Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 2, Article 48, 2006

EXPLICIT BOUNDS ON SOME NONLINEAR INTEGRAL INEQUALITIES

YOUNG-HO KIM

DEPARTMENT OF APPLIED MATHEMATICS,
CHANGWON NATIONAL UNIVERSITY,
CHANGWON 641-773, KOREA
yhkim@sarim.changwon.ac.kr

Received 24 September, 2005; accepted 09 November, 2005
Communicated by D. Hinton

ABSTRACT. In this paper, some new nonlinear integral inequalities involving functions of one and two independent variables which provide explicit bounds on unknown functions are established. We also present some of its applications to the qualitative study of retarded differential equations.

Key words and phrases: Integral inequalities, Retarded integral inequality, Retarded differential equations, Partial delay differential equations.

2000 *Mathematics Subject Classification.* 26D10, 26D15, 34K10, 35B35.

1. INTRODUCTION

Nonlinear differential equations whose solutions cannot be found explicitly arise in essentially every branch of modern science, engineering and mathematics. One of the most useful methods available for studying a nonlinear system of ordinary differential equations is to compare it with a single first-order equation derived naturally from some bounds on the system. However, the bounds provided by the comparison method are sometimes difficult or impossible to calculate explicitly. In fact, in many applications explicit bounds are more useful while studying the behavior of solutions of such systems. Another basic tool, which is typical among investigations on this subject, is the use of nonlinear integral inequalities which provide explicit bounds on the unknown functions. Over the last scores of years several new nonlinear integral inequalities have been developed in order to study the behavior of solutions of such systems.

One of the most useful inequalities in the development of the theory of differential equations is given in the following theorem. *If* u, f *are nonnegative continuous functions on* $\mathbb{R}_+ = [0, \infty)$, $u_0 \geq 0$ *is a constant and*

$$u^2(t) \leq u_0^2 + 2\int_0^t f(s)u(s)ds$$

ISSN (electronic): 1443-5756

This research is financially supported by Changwon National University in 2005.

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for $t \in \mathbb{R}_+$, *then*

$$u(t) \leq u_0 \int_0^t f(s)ds, \quad t \in \mathbb{R}_+.$$

It appears that this inequality was first considered by Ou-Iang [5], while investigating the boundedness of certain solutions of certain second-order differential equations.

In the past few years this inequality given in [5] has been used considerably in the study of qualitative properties of the solutions of certain abstract differential, integral and partial differential equations.

Recently, Pachpatte in [9] obtained a useful upper bound on the following inequality:

$$u^p(t) \leq c + p \sum_{i=1}^{n} \int_{\alpha_i(t_0)}^{\alpha_i(t)} [a_i(s)u^p(s) + b_i(s)u(s)]ds, \tag{1.1}$$

and its variants, under some suitable conditions on the functions involved in (1.1), including the constant $p > 1$. In fact, the results given in [9] are generalized versions of the inequalities established by Lipovan in [4], Qu-Iang in [5] and Pachpatte in [6].

The main purpose of this paper is to obtain explicit bounds on the following retarded integral inequality

$$u(t) \leq c + \sum_{i=1}^{n} \left[\int_{t_0}^{t} a_i(s)u^p(s) + \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(s)u^p(s)ds \right], \tag{1.2}$$

and its variants, under some suitable conditions on the functions involved in (1.2), including the constant $p \geq 0$, $p \neq 1$, or $p > 1$. We also prove the two independent variable generalization of the result and present some applications of those to the global existence of solutions of differential equations with time delay.

2. THE INTEGRAL INEQUALITIES

We shall introduce some notation, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0, \infty)$, $I = [t_0, T)$ are the given subsets of $\mathbb{R}$. The first order derivative of a function $z(t)$ with respect to t will be denoted by $z'(t)$. Let $C(M, N)$ denote the class of continuous functions from the set M to the set N. In the following theorems we prove some nonlinear integral inequalities.

Theorem 2.1. *Let* $u, a_i, b_i \in C(I, \mathbb{R}_+)$, $\alpha_i \in C^1(I, I)$ *be nondecreasing with* $\alpha_i(t) \leq t$ *on* I, *for* $i = 1, 2, \ldots, n$. *Let* $p \geq 0, p \neq 1$, *and* $c \geq 0$ *be constants. If*

$$u(t) \leq c + \sum_{i=1}^{n} \left[\int_{t_0}^{t} a_i(s)u^p(s)\, ds + \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(s)u^p(s)\, ds \right] \tag{2.1}$$

for $t \in I$, *then*

$$u(t) \leq \left[c^q + q \sum_{i=1}^{n} \left(\int_{t_0}^{t} a_i(s)\, ds + \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(s)\, ds \right) \right]^{\frac{1}{q}} \tag{2.2}$$

for $t \in [t_0, T_1)$, *where* $q = 1 - p$ *and* T_1 *is chosen so that the expression inside* $[\ldots]$ *is positive on the subinterval* $[t_0, T_1)$.

Proof. From the hypotheses we observe that $\alpha'(t) \geq 0$ for $t \in I$. Let $c \geq 0$ and define a function $z(t)$ by the right-hand side of (2.1). Then, $z(t_0) = c$, $z(t)$ is nondecreasing for $t \in I$,

$u(t) \le z(t)$, and

$$\begin{aligned} z'(t) &= \sum_{i=1}^{n}[a_i(t)u^p(t) + b_i(\alpha_i(t))u^p(\alpha_i(t))\alpha_i'(t)] \\ &\le \sum_{i=1}^{n}[a_i(t)z^p(t) + b_i(\alpha_i(t))z^p(\alpha_i(t))\alpha_i'(t)] \\ &\le \sum_{i=1}^{n}[a_i(t) + b_i(\alpha_i(t))\alpha_i'(t)][z(t)]^p. \end{aligned}$$

By making the constant $q = 1 - p$ and using the function $z_1(t) = z^q(t)/q$ we get

$$z_1'(t) \le \sum_{i=1}^{n}[a_i(t) + b_i(\alpha_i(t))\alpha_i'(t)]. \tag{2.3}$$

By taking $t = s$ in (2.3) and integrating it with respect to s from t_0 to t, $t \in I$, we obtain

$$\int_{t_0}^{t} z_1'(s)\,ds \le \sum_{i=1}^{n}\left[\int_{t_0}^{t} a_i(s)\,ds + \int_{t_0}^{t} b_i(\alpha_i(s))\alpha_i'(s)\,ds\right]. \tag{2.4}$$

Integrating, making the change of the function on the left side in (2.4) and rewriting yields

$$\frac{z^q(t)}{q} \le \frac{c^q}{q} + \sum_{i=1}^{n}\left[\int_{t_0}^{t} a_i(s)\,ds + \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(s)\,ds\right]$$

for $t \in I$. It follows that

$$z(t) \le \left[c^q + q\sum_{i=1}^{n}\left(\int_{t_0}^{t} a_i(s)\,ds + \int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(s)\,ds\right)\right]^{\frac{1}{q}} \tag{2.5}$$

for $t \in [t_0, T_1)$, where T_1 is chosen so that the expression inside $[\dots]$ is positive on the subinterval $[t_0, T_1)$. Using (2.5) in $u(t) \le z(t)$ we get the inequality in (2.2). □

Corollary 2.2. *Let $u, b_i \in C(I, \mathbb{R}_+)$, $\alpha_i \in C^1(I, I)$ be nondecreasing with $\alpha_i(t) \le t$ on I for $i = 1, \dots, n$, and let $p \ge 0, p \ne 1$, and $c \ge 0$ be constants. If*

$$u(t) \le c + \sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(s)u^p(s)\,ds$$

for $t \in I$, then

$$u(t) \le \left[c^q + q\sum_{i=1}^{n}\int_{\alpha_i(t_0)}^{\alpha_i(t)} b_i(s)\,ds\right]^{\frac{1}{q}}$$

for $t \in [t_0, T_1)$, where $q = 1 - p$ and T_1 is chosen so that the expression inside $[\dots]$ is positive on the subinterval $[t_0, T_1)$.

The following theorem deals with the constant $1 < p < \infty$ versions of the inequalities established in Theorem 2.1.

Theorem 2.3. *Let $u, a, b_i, c_i \in C(I, \mathbb{R}_+)$, $\alpha_i \in C^1(I, I)$ be nondecreasing with $\alpha_i(t) \le t$ on I, for $i = 1, 2, \dots, n$. Also, let $a(t)$ be nondecreasing in t, $t \in I$ and $p > 1$ be constants. If*

$$u(t) \le a(t) + \sum_{i=1}^{n}\left[\int_{t_0}^{t} b_i(s)u^p(s)\,ds + \int_{\alpha_i(t_0)}^{\alpha_i(t)} c_i(s)u^p(s)\,ds\right] \tag{2.6}$$

for $t \in I$, *then*

$$u(t) \le a(t)\left[1-(p-1)a^{p-1}(t)\sum_{i=1}^{n}\left(\int_{t_0}^{t} b_i(s)\,ds + \int_{\alpha_i(t_0)}^{\alpha_i(t)} c_i(s)\,ds\right)\right]^{\frac{1}{1-p}} \tag{2.7}$$

for $t \in [t_0, T)$, *where*

$$T = \sup\left\{t \in I : (p-1)a^{p-1}(t)\sum_{i=1}^{n}\left(\int_{t_0}^{t} b_i(s)\,ds + \int_{\alpha_i(t_0)}^{\alpha_i(t)} c_i(s)\,ds\right) < 1\right\}.$$

Proof. From the hypotheses we observe that $\alpha'(t) \ge 0$ for $t \in I$. Define a function $v(t)$ by

$$v(t) = \sum_{i=1}^{n}\left[\int_{t_0}^{t} b_i(s)u^p(s)\,ds + \int_{\alpha_i(t_0)}^{\alpha_i(t)} c_i(s)u^p(s)\,ds\right].$$

Then, $v(t_0) = 0$, $v(t)$ is nondecreasing for $t \in I$, $u(t) \le a(t) + v(t)$, and

$$\begin{aligned} v'(t) &\le \sum_{i=1}^{n}[b_i(t) + c_i(\alpha_i(t))\alpha_i'(t)][a(t)+z(t)]^p \\ &\le R(t)[a(t)+v(t)], \end{aligned}$$

where

$$R(t) = \sum_{i=1}^{n}[b_i(t) + c_i(\alpha_i(t))\alpha_i'(t)][a(t)+z(t)]^{p-1}.$$

Now by the comparison result for linear differential inequalities(see [1, Lemma 1.1., p. 2]), this implies that

$$\begin{aligned} v(t) &\le \int_{t_0}^{t} R(s)a(s)\exp\left(\int_s^t R(\tau)\,d\tau\right)\,ds \\ &\le a(t)\left[\int_{t_0}^{t} R(s)\exp\left(\int_s^t R(\tau)\,d\tau\right)\,ds\right] \end{aligned} \tag{2.8}$$

for $s \ge t_0$. By integrating on the right hand side in (2.8) we get

$$v(t) + a(t) \le a(t)\exp\left(\int_{t_0}^{t} R(\tau)\,d\tau\right). \tag{2.9}$$

From (2.9) we successively obtain

$$[v(t)+a(t)]^{p-1} \le a^{p-1}(t)\exp\left(\int_{t_0}^{t}(p-1)R(\tau)\,d\tau\right),$$

$$R(t) \le a^{p-1}(t)\exp\left(\int_{t_0}^{t}(p-1)R(\tau)\,d\tau\right)\sum_{i=1}^{n}[b_i(t) + c_i(\alpha_i(t))\alpha_i'(t)]$$

and

$$\begin{aligned} A(t) &= (p-1)R(t) \\ &\le (p-1)a^{p-1}(t)\exp\left(\int_{t_0}^{t} A(\tau)\,d\tau\right)\sum_{i=1}^{n}[b_i(t) + c_i(\alpha_i(t))\alpha_i'(t)]. \end{aligned}$$

Consequently, we get

$$A(t)\exp\left(-\int_{t_0}^{t} A(\tau)\,d\tau\right) \le (p-1)a^{p-1}(t)\sum_{i=1}^{n}[b_i(t) + c_i(\alpha_i(t))\alpha_i'(t)],$$

or

$$\text{(2.10)} \qquad \frac{d}{dt}\left[-\exp\left(-\int_{t_0}^{t} A(\tau)\,d\tau\right)\right] \le (p-1)a^{p-1}(t)\sum_{i=1}^{n}[b_i(t)+c_i(\alpha_i(t))\alpha_i'(t)],$$

By taking $t=s$ in (2.10) and integrating it with respect to s from t_0 to t, $t\in I$, we obtain

$$\text{(2.11)} \qquad 1-\exp\left(-\int_{t_0}^{t} A(\tau)\,d\tau\right) \le (p-1)a^{p-1}(t)\sum_{i=1}^{n}\left[\int_{t_0}^{t} b_i(s)\,ds+\int_{t_0}^{t} c_i(\alpha_i(s))\alpha_i'(s)\,ds\right].$$

Making the change of the variables on the right side in (2.11) and rewriting yields

$$\exp\left(\int_{t_0}^{t} R(\tau)\,d\tau\right) \le \left[1-(p-1)a^{p-1}(t)\sum_{i=1}^{n}\left(\int_{t_0}^{t} b_i(s)\,ds+\int_{\alpha_i(t_0)}^{\alpha_i(t)} c_i(s)\,ds\right)\right]^{\frac{1}{1-p}}$$

for $t\in[t_0,T)$, where T is chosen so that the expression inside $[\dots]$ is positive in the subinterval $[t_0,T)$. This, together with (2.9) and $u(t)\le a(t)+v(t)$, gives the inequality in (2.7). □

In the following theorem we establish two independent-variable versions of Theorem 2.1 which can be used in the qualitative analysis of hyperbolic partial differential equations with retarded arguments. Let $\triangle = I_1\times I_2$, where $I_1=[x_0,X)$, $I_2=[y_0,Y)$ are the given subsets of the real numbers, $\mathbb{R}$. The first order partial derivatives of a function $z(x,y)$ defined for $x,y\in\mathbb{R}$ with respect to x and y are denoted by $\partial z(x,y)/\partial x$ and $\partial z(x,y)/\partial y$ respectively.

Theorem 2.4. *Let $u,a_i,b_i\in C(\triangle,\mathbb{R}_+)$, $\alpha_i\in C^1(I_1,I_1)$, $\beta_i\in C^1(I_2,I_2)$ be nondecreasing with $\alpha_i(x)\le x$ on I_1, $\beta_i(y)\le y$ on I_2, for $i=1,2,\dots,n$. Let $p\ge 0, p\ne 1$, and $c\ge 0$ be constants. If*

$$\text{(2.12)} \qquad u(x,y)\le c+\sum_{i=1}^{n}\left(\int_{x_0}^{x}\int_{y_0}^{y} a_i(s,t)u^p(s,t)\,dt\,ds + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)u^p(s,t)\,dt\,ds\right)$$

for $(x,y)\in I_1\times I_2$, then

$$\text{(2.13)} \qquad u(x,y)\le\left[c^q+q\sum_{i=1}^{n}\left(\int_{x_0}^{x}\int_{y_0}^{y} a_i(s,t)\,dt\,ds + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\,dt\,ds\right)\right]^{\frac{1}{q}}$$

for $(x,y)\in[x_0,X)\times[y_0,Y)$, where $q=1-p$ and X,Y are chosen so that the expression inside $[\dots]$ is positive on the subintervals $[x_0,X)$ and $[y_0,Y)$.

Proof. The details of the proof of Theorem 2.4 follow by an argument similar to that in the proof of Theorem 2.1 with suitable changes. From the hypotheses we observe that $\alpha'(x)\ge 0$ for $x\in I_1$ and $\beta'(y)\ge 0$ for $y\in I_2$. Let $c\ge 0$ and define a function $z(x,y)$ by the right-hand side of (2.12). Then, $z(x_0,y)=z(x,y_0)=c$, $z(x,y)$ is nondecreasing for $(x,y)\in\triangle$, $u(x,y)\le z(x,y)$, and

$$\frac{\partial}{\partial x}z(x,y)\le\sum_{i=1}^{n}\left[\int_{y_0}^{y} a_i(x,t)\,dt+\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t)\alpha_i'(x)\,dt\right][z(x,y)]^p.$$

By making the constant $q = 1 - p$ and using the function $v(x,y) = z^q(x,y)/q$ we get

$$\frac{\partial}{\partial x} v(x,y) \le \sum_{i=1}^{n} \left[\int_{y_0}^{y} a_i(x,t)\,dt + \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t)\alpha_i'(x)\,dt \right]. \tag{2.14}$$

By taking $x = s$ in (2.14) and integrating it with respect to s from x_0 to x, $x \in I_1$, we obtain

$$\int_{x_0}^{x} \frac{\partial}{\partial s} v(s,y)\,ds \le \sum_{i=1}^{n} \left[\int_{x_0}^{x}\int_{y_0}^{y} a_i(s,t)\,dt\,ds + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\,dt\,ds \right]. \tag{2.15}$$

Integrating with respect to s from x_0 to x, making the change of the function on the left side in (2.15) and rewriting yields

$$\frac{z^q(x,y)}{q} \le \frac{c^q}{q} + \sum_{i=1}^{n} \left[\int_{x_0}^{x}\int_{y_0}^{y} a_i(s,t)\,dt\,ds + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\,dt\,ds \right]$$

for $(x,y) \in \triangle$. This implies

$$z(x,y) \le \left[c^q + q\sum_{i=1}^{n} \left(\int_{x_0}^{x}\int_{y_0}^{y} a_i(s,t)\,dt\,ds + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\,dt\,ds \right) \right]^{\frac{1}{q}} \tag{2.16}$$

for $(x,y) \in [x_0,X) \times [y_0,Y)$, where X, Y are chosen so that the expression inside $[\dots]$ is positive on the subintervals $[x_0,X)$ and $[y_0,Y)$. Using (2.16) in $u(x,y) \le z(x,y)$ we get the inequality in (2.13). $\square$

The following theorem deals with the constant $1 < p < \infty$ versions of the inequalities established in Theorem 2.4.

Theorem 2.5. *Let $u, a_i, b_i \in C(\triangle, \mathbb{R}_+)$, $\alpha_i \in C^1(I_1, I_1)$, $\beta_i \in C^1(I_2, I_2)$ be nondecreasing with $\alpha_i(x) \le x$ on I_1, $\beta_i(y) \le y$ on I_2, for $i = 1, 2, \dots, n$. Let $a(x,y)$ be nondecreasing in $(x,y) \in \triangle$ and $1 < p < \infty$ be constant. If*

$$u(x,y) \le a(x,y) + \sum_{i=1}^{n} \left[\int_{x_0}^{x}\int_{y_0}^{y} b_i(s,t)u^p(s,t)\,dt\,ds \right. \\ \left. + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} c_i(s,t)u^p(s,t)\,dt\,ds \right] \tag{2.17}$$

for $(x,y) \in I_1 \times I_2$, then

$$u(x,y) \le a(x,y) \left[1 - (p-1)a^{p-1}(x,y) \sum_{i=1}^{n} \Psi_i(x,y) \right]^{\frac{1}{1-p}} \tag{2.18}$$

for $(x,y) \in \triangle_1$, where

$$\Psi_i(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} b_i(s,t)\,dt\,ds + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} c_i(s,t)\,dt\,ds$$

and

$$\triangle_1 = \sup\left\{ (x,y) \in \triangle : (p-1)a^{p-1}(x,y)\sum_{i=1}^{n} \xi_i(x,y) < 1 \right\}.$$

Proof. The details of the proof of Theorem 2.5 follows by an argument similar to that in the proof of Theorem 2.3 with suitable changes. From the hypotheses we observe that $\alpha'(x) \geq 0$ for $x \in I_1$ and $\beta'(y) \geq 0$ for $y \in I_2$. Define a function $v(x,y)$ by

$$v(x,y) = \sum_{i=1}^{n} \left[\int_{x_0}^{x} \int_{y_0}^{y} b_i(s,t) u^p(s,t)\, dt\, ds + \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} c_i(s,t) u^p(s,t)\, dt\, ds \right].$$

Then, $v(x_0,y) = v(x,y_0) = 0$, $v(x,y)$ is nondecreasing for $(x,y) \in \triangle$, $u(x,y) \leq a(x,y) + v(x,y)$, and

$$\frac{\partial}{\partial x} v(x,y) \leq R(x,y)[a(x,y) + v(x,y)],$$

where

$$R(x,y) = \sum_{i=1}^{n} \left[\int_{y_0}^{y} b_i(x,t)\, dt + \int_{\beta_i(y_0)}^{\beta_i(y)} c_i(\alpha(x),t)\alpha'(x)\, dt \right] [a(x,y) + v(x,y)]^{p-1}.$$

Now by keeping y fixed and using the comparison result for linear differential inequalities (see [1, Lemma 1.1., p. 2]), this implies that

$$v(x,y) \leq \int_{x_0}^{x} R(s,y) a_i(s,y) \exp\left(\int_{s}^{x} R(\tau,y)\, d\tau \right) ds \tag{2.19}$$

for $s \geq x_0$. By integrating on the right hand side in (2.19) we get

$$v(x,y) + a(x,y) \leq a(x,y) \exp\left(\int_{x_0}^{x} R(\tau,y)\, d\tau \right). \tag{2.20}$$

From (2.20) we successively obtain

$$[v(x,y) + a(x,y)]^{p-1} \leq a^{p-1}(x,y) \exp\left(\int_{x_0}^{x} (p-1) R(\tau,y)\, d\tau \right),$$

$$\begin{aligned} A(x,y) &= (p-1)R(x,y) \\ &\leq (p-1) a^{p-1}(x,y) \exp\left(\int_{x_0}^{x} R(\tau,y)\, d\tau \right) \sum_{i=1}^{n} \psi_i(x,y), \end{aligned}$$

where

$$\psi_i(x,y) = \int_{y_0}^{y} b_i(x,t)\, dt + \int_{\beta_i(y_0)}^{\beta_i(y)} c_i(\alpha(x),t)\alpha'(x)\, dt.$$

Consequently, we have

$$\frac{\partial}{\partial x}\left[-\exp\left(-\int_{x_0}^{x} A(\tau,y)\, d\tau \right) \right] \leq (p-1) a^{p-1}(x,y) \sum_{i=1}^{n} \psi_i(x,y). \tag{2.21}$$

By taking $x = s$ in (2.21) and integrating it with respect to s from x_0 to x, $x \in I_1$, we obtain

$$1 - \exp\left(-\int_{x_0}^{x} A(\tau,y)\, d\tau \right) \leq (p-1) a^{p-1}(x,y) \sum_{i=1}^{n} \Psi_i(x,y), \tag{2.22}$$

where

$$\Psi_i(x,y) = \int_{x_0}^{x} \int_{y_0}^{y} b_i(s,t)\, dt\, ds + \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} c_i(s,t)\, dt\, ds.$$

Making the change of the function on the inequality (2.22) and rewriting yields

$$\exp\left(\int_{x_0}^{x} R(\tau, y)\, d\tau\right) \leq \left[1 - (p-1)a^{p-1}(x,y)\sum_{i=1}^{n}\Psi_i(x,y)\right]^{\frac{1}{1-p}}$$

for $(x, y) \in \triangle_1$, where $\triangle_1$ is chosen so that the expression inside $[\dots]$ is positive in the subinterval $\triangle_1$. This, together with (2.20) and $u(x, y) \leq a(x, y) + v(x, y)$, gives the inequality in (2.18). □

3. APPLICATIONS

In this section we will show that our results are useful in proving the global existence of solutions to certain differential equations with time delay. First consider the functional differential equation involving several retarded arguments with the initial condition

$$\begin{cases} x'(t) = F(t, x(t), x(t-h_1(t)), \dots, x(t-h_n(t)), & t \in I, \\ x(t_0) = x_0, \end{cases} \tag{3.1}$$

where x_0 is constant, $F \in C(I \times \mathbb{R}^{n+1}, \mathbb{R})$ and for $i = 1, \dots, n$, let $h_i \in C^1(I, \mathbb{R}_+)$ be nonincreasing and such that $t - h_i(t) \geq 0$, $x - h_i(t) \in C^1(I, I)$, $h_i'(t) < 1$, and $h(t_0) = 0$. The following theorem deals with a bound on the solution of the problem (3.1).

Theorem 3.1. *Assume that $F : I \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous function for which there exist continuous nonnegative functions $a_i(t)$, $b_i(t)$ for $t \in I$ such that*

$$|F(t, v, u_1, \dots, u_n)| \leq \sum_{i=1}^{n}[a_i(t)\,|v|^p + b_i(t)\,|u_i|^p], \tag{3.2}$$

where $p \geq 0$, $p \neq 1$ is constant, and let

$$M_i = \max_{t\in I}\frac{1}{1-h_i'(x)}, \qquad i = 1, \dots, n. \tag{3.3}$$

If $x(t)$ is any solution of the problem (3.1), then

$$|x(t)| \leq \left[|x_0|^q + q\sum_{i=1}^{n}\left(\int_{t_0}^{t} a_i(\sigma)\, d\sigma + \int_{t_0}^{t-h_i(t)} \bar{b}_i(\sigma)\, d\sigma\right)\right]^{\frac{1}{q}}$$

for $t \in I$, where $\bar{b}_i(\sigma) = M_i b_i(\sigma + h_i(s))$, $\sigma, s \in I$.

Proof. The solution $x(t)$ of the problem (3.1) can be written as

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x(s), x(s-h_1(s)), \dots, x(s-h_n(s))\, ds. \tag{3.4}$$

From (3.2), (3.3), (3.4) and making the change of variables we have

$$\begin{aligned} |x(t)| &\leq |x_0| + \sum_{i=1}^{n}\left(\int_{t_0}^{t} a_i(s)|x(s)|^p\, ds + \int_{t_0}^{t} b_i(s)|x(s-h_i(s))|^p\, ds\right) \\ &\leq |x_0| + \sum_{i=1}^{n}\left(\int_{t_0}^{t} a_i(s)|x(s)|^p\, ds + \int_{t_0}^{t-h_i(t)} \bar{b}_i(\sigma)|x(\sigma)|^p\, d\sigma\right) \end{aligned} \tag{3.5}$$

for $t \in I$, where $\bar{b}_i(\sigma) = M_i b_i(\sigma + h_i(s))$, $\sigma, s \in I$. Now a suitable application of the inequality in Theorem 2.1 to (3.5) yields the result. □

Remark 3.2.

(i) For $1 < p < \infty$, a suitable application of the inequality in Theorem 2.3 to (3.5) yields the following result

$$|x(t)| \leq \left[|x_0| - (p-1)|x_0|^p \sum_{i=1}^{n} \left(\int_{t_0}^{t} a_i(s)\,ds + \int_{t_0}^{t-h_i(t)} \overline{b}_i(\sigma)\,d\sigma \right) \right]^{\frac{1}{1-p}}.$$

This shows in particular that the solution $x(t)$ is bounded on $[t_0, T)$, where T is chosen so that the expression inside $[\dots]$ is positive in the subinterval $[t_0, T)$.

(ii) Consider the functional differential equation

$$\text{(3.6)} \qquad \begin{cases} x'(t) = F(t, x(t-h_1(t)), \dots, x(t-h_n(t)), & t \in I, \\ x(t_0) = x_0. \end{cases}$$

Assume that $F : I \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous function for which there exist continuous nonnegative functions $b_i(t)$ for $t \in I$ such that

$$\text{(3.7)} \qquad |F(t, u_1, \dots, u_n)| \leq \sum_{i=1}^{n} b_i(t)\,|u_i|\,,$$

where $p \geq 0$, $p \neq 1$ is constant. Let M_i be a function defined by (3.3). If $x(t)$ is any solution of (3.6), the solution $x(t)$ can be written as

$$\text{(3.8)} \qquad x(t) = x_0 + \int_{t_0}^{t} F(s, x(s-h_1(s)), \dots, x(s-h_n(s))\,ds.$$

From (3.7), (3.8) and making the change of variables we have

$$\text{(3.9)} \qquad |x(t)| \leq |x_0| + p \sum_{i=1}^{n} \int_{t_0}^{t-h_i(t)} \overline{b}_i(\sigma)|x(\sigma)|^p\,d\sigma$$

for $t \in I$, where $\overline{b}_i(\sigma) = M_i b_i(\sigma + h_i(s))$, $\sigma, s \in I$. Now a suitable application of the inequality in Corollary 2.2 to (3.8) yields

$$|x(t)| \leq \left[|x_0|^q + q \sum_{i=1}^{n} \int_{t_0}^{t-h_i(t)} \overline{b}_i(\sigma)\,d\sigma \right]^{\frac{1}{q}}$$

for $t \in I$, where $q = 1-p$, $\overline{b}_i(\sigma) = M_i b_i(\sigma + h_i(s))$, $\sigma, s \in I$.

In the following we present an application of the inequality given in Section 2 to study the boundedness of the solutions of the initial boundary value problem for hyperbolic partial delay differential equations of the form

$$\text{(3.10)} \qquad \begin{cases} \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x} z(x,y)\right] \\ = F(x, y, z(x,y), z(x-h_1(x), y-g_1(y)), \dots, z(x-h_n(x), y-g_n(y)), \\ z(x, y_0) = e_1(x), \quad z(x_0, y) = e_2(y), \quad e_1(x_0) = e_2(y_0) = 0, \end{cases}$$

where $p > 0, p \neq 1$ is constant, $F \in C(\triangle \times \mathbb{R}^{n+1}, \mathbb{R})$, $e_1 \in C^1(I_1, \mathbb{R})$, $e_2 \in C^1(I_2, \mathbb{R})$, and $h_i \in C^1(I_1, \mathbb{R}_+)$, $g_i \in C^1(I_2, \mathbb{R}_+)$ are nonincreasing and such that $x - h_i(x) \geq 0$, $x - h_i(x) \in C^1(I_1, I_1)$, $y - g_i(y) \geq 0$, $y - h_i(y) \in C^1(I_2, I_2)$, $h_i'(t) < 1$, $g_i'(t) < 1$, and $h_i(x_0) = g_i(y_0) = 0$ for $i = 1, \dots, n$.

Theorem 3.3. *Assume that $F : \triangle \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous function for which there exists continuous nonnegative functions $a_i(x,y), b_i(x,y)$ for $i = 1,\dots,n$; $x \in I_1, y \in I_2$ such that*

$$\begin{cases} |F(x,y,v,u_1,\dots,u_n)| \le \sum_{i=1}^{n}[a_i(x,y)\,|v|^p + b_i(x,y)\,|u_i|^p], \\ |e_1(x)+e_2(y)| \le c \end{cases} \tag{3.11}$$

for $c \ge 0$, $p \ge 0$, $p \ne 1$, and let

$$M_i = \max_{x\in I_1} \frac{1}{1-h_i'(x)}, \quad N_i = \max_{y\in I_2} \frac{1}{1-g_i'(y)}, \quad i = 1,\dots,n. \tag{3.12}$$

If $z(x,y)$ is any solution of the problem (3.10), then

$$|z(x,y)| \le \left[c^q + q\sum_{i=1}^{n}\left(\int_{x_0}^{x}\int_{y_0}^{y} a_i(\sigma,\tau)\,d\tau\,d\sigma + \int_{x_0}^{\phi(x)}\int_{y_0}^{\psi(y)} \overline{b}_i(\sigma,\tau)]\,d\tau\,d\sigma\right)\right]^{\frac{1}{q}}$$

for $(x,y) \in \triangle_1$, where $q = 1-p$, $\phi(x) = x - h_i(x)$, $\psi(y) = y - g_i(y)$ and $\overline{b}(\sigma,\tau) = M_iN_ib_i(\sigma + h_i(s), \tau + g_i(t))$, $\sigma, s \in I_1$, $\tau, t \in I_2$.

Proof. It is easy to see that the solution $z(x,y)$ of the problem (3.10) satisfies the equivalent integral equation

$$z(x,y) = e_1(x) + e_2(y) + \int_{x_0}^{x}\int_{y_0}^{y} F(s,t,z(s,t),z(s-h_1(s),t-g_1(t)), \dots, z(s-h_n(s),t-g_n(t))\,dt\,ds. \tag{3.13}$$

From (3.11), (3.13) and making the change of variables, we have

$$|z(x,y)| \le c + \int_{x_0}^{x}\int_{y_0}^{y}\sum_{i=1}^{n}\big(a_i\,|z(s,t)|^p + b_i\,|z(s-h_i(s),t-g_i(t))|^p\big)\,ds\,dt. \tag{3.14}$$

Now a suitable application of the inequality given in Theorem 2.4 to (3.14) yields the desired result. □

Remark 3.4. For $1 < p < \infty$, a suitable application of the inequality in Theorem 2.5 to (3.14) yields the following result

$$|z(x,y)| \le \left[c - c^p(p-1)\sum_{i=1}^{n}\overline{\Psi}_i(x,y)\right]^{\frac{1}{1-p}},$$

where

$$\overline{\Psi}_i(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} a_i(\sigma,\tau)\,d\tau\,d\sigma + \int_{x_0}^{\phi(x)}\int_{y_0}^{\psi(y)} \overline{b}_i(\sigma,\tau)\,d\tau\,d\sigma.$$

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