**Journal of Inequalities in Pure and Applied Mathematics**

# ON YOUNG'S INEQUALITY

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volume 7, issue 5, article 164,
2006.

*Received 24 October, 2006;
accepted 20 November, 2006.*

*Communicated by:* *P.S. Bullen*

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ISSN (electronic): 1443-5756
289-06

## Abstract

In this note we offer two short proofs of Young's inequality and prove its reverse.

*2000 Mathematics Subject Classification:* 26D15.
*Key words:* Young's inequality, convex function.

The famous Young's inequality states that

**Theorem 1.** *If $f : [0, A] \to \mathbb{R}$ is continuous and a strictly increasing function satisfying $f(0) = 0$ then for every positive $0 < a \leq A$ and $0 < b \leq f(A)$,*

$$\int_0^a f(t)dt + \int_0^b f^{-1}(t)dt \geq ab \tag{1}$$

*holds with equality if and only if $b = f(a)$.*

This theorem has an easy geometric interpretation. It is so easy that some monographs simply refer to it omitting the proof ([5]) or give the idea of a proof disregarding the details ([4]). Some authors make additional assumptions to simplify the proof ([3]) while some others obtain the Young inequality as a special case of quite complicated theorems ([2]). An overview of available proofs and a complete proof of Theorem 1 can be found in [1]. In this note we offer two simple proofs of Young's inequality and present its reverse version.

The proofs are based on the following

**Lemma 2.** *If $f$ satisfies the assumptions of Theorem 1, then*

$$\int_0^a f(t)dt + \int_0^{f(a)} f^{-1}(t)dt = af(a). \tag{2}$$

The graph of $f$ divides the rectangle with diagonal $(0,0)-(a,f(a))$ into lower and upper parts, and the integrals represent their respective areas. Of course this is just a geometric idea, so at the end of this note we give the formal proof of Lemma 2 (another proof can be found in [1]).

The first proof is based on the fact that the graph of a convex function lies above its supporting line.

*First proof of Theorem 1.* As $f$ is strictly increasing its antiderivative is strictly convex. Hence for every $0 < c \neq a < A$ we have

$$\int_0^a f(t)dt > \int_0^c f(t)dt + f(c)(a-c).$$

In particular for $c = f^{-1}(b)$ we obtain

$$\int_0^a f(t)dt > \int_0^{f^{-1}(b)} f(t)dt + ab - bf^{-1}(b).$$

Applying now Lemma 2 to the function $f^{-1}$ we see that the right hand side of the last inequality equals $ab - \int_0^b f^{-1}(t)dt$ and the proof is complete. $\square$

The second proof uses the Mean Value Theorem.

*Second proof of Theorem 1.* Since $f$ is strictly decreasing, we have

$$f(a) < \frac{\int_0^{f^{-1}(b)} f(t)dt - \int_0^a f(t)dt}{f^{-1}(b) - a} < f(f^{-1}(b)) = b \tag{3}$$

if $a < f^{-1}(b)$ and reverse inequalities if $a > f^{-1}(b)$.

Replacing $\int_0^{f^{-1}(b)} f(t)dt$ by $bf^{-1}(b) - \int_0^b f^{-1}(t)dt$ and simplifying we obtain in both cases

$$ab < \int_0^a f(t)dt + \int_0^b f^{-1}(t)dt < af(a) + f^{-1}(b)(b - f(a)).$$

□

**Theorem 3 (Reverse Young's Inequality).** *Under the assumptions of Theorem 1, the inequality*

$$\min\left\{1, \frac{b}{f(a)}\right\} \int_0^a f(t)dt + \min\left\{1, \frac{a}{f^{-1}(b)}\right\} \int_0^b f^{-1}(t)dt \le ab$$

*holds with equality if and only if $b = f(a)$.*

*Proof.* The function $F(x) = \int_0^x f(t)dt$ is strictly convex.

If $a < f^{-1}(b)$, this yields

$$\begin{aligned} F(a) &< \frac{a}{f^{-1}(b)} F(f^{-1}(b)) \\ &= \frac{a}{f^{-1}(b)} \left[ bf^{-1}(b) - \int_0^b f^{-1}(t)dt \right] \end{aligned}$$

$$= ab - \frac{a}{f^{-1}(b)} \int_0^b f^{-1}(t)dt,$$

so

$$\int_0^a f(t)dt + \frac{a}{f^{-1}(b)} \int_0^b f^{-1}(t)dt < ab.$$

If $a > f^{-1}(b)$, we apply the same reasoning to the function $G(x) = \int_0^x f^{-1}(t)dt$, obtaining

$$\frac{b}{f(a)} \int_0^a f(t)dt + \int_0^b f^{-1}(t)dt < ab.$$

□

*Proof of Lemma 2.* Let $0 = x_0 < x_1 < \cdots < x_n = a$ be a partition of the interval $[0, a]$ and let $y_i = f(x_i)$ and $\Delta x_i = x_i - x_{i-1}$.

$\underline{S}(f, \mathbf{x}) = \sum_{i=1}^n f(x_{i-1})\Delta x_i$ and $\overline{S}(f, \mathbf{x}) = \sum_{i=1}^n f(x_i)\Delta x_i$ are lower and upper Riemann sums for $f$ corresponding to the partition $\mathbf{x}$.

For $\varepsilon > 0$ select $\mathbf{x}$ in such a way that $\Delta y_i < \varepsilon/a$. Then

$$\overline{S}(f, \mathbf{x}) - \underline{S}(f, \mathbf{x}) = \overline{S}(f^{-1}, \mathbf{y}) - \underline{S}(f^{-1}, \mathbf{y}) = \sum_{i=1}^n \Delta x_i \Delta y_i < \varepsilon.$$

We have

$$\begin{aligned} af(a) &= \sum_{i=1}^n \Delta x_i \sum_{j=1}^n \Delta y_j = \sum_{i=1}^n \Delta x_i \left( \sum_{j=1}^i \Delta y_j + \sum_{j=i+1}^n \Delta y_j \right) \\ &= \sum_{i=1}^n y_i \Delta x_i + \sum_{i=1}^n \Delta x_i \sum_{j=i+1}^n \Delta y_j \end{aligned}$$

$$
\begin{aligned}
&= \overline{S}(f,\mathbf{x}) + \sum_{j=2}^{n} \Delta y_j \sum_{i=1}^{j-1} \Delta x_i \\
&= \overline{S}(f,\mathbf{x}) + \underline{S}(f^{-1},\mathbf{y}),
\end{aligned}
$$

so

$$
\begin{aligned}
&\left| a f(a) - \int_0^a f(t)dt - \int_0^{f(a)} f^{-1}(t)dt \right| \\
&\quad = \left| \overline{S}(f,\mathbf{x}) - \int_0^a f(t)dt + \underline{S}(f^{-1},\mathbf{y}) - \int_0^{f(a)} f^{-1}(t)dt \right| \\
&\quad \le \overline{S}(f,\mathbf{x}) - \underline{S}(f,\mathbf{x}) + \overline{S}(f^{-1},\mathbf{y}) - \underline{S}(f^{-1},\mathbf{y}) < 2\varepsilon.
\end{aligned}
$$

□

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