'(x) &= \frac{2}{x} + \frac{4x^3(x^4-1)}{(x^4+1)^2} - \frac{4x^3}{x^4+1} + \frac{2(5x^4+1)(x^2-1)^3}{3x^2(x^4+1)^2} - \frac{4(x^2-1)^2}{(x^4+1)} \\ &= \frac{2}{3x^2(x^4+1)^2}\big[3x(x^4+1)^2 - 12x^5 \\ &\qquad + (5x^4+1)(x^2-1)^3 - 6x^2(x^2-1)^2(x^4+1)\big] \\ &= -\frac{2(x^2-1)(x-1)^3}{3x^2(x^4+1)^2}(x^5+x^3+x^2+1). \end{aligned}$$

Thus, since $g(1) = 0$, we have $g(x) < 0$ for $x > 1$. Hence, (7.26) holds.

Step 9 (total estimation of (7.12) for $s > 7$). Now we present the final estimation. From the assumption $s > 7$, we have $\lceil s\rceil > \lceil r\rceil \geq 7$. Substituting $n = 5$ in (7.15), we define

$$\begin{aligned} \varepsilon(5) &= 1 + \frac{1}{2} + \frac{1}{\sqrt{3}} + \frac{1}{3} + \frac{2}{\sqrt{7}} + \frac{1}{4} \\ &\qquad + \frac{2}{\sqrt{13}} + \frac{1}{\sqrt{12}} + \frac{1}{5} + \frac{2}{\sqrt{21}} + \frac{2}{\sqrt{19}} - 5\ln 3 \\ &= -0.1378\ldots. \end{aligned}$$

Since ${f_1}'$ is convex on $\left[\frac{s}{\sqrt{2}}, s\right]$, from Lemma 7.2, (7.25), and (7.26), we have

$$\begin{aligned} \sum_{i=\lceil r\rceil}^{\lceil s\rceil-1} {f_1}'(i) &\leq \int_{\lceil r\rceil-\frac{1}{2}}^{s-\frac{1}{2}} {f_1}'(x)dx \\ &< \frac{2\left(s-\frac{1}{2}\right)^{\frac{1}{2}}}{3s} - r\ln 3 + r - (\ln 3 - 2)\left(\lceil r\rceil - r - \frac{1}{2}\right). \end{aligned} \tag{7.27}$$

Finally, from (7.12), (7.16), (7.23), and (7.27), we obtain

$$\begin{aligned} \sum_{\mathbf{x}\in\Lambda_1\cap B_r}\left[\frac{1}{|\mathbf{x}|} - \frac{2}{r}\right] &< \varepsilon(5) + (\lceil r\rceil - 1)\ln 3 - \frac{(\lceil r\rceil-1)\lceil r\rceil}{r} + \frac{(\lceil s\rceil - \lceil r\rceil)}{r} + \sum_{i=\lceil r\rceil}^{\lceil s\rceil-1} {f_1}'(i) \\ &< -\frac{(\lceil r\rceil - r)^2}{r} + \frac{\lceil s\rceil}{r} - \frac{1}{2}\ln 3 - 1 + \frac{2\left(s-\frac{1}{2}\right)^{\frac{1}{2}}}{3s} + \varepsilon(5) \\ &\leq \frac{2}{\sqrt{3}s} + \frac{2}{\sqrt{3}} - \frac{1}{2}\ln 3 - 1 + \frac{2\left(s-\frac{1}{2}\right)^{\frac{1}{2}}}{3s} + \varepsilon(5) \\ &< \frac{2}{7\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{1}{2}\ln 3 - 1 + \frac{2(7-\frac{1}{2})^{\frac{1}{2}}}{3\cdot 7} + \varepsilon(5) \\ &= -0.1246\ldots \\ &< 0. \end{aligned}$$

This concludes the proof of Theorem 7.1. □

In Gauss's problem, let $G(r)$ denote the number of triangular lattice points lying truly inside a circle of radius r centered at the origin. Then, since $G(r) = 6\sum_{\mathbf{x}\in\Lambda_1\cap B_r} +1$, from Theorem 7.1, we obtain the relation between Gauss's problem and the lattice sum of finite type, given as follows:

$$G(r) > 3r\sum_{\mathbf{x}\in\Lambda_1\cap B_r}\frac{1}{|\mathbf{x}|}+1\left(=3r\sum_{i=1}^{\lceil r^2\rceil-1}\frac{N'(i)}{\sqrt{i}}+1\right).$$

If we assume that $G(r)$ also contains the triangular lattice points that lie just on the circle, then by redefining $B_r = \{\mathbf{x} : |\mathbf{x}| \le r\}$, we obtain the same inequality.

Similarly, we obtain Theorem 7.3. The logic of the proof is mainly same as that of the proof of Theorem 7.1. In the proof of Theorem 7.3, we omit the proofs for some increasing or convex properties of functions, which can be proved similar to the manner followed in Theorem 7.1.

Theorem 7.3. *Let $r > \sqrt{7}$. Then, for the triangular lattice points Λ_1 defined by (2.1),*

$$\sum_{\mathbf{x}\in\Lambda_1\cap B_r}\left[\frac{1}{|\mathbf{x}|}-\frac{3}{r^2}|\mathbf{x}|\right]<0 \tag{7.28}$$

holds, where $B_r = \{\mathbf{x} : |\mathbf{x}| < r\}$. Moreover, (7.28) is equivalent to (5.3) for $n \ge 4$ and

$$\sum_{i=1}^{n-1}\left[\frac{1}{\sqrt{i}}-\frac{3\sqrt{i}}{n}\right]N'(i)<0 \tag{7.29}$$

for $n > 7$ with $N'(n) \neq 0$.

Proof. By referring to Theorem 7.1, the proof of the equivalence of (7.28), (5.3), and (7.29) can be obtained in the same manner as that followed in (7.4), (5.2), and (7.5), and the estimation can be carried out on each of the regions A and B. In the following steps, we estimate the inequality (7.28).

Step 1 (estimation of (7.28) related to A). Let $s > 7$. For $0 \le x \le i$, let $h_i(x)$ be defined by (7.14) and

$$l_i(x)=\frac{2x-i}{4}[i^2-ix+x^2]^{\frac{1}{2}}+\frac{3i^2}{8}\ln\left|\frac{2x-i}{2}+[i^2-ix+x^2]^{\frac{1}{2}}\right|.$$

Then,

$$l_i{}'(x)=[i^2-ix+x^2]^{\frac{1}{2}}.$$

Here, $l_i{}'(x)$ is strictly convex on $[0,i]$, and $l_i{}'(x) = l_i{}'(i-x)$ holds. Hence, we have

$$\begin{aligned}\sum_{j=0}^{i-1}l_i{}'(j)&=\frac{1}{2}\sum_{j=0}^{i-1}\left[l_i{}'(j)+l_i{}'(j+1)\right]>\int_0^i l_i{}'(x)dx\\&=l_i(i)-l_i(0)=\left(\frac{1}{2}+\frac{3}{8}\ln 3\right)i^2.\end{aligned}$$

For $1 \le n < \lceil r \rceil - 1$, let $\varepsilon(n)$ be defined by (7.15). Then, we obtain the estimation on A:

$$\begin{aligned}
(7.30)\qquad &\sum_{i=1}^{\lceil r \rceil -1} \sum_{j=0}^{i-1} \left[h_i{}'(j) - \frac{3}{r^2} l_i{}'(j) \right] \\
&= \sum_{i=1}^{n} \sum_{j=0}^{i-1} h_i{}'(j) + \sum_{i=n+1}^{\lceil r \rceil -1} \sum_{j=0}^{i-1} h_i{}'(j) - \sum_{i=1}^{\lceil r \rceil -1} \sum_{j=0}^{i-1} \frac{3}{r^2} l_i{}'(j) \\
&< \varepsilon(n) + n \ln 3 + (\lceil r \rceil - n - 1) \ln 3 - \sum_{i=1}^{\lceil r \rceil -1} \frac{3}{r^2} \left(\frac{1}{2} + \frac{3}{8} \ln 3 \right) i^2 \\
&= \varepsilon(n) + (\lceil r \rceil - 1) \ln 3 - \frac{1}{4r^2} \left(1 + \frac{3}{4} \ln 3 \right) (2\lceil r \rceil^3 - 3\lceil r \rceil^2 + \lceil r \rceil).
\end{aligned}$$

Step 2 (estimation of (7.28) related to B for $j = \lfloor k_i \rfloor, \dots, i - \lfloor k_i \rfloor$). Suppose that $\lfloor k_i \rfloor \le \frac{i}{2}$. Since $h_i{}'$ is strictly concave, $l_i{}'$ is strictly convex, and $\lfloor k_i \rfloor \le i - \lfloor k_i \rfloor$, we get

$$\begin{aligned}
(7.31)\qquad \sum_{j=\lfloor k_i \rfloor}^{i-\lfloor k_i \rfloor} \left[h_i{}'(j) - \frac{3}{r^2} l_i{}'(j) \right] &< h_i{}'(\lfloor k_i \rfloor) + h_i(i - \lfloor k_i \rfloor) - h_i(\lfloor k_i \rfloor) \\
&\quad - \frac{3}{r^2} \big(l_i{}'(\lfloor k_i \rfloor) + l_i(i - \lfloor k_i \rfloor) - l_i(\lfloor k_i \rfloor) \big).
\end{aligned}$$

If $\lfloor k_i \rfloor$ on the right-hand side of (7.31) is replaced with x, the value of the term on the right-hand side of (7.31) increases with x. Moreover, again since $l_i{}'$ is strictly convex, for any $t \in \mathbb{R}$ with $0 \le t \le i - 1$, we have

$$\begin{aligned}
(7.32)\qquad &l_i(i - t + 1) - l_i(i - t) + l_i(t) - l_i(t - 1) - [l_i{}'(t - 1) + l_i{}'(t)] \\
&= 2 \int_{t-1}^{t} l_i{}'(x) dx - [l_i{}'(t - 1) + l_i{}'(t)] < 0.
\end{aligned}$$

From (7.31) and (7.32) and the increase in the value of the terms on the right-hand side of (7.31), we get

$$\begin{aligned}
\sum_{j=\lfloor k_i \rfloor}^{i-\lfloor k_i \rfloor} \left[h_i{}'(j) - \frac{3}{r^2} l_i{}'(j) \right] &< -h_i{}'(k_i - 1) + h_i(i - k_i + 1) - h_i(k_i - 1) \\
&\quad - \frac{3}{r^2} \big(-l_i{}'(k_i - 1) + l_i(i - k_i + 1) - l_i(k_i - 1) \big).
\end{aligned}$$

Thus, from (7.10), (7.20), (7.21), and the definition of l_i and $l_i{}'$, we obtain

$$\begin{aligned}
(7.33)\qquad &\sum_{j=\lfloor k_i \rfloor}^{i-\lfloor k_i \rfloor} \left[h_i{}'(j) - \frac{3}{r^2} l_i{}'(j) \right] \\
&< \frac{4}{\sqrt{3}s} + 2 \ln \left| \frac{i}{s - \sqrt{s^2 - i^2}} \right| - \frac{3\sqrt{s^2 - i^2}}{s} - \frac{3i^2}{s^2} \ln \left| \frac{i}{s - \sqrt{s^2 - i^2}} \right|.
\end{aligned}$$

In fact, (7.33) also holds in the case of $\lfloor k_i \rfloor > \frac{i}{2}$; this is similar to step 7 of Theorem 7.1.

Next, we consider the properties of the right-hand side of (7.33). For $0 \le x \le s$, let

$$f_2(x) = 2x \ln \left| \frac{x}{s - \sqrt{s^2 - x^2}} \right| - \frac{x\sqrt{s^2 - x^2}}{s} - \frac{x^3}{s^2} \ln \left| \frac{x}{s - \sqrt{s^2 - x^2}} \right|.$$

Then,

$$f_2{}'(x) = 2\ln\left|\frac{x}{s-\sqrt{s^2-x^2}}\right| - \frac{3\sqrt{s^2-x^2}}{s} - \frac{3x^2}{s^2}\ln\left|\frac{x}{s-\sqrt{s^2-x^2}}\right|.$$

It can be verified that f_2 is strictly convex on $\left[\frac{s}{\sqrt{2}}, s\right]$ and $f_2{}'$ is strictly convex on $[0, s]$ and increasing on $\left[\frac{s}{\sqrt{2}}, s\right]$. Since $f_2{}'(s) = 0$, $f_2{}'(x) \le 0$ also holds on $\left[\frac{s}{\sqrt{2}}, s\right]$.

Moreover, we have $f_2(x) \le 2f_1(x)$, where f_1 is defined by (7.24). To obtain the proof of this inequality, let

$$y^{\frac{1}{2}} = \frac{x}{s-\sqrt{s^2-x^2}},$$

where $y \ge 1$ because $0 \le x \le s$. Then, we can write

$$\begin{aligned} 2f_1(x) - f_2(x) &= 2x\ln\left|\frac{x}{s-\sqrt{s^2-x^2}}\right| - \frac{3x\sqrt{s^2-x^2}}{s} + \frac{x^3}{s^2}\ln\left|\frac{x}{s-\sqrt{s^2-x^2}}\right| \\ &= \ln y - 3\left(\frac{y-1}{y+1}\right) + 2\frac{y}{(y+1)^2}\ln y \\ &= \frac{(y^2+4y+1)^2}{(y+1)^2}\left|\ln y - \frac{3(y^2-1)}{(y^2+4y+1)^2}\right|. \end{aligned}$$

Here, assuming

$$g(x) = \ln x - \frac{3(x^2-1)}{(x^2+4x+1)^2},$$

we get

$$g'(x) = \frac{(x-1)^4}{x(x^2+4x+1)^2}.$$

Hence, g is increasing on $x \ge 1$ with $g(1) = 0$. Thus, $2f_1(x) - f_2(x) \ge 0$ for $0 \le x \le s$.

Considering that the function f_2 is strictly convex on $\left[\frac{s}{\sqrt{2}}, s\right]$ and the interval $\left[\min\{\lceil r\rceil - \frac{1}{2}, r\}, \max\{\lceil r\rceil - \frac{1}{2}, r\}\right]$ is contained in $\left[\frac{s}{\sqrt{2}}, s\right]$, we have

$$\begin{aligned} f_2\left(\lceil r\rceil - \frac{1}{2}\right) &> f_2(r) + f_2{}'(r)\left(\lceil r\rceil - r - \frac{1}{2}\right) \\ &= \frac{5}{8}r\ln 3 - \frac{1}{2}r - \left(\frac{1}{8}\ln 3 + \frac{3}{2}\right)\left(\lceil r\rceil - r - \frac{1}{2}\right). \end{aligned} \tag{7.34}$$

In addition, from $f_2(x) \le 2f_1(x)$ and (7.26), we have

$$f_2\left(s - \frac{1}{2}\right) < \frac{4\left(s-\frac{1}{2}\right)^{\frac{1}{2}}}{3s}. \tag{7.35}$$

Step 3 (total estimation of (7.28) for $s \ge 21$*).* Let $s \ge 21$. Since $f_2{}'$ is convex on $\left[\frac{s}{\sqrt{2}}, s\right]$, from Lemma 7.2, (7.34), and (7.35), we have

$$\begin{aligned} \sum_{i=\lceil r\rceil}^{\lceil s\rceil - 1} f_2{}'(i) &\le \int_{\lceil r\rceil - \frac{1}{2}}^{s-\frac{1}{2}} f_2{}'(x)dx \\ &< \frac{4\left(s-\frac{1}{2}\right)^{\frac{1}{2}}}{3s} - \frac{5}{8}r\ln 3 + \frac{1}{2}r + \left(\frac{1}{8}\ln 3 + \frac{3}{2}\right)\left(\lceil r\rceil - r - \frac{1}{2}\right). \end{aligned} \tag{7.36}$$

Thus, from (7.11), (7.30), (7.33), and (7.36), for $s \geq 21$, we obtain

$$\sum_{\mathbf{x}\in\Lambda_1\cap B_r}\left[\frac{1}{|\mathbf{x}|}-\frac{3}{r^2}|\mathbf{x}|\right] < g(r)+\varepsilon(5)+\frac{2\lceil s\rceil}{r}+\frac{4\left(s-\frac{1}{2}\right)^{\frac{1}{2}}}{3s}, \tag{7.37}$$

where

$$g(r) = (\lceil r\rceil - 1)\ln 3 - \frac{1}{4r^2}\left(1+\frac{3}{4}\ln 3\right)(2\lceil r\rceil^3 - 3\lceil r\rceil^2 + \lceil r\rceil) - \frac{2\lceil r\rceil}{r}$$
$$- \frac{5}{8}r\ln 3 + \frac{1}{2}r + \left(\frac{1}{8}\ln 3 + \frac{3}{2}\right)\left(\lceil r\rceil - r - \frac{1}{2}\right).$$

Here, by using the substitution $\alpha = \lceil r\rceil - r$, we have

$$\begin{aligned}
g(r) &= \frac{\ln 3}{16r^2}(-8r^2 - 3r - 18\lceil r\rceil\alpha^2 + 18\lceil r\rceil\alpha + 12\alpha^3 - 9\alpha^2 - 3\alpha) \\
&\quad + \frac{1}{4r^2}(-8r^2 - r - 6\lceil r\rceil\alpha^2 + 4\alpha^3 + 5\alpha^2 - \alpha) \\
&< \frac{\ln 3}{16r^2}(-8r^2 - 2r) + \frac{\ln 3}{16r^2}(-r - 18\lceil r\rceil\alpha^2 + 18\lceil r\rceil\alpha + 12\alpha^3 - 9\alpha^2 - 3\alpha) \\
&\quad + \frac{1}{4r^2}(-8r^2 - r) + \frac{3\ln 3}{16r^2}(-6\lceil r\rceil\alpha^2 - 2\lceil r\rceil\alpha + 4\alpha^3 + 5\alpha^2 - \alpha) \\
&= \ln 3\left(-\frac{1}{2} - \frac{1}{8r}\right) - 2 - \frac{1}{4r} + \frac{\ln 3}{16r^2}\left(-\lceil r\rceil(6\alpha - 1)^2 + 24\alpha^3 + 6\alpha^2 - 5\alpha\right) \\
&\leq \ln 3\left(-\frac{1}{2} - \frac{1}{8r}\right) - 2 - \frac{1}{4r} + \frac{\ln 3}{16r^2}\left(-(6\alpha - 1)^2 + 24\alpha^3 + 6\alpha^2 - 5\alpha\right) \\
&= \ln 3\left(-\frac{1}{2} - \frac{1}{8r}\right) - 2 - \frac{1}{4r} + \frac{\ln 3}{16r^2}(\alpha - 1)\left(24\left(\alpha - \frac{1}{8}\right)^2 + \frac{5}{8}\right) \\
&< \ln 3\left(-\frac{1}{2} - \frac{1}{8r}\right) - 2 - \frac{1}{4r},
\end{aligned} \tag{7.38}$$

where the first inequality holds since $-\frac{1}{4} < -\frac{3\ln 3}{16} = -0.206\ldots$ and

$$\begin{aligned}
6\lceil r\rceil\alpha^2 + 2\lceil r\rceil\alpha - 4\alpha^3 - 5\alpha^2 + \alpha &\geq 6\alpha^2 + 2\alpha - 4\alpha^3 - 5\alpha^2 + \alpha \\
&= \alpha(4\alpha + 3)(1 - \alpha) \geq 0.
\end{aligned}$$

Finally, from (7.37), (7.38), and $\varepsilon(5) = -0.1378\ldots$, for $s \geq 21$, we obtain

$$\begin{aligned}
&\sum_{\mathbf{x}\in\Lambda_1\cap B_r}\left[\frac{1}{|\mathbf{x}|} - \frac{3|\mathbf{x}|}{r^2}\right] \\
&< -\ln 3\left(\frac{1}{2} + \frac{1}{4\sqrt{3}s}\right) - 2 - \frac{1}{2\sqrt{3}s} + \varepsilon(5) + \frac{4}{\sqrt{3}} + \frac{4}{\sqrt{3}s} + \frac{4\left(s - \frac{1}{2}\right)^{\frac{1}{2}}}{3s} \\
&= -2 - \frac{1}{2}\ln 3 + \frac{4}{\sqrt{3}} + \frac{1}{\sqrt{3}}\left(\frac{7}{2} - \frac{1}{4}\ln 3\right)\frac{1}{s} + \varepsilon(5) + \frac{4\left(s - \frac{1}{2}\right)^{\frac{1}{2}}}{3s} \\
&\leq -2 - \frac{1}{2}\ln 3 + \frac{4}{\sqrt{3}} + \frac{1}{\sqrt{3}}\left(\frac{7}{2} - \frac{1}{4}\ln 3\right)\frac{1}{21} + \varepsilon(5) + \frac{4(21 - \frac{1}{2})^{\frac{1}{2}}}{3\cdot 21} \\
&= -0.0016\ldots \\
&< 0.
\end{aligned}$$

Step 4 (proof for $s < 21$). For $s < 21$, it is straightforward to check the required inequality (7.29) by carrying out direct calculations. Using the same argument as in step 1 of Theorem 7.1 and considering that

$$r = \frac{\sqrt{3}}{2}s \leq \frac{\sqrt{3}}{2}21 = \sqrt{330.75},$$

it is sufficient to verify (7.29) for $n \in \mathbb{N}$ satisfying $7 < n \leq 331$ and $N'(n) \neq 0$. Let

$$v(n) = 10 \cdot \sum_{i=1}^{n-1} \left[\frac{3\sqrt{i}}{n} - \frac{1}{\sqrt{i}} \right] N'(i).$$

Table 7.1 shows the approximations of the calculated values of $v(n)$. From this result, (7.29) holds for each $7 < n \leq 331$ with $N'(n) \neq 0$.

This concludes the proof of Theorem 7.3. □

n	N'	v	n	N'	v	n	N'	v	n	N'	v	n	N'	v	n	N'	v
1	1	—	48	1	1.7	103	2	4.8	163	2	4.8	225	1	7.2	291	2	4.9
3	1	0	49	3	3.0	108	1	2.9	169	3	2.4	228	2	6.1	292	2	6.6
4	1	4.7	52	2	6.3	109	2	3.7	171	2	5.2	229	2	8.0	300	1	3.3
7	2	-0.5	57	2	3.6	111	2	5.3	172	2	7.3	237	2	4.2	301	4	3.8
9	1	5.1	61	2	2.7	112	2	7.9	175	2	7.6	241	2	3.8	304	2	6.3
12	1	0.9	63	2	4.8	117	2	5.9	181	2	5.1	243	1	4.8	307	2	6.5
13	2	3.5	64	1	8.2	121	1	5.2	183	2	6.3	244	2	5.3	309	2	7.4
16	1	4.3	67	2	6.2	124	2	3.8	189	2	4.0	247	4	5.6	313	2	6.9
19	2	1.1	73	2	2.4	127	2	4.2	192	1	4.3	252	2	6.8	316	2	7.1
21	2	4.8	75	1	4.3	129	2	5.6	193	2	4.9	256	1	6.2	324	1	4.0
25	1	3.1	76	2	5.3	133	4	4.9	196	3	5.2	259	4	5.2	325	2	4.4
27	1	2.6	79	2	5.7	139	2	5.5	199	2	6.8	268	2	3.5	327	2	5.3
28	2	4.3	81	1	7.4	144	1	3.8	201	2	7.9	271	2	3.8	331	2	4.9
31	2	5.0	84	2	5.6	147	3	2.6	208	2	4.8	273	4	4.8			
36	1	1.7	91	4	1.1	148	2	6.6	211	2	5.1	277	2	6.7			
37	2	3.2	93	2	7.0	151	2	6.8	217	4	2.9	279	2	7.6			
39	2	5.8	97	2	6.0	156	2	5.2	219	2	6.7	283	2	7.1			
43	2	4.4	100	1	6.4	157	2	7.4	223	2	6.2	289	1	5.2			

Table 7.1: List of values $n \in \mathbb{N}$, $N'(n)$, and $v(n)$ restricted to $1 \leq n \leq 331$ and $N'(n) \neq 0$.

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