

Z. ABDULHADI AND Y. ABU MUHANNA

Department of Mathematics
American University of Sharjah
Sharjah, Box 26666, UAE

EMail: zahadi@aus.ac.ae

EMail: ymuhanna@aus.ac.ae



volume 7, issue 4, article 123,
2006.

*Received 20 September, 2005;
accepted 13 December, 2005.*

Communicated by: Q.I. Rahman

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

In this paper, we consider univalent log-harmonic mappings of the form $f = zh\bar{g}$ defined on the unit disk U which are starlike of order α . Representation theorems and distortion theorem are obtained.

2000 Mathematics Subject Classification: Primary 30C35, 30C45; Secondary 35Q30.

Key words: log-harmonic, Univalent, Starlike of order α .

Contents

1	Introduction	3
2	Representation Theorems	5
3	Distortion Theorem	9
	References	



Starlike log-harmonic Mappings of Order α

Z. AbdulHadi and Y. Abu Muhanna

Title Page

Contents



Go Back

Close

Quit

Page 2 of 13

1. Introduction

Let $H(U)$ be the linear space of all analytic functions defined on the unit disk $U = \{z : |z| < 1\}$. A log-harmonic mapping is a solution of the nonlinear elliptic partial differential equation

$$(1.1) \quad \frac{\overline{f_z}}{f} = a \frac{f_z}{f},$$

where the second dilation function $a \in H(U)$ is such that $|a(z)| < 1$ for all $z \in U$. It has been shown that if f is a non-vanishing log-harmonic mapping, then f can be expressed as

$$f(z) = h(z)\overline{g(z)},$$

where h and g are analytic functions in U . On the other hand, if f vanishes at $z = 0$ but is not identically zero, then f admits the following representation

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)},$$

where $\text{Re } \beta > -1/2$, and h and g are analytic functions in U , $g(0) = 1$ and $h(0) \neq 0$ (see [3]). Univalent log-harmonic mappings have been studied extensively (for details see [1] – [5]).

Let $f = z|z|^{2\beta}h\overline{g}$ be a univalent log-harmonic mapping. We say that f is a starlike log-harmonic mapping of order α if

$$(1.2) \quad \frac{\partial \arg f(re^{i\theta})}{\partial \theta} = \text{Re } \frac{zf_z - \overline{z}f_{\overline{z}}}{f} > \alpha, \quad 0 \leq \alpha < 1$$



Starlike log-harmonic Mappings
of Order α

Z. AbdulHadi and Y. Abu Muhanna

Title Page

Contents



Go Back

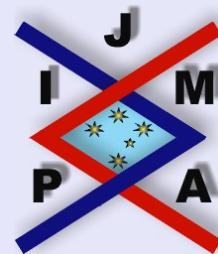
Close

Quit

Page 3 of 13

for all $z \in U$. Denote by $ST_{Lh}(\alpha)$ the set of all starlike log-harmonic mappings of order α . If $\alpha = 0$, we get the class of starlike log-harmonic mappings. Also, let $ST(\alpha) = \{f \in ST_{Lh}(\alpha) \text{ and } f \in H(U)\}$. If $f \in ST_{Lh}(0)$ then $F(\zeta) = \log(f(e^\zeta))$ is univalent and harmonic on the half plane $\{\zeta : \operatorname{Re}\{\zeta\} < 0\}$. It is known that F is closely related with the theory of nonparametric minimal surfaces over domains of the form $-\infty < u < u_0(v)$, $u_0(v + 2\pi) = u_0(v)$, (see [7]).

In Section 2 we include two representation theorems which establish the linkage between the classes $ST_{Lh}(\alpha)$ and $ST(\alpha)$. In Section 3 we obtain a sharp distortion theorem for the class $ST_{Lh}(\alpha)$.



**Starlike log-harmonic Mappings
of Order α**

Z. AbdulHadi and Y. Abu Muhanna

Title Page

Contents



Go Back

Close

Quit

Page 4 of 13

2. Representation Theorems

In this section, we obtain two representation theorems for functions in $ST_{Lh}(\alpha)$. In the first one we establish the connection between the classes $ST_{Lh}(\alpha)$ and $ST(\alpha)$. The second one is an integral representation theorem.

Theorem 2.1. *Let $f(z) = zh(z)\overline{g(z)}$ be a log-harmonic mapping on U , $0 \notin hg(U)$. Then $f \in ST_{Lh}(\alpha)$ if and only if $\varphi(z) = \frac{zh(z)}{g(z)} \in ST(\alpha)$.*

Proof. Let $f(z) = zh(z)\overline{g(z)} \in ST_{Lh}(\alpha)$, then it follows that

$$\begin{aligned} \frac{\partial \arg f(re^{i\theta})}{\partial \theta} &= \operatorname{Re} \frac{zf_z - \bar{z}f_{\bar{z}}}{f} \\ &= \operatorname{Re} \left(1 + \frac{zh'}{h} - \frac{\bar{z}g'}{\bar{g}} \right) \\ &= \operatorname{Re} \left(1 + \frac{zh'}{h} - \frac{zg'}{g} \right) > \alpha. \end{aligned}$$

Setting

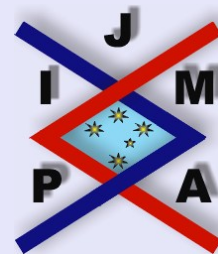
$$\varphi(z) = \frac{zh(z)}{g(z)},$$

we obtain

$$\operatorname{Re} \frac{zf_z - \bar{z}f_{\bar{z}}}{f} = \operatorname{Re} \frac{z\varphi'}{\varphi} > \alpha.$$

Since f is univalent, we know that $0 \notin f_z(U)$. Furthermore,

$$\varphi \circ f^{-1}(w) = q_1(w) = w|g \circ f^{-1}(w)|^{-2},$$



Starlike log-harmonic Mappings
of Order α

Z. AbdulHadi and Y. Abu Muhanna

Title Page

Contents



Go Back

Close

Quit

Page 5 of 13

is locally univalent on $f(U)$.

Indeed, we have $\frac{z\varphi'(z)}{\varphi(z)} = (1 - a(z))\frac{zf_z}{f} \neq 0$ for all $z \in U$. From Lemma 2.3 in [4] we conclude that φ is univalent on U . Hence $\varphi \in ST(\alpha)$.

Conversely, let $\varphi \in ST(\alpha)$ and $a \in H(U)$ such that $|a(z)| < 1$ for all $z \in U$ be given. We consider

$$(2.1) \quad g(z) = \exp \left(\int_0^z \frac{a(s)\varphi'(s)}{\varphi(s)(1-a(s))} ds \right),$$

where $\frac{z\varphi'(z)}{\varphi(z)} = (1 - \alpha)p(z) + \alpha$, and $p \in H(U)$ such that $p(0) = 1$ and $\operatorname{Re}(p) > 0$.

Also, let

$$h(z) = \frac{\varphi(z)g(z)}{z}$$

and

$$(2.2) \quad f(z) = zh(z)\overline{g(\bar{z})} = \varphi(z)|g(z)|^2.$$

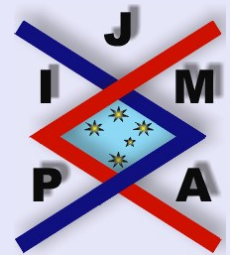
Then h and g are non-vanishing analytic functions defined on U , normalized by $h(0) = g(0) = 1$ and f is a solution of (1.1) with respect to a .

Simple calculations give that

$$\frac{\partial \arg f(re^{i\theta})}{\partial \theta} = \operatorname{Re} \frac{zf_z - \bar{z}f_{\bar{z}}}{f} = \operatorname{Re} \frac{z\varphi'(z)}{\varphi(z)} > \alpha.$$

Using the same argument we conclude that

$$f \circ \varphi^{-1}(w) = q_2(w) = w|g \circ \varphi^{-1}(w)|^2$$



**Starlike log-harmonic Mappings
of Order α**

Z. AbdulHadi and Y. Abu Muhanna

Title Page

Contents



Go Back

Close

Quit

Page 6 of 13

is locally univalent on $\varphi(U)$ and that f is univalent from Lemma 2.3 in [4]. It follows that $f \in ST_{Lh}(\alpha)$, which completes the proof of Theorem 2.1. \square

The next result is an integral representation for $f \in ST_{Lh}(\alpha)$ for the case $a(0) = 0$. For $\varphi \in ST(\alpha)$, we have

$$\frac{z\varphi'(z)}{\varphi(z)} = (1 - \alpha)p(z) + \alpha,$$

where $p \in H(U)$ is such that $p(0) = 1$ and $\operatorname{Re}(p) > 0$. Hence, there is a probability measure μ defined on the Borel σ -algebra of ∂U such that

$$(2.3) \quad \frac{z\varphi'(z)}{\varphi(z)} = (1 - \alpha) \int_{\partial U} \frac{1 + \zeta z}{1 - \zeta z} d\mu(\zeta) + \alpha,$$

and therefore,

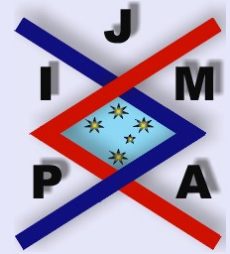
$$(2.4) \quad \varphi(z) = z \exp \left(-2(1 - \alpha) \int_{\partial U} \log(1 - \zeta z) d\mu(\zeta) \right).$$

On the other hand, let $a \in H(U)$ be such that $|a(z)| < 1$ for all $z \in U$ and $a(0) = 0$. Then there is a probability measure ν defined on the Borel σ -algebra of ∂U such that

$$(2.5) \quad \frac{a(z)}{1 - a(z)} = \int_{\partial U} \frac{\xi z}{1 - \xi z} d\nu(\xi).$$

Substituting (2.3), (2.4), and (2.5), into (2.1) and (2.2) we get

$$f(z) = z \exp \left(-2(1 - \alpha) \int_{\partial U} \log(1 - \zeta z) d\mu(\zeta) \right) + L(z),$$



**Starlike log-harmonic Mappings
of Order α**

Z. AbdulHadi and Y. Abu Muhanna

Title Page

Contents



Go Back

Close

Quit

Page 7 of 13

where

$$L(z) = \int_{\partial U_x \partial U} \left[\int_0^z \frac{\xi}{1-\xi s} \left[(1-\alpha) \frac{1+\zeta s}{1-\zeta s} + \alpha \right] ds \right] d\mu(\zeta) d\nu(\xi).$$

Integrating and simplifying implies the following theorem:

Theorem 2.2. $f = zh\bar{g} \in ST_{Lh}(\alpha)$ with $a(0) = 0$ if and only if there are two probability measures μ and ν such that

$$f(z) = z \exp \left(\int_{\partial U_x \partial U} K(z, \zeta, \xi) d\mu(\zeta) d\nu(\xi) \right),$$

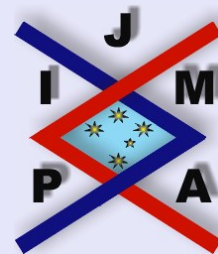
where

$$K(z, \zeta, \xi) = (1-\alpha) \log \left(\frac{1+\bar{\zeta}z}{1-\zeta z} \right) + T(z, \zeta, \xi);$$

(2.6) $T(z, \zeta, \xi)$

$$= \left\{ \begin{array}{ll} \begin{array}{l} -2(1-\alpha) \operatorname{Im} \left(\frac{\zeta+\xi}{\zeta-\xi} \right) \arg \left(\frac{1-\xi z}{1-\zeta z} \right) \\ -2\alpha \log |1-\xi z|; \end{array} & \text{if } |\zeta| = |\xi| = 1, \zeta \neq \xi \\ \begin{array}{l} (1-\alpha) \operatorname{Re} \left(\frac{4\zeta z}{1-\zeta z} \right) \\ -2\alpha \log |1-\zeta z|; \end{array} & \text{if } |\zeta| = |\xi| = 1, \zeta = \xi \end{array} \right\}.$$

Remark 1. Theorem 2.2 can be used in order to solve extremal problems for the class $ST_{Lh}(\alpha)$ with $a(0) = 0$. For example see Theorem 3.1.



Starlike log-harmonic Mappings
of Order α

Z. AbdulHadi and Y. Abu Muhanna

Title Page

Contents



Go Back

Close

Quit

Page 8 of 13

3. Distortion Theorem

The following is a distortion theorem for the class $ST_{Lh}(\alpha)$ with $a(0) = 0$.

Theorem 3.1. Let $f(z) = zh(z)\overline{g(z)} \in ST_{Lh}(\alpha)$ with $a(0) = 0$. Then for $z \in U$ we have

$$(3.1) \quad \frac{|z|}{(1+|z|)^{2\alpha}} \exp\left((1-\alpha)\frac{-4|z|}{1+|z|}\right) \leq |f(z)| \leq \frac{|z|}{(1-|z|)^{2\alpha}} \exp\left((1-\alpha)\frac{4|z|}{1-|z|}\right).$$

The equalities occur if and only if $f(z) = \bar{\zeta}f_0(\zeta z)$, $|\zeta| = 1$, where

$$(3.2) \quad f_0(z) = z \left(\frac{1-\bar{z}}{1-z} \right) \frac{1}{(1-\bar{z})^{2\alpha}} \exp\left((1-\alpha) \operatorname{Re} \frac{4z}{1-z}\right).$$

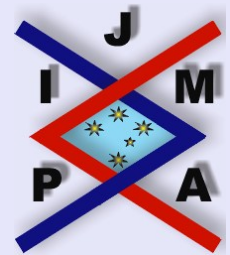
Proof. Let $f(z) = zh(z)\overline{g(z)} \in ST_{Lh}(\alpha)$ with $a(0) = 0$. It follows from (2.1) and (2.2) that f admits the representation

$$(3.3) \quad f(z) = \varphi(z) \exp\left(2 \operatorname{Re} \int_0^z \frac{a(s)\varphi'(s)}{\varphi(s)(1-a(s))} ds\right),$$

where $\varphi \in ST(\alpha)$ and $a \in H(U)$ such that $|a(z)| < 1$ for all $z \in U$.

For $|z| = r$, the well known facts

$$\left| \frac{z\varphi'(z)}{\varphi(z)} \right| \leq (1-\alpha) \frac{1+r}{1-r} + \alpha,$$



Starlike log-harmonic Mappings
of Order α

Z. AbdulHadi and Y. Abu Muhanna

Title Page

Contents

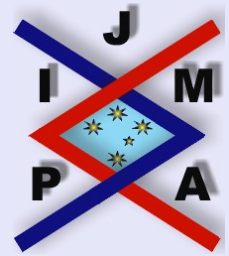


Go Back

Close

Quit

Page 9 of 13



Title Page

Contents



Go Back

Close

Quit

Page 10 of 13

$$\left| \frac{a(z)}{z(1-a(z))} \right| \leq \frac{1}{1-r},$$

and

$$|\varphi(z)| \leq \frac{r}{(1-r)^{2(1-\alpha)}},$$

imply that

$$\begin{aligned} |f(z)| &\leq \frac{r}{(1-r)^{2(1-\alpha)}} \exp \left(2 \int_0^r \frac{1}{1-t} \left((1-\alpha) \frac{1+t}{1-t} + \alpha \right) dt \right) \\ &= \frac{r}{(1-r)^{2\alpha}} \exp \left((1-\alpha) \frac{4r}{1-r} \right). \end{aligned}$$

Equality occurs if and only if, $a(z) = \zeta z$ and $\varphi(z) = \frac{z}{(1-\zeta z)^{2-2\alpha}}$, $|\zeta| = 1$, which leads to $f(z) = \bar{\zeta} f_0(\zeta z)$.

For the left-hand side, we have

$$f(z) = z \exp \left(\int_{\partial U_{x\partial U}} K(z, \zeta, \xi) d\mu(\zeta) d\nu(\xi) \right),$$

where

$$K(z, \zeta, \xi) = (1-\alpha) \log \left(\frac{1+\bar{\zeta}z}{1-\zeta z} \right) + T(z, \zeta, \xi);$$

$$T(z, \zeta, \xi) = \left\{ \begin{array}{ll} -2(1-\alpha) \operatorname{Im} \left(\frac{\zeta+\xi}{\zeta-\xi} \right) \arg \left(\frac{1-\xi z}{1-\zeta z} \right) \\ \quad -2\alpha \log |1-\xi z|; & \text{if } |\zeta| = |\xi| = 1, \zeta \neq \xi \\ (1-\alpha) \operatorname{Re} \left(\frac{4\zeta z}{1-\zeta z} \right) - 2\alpha \log |1-\zeta z|; & \text{if } |\zeta| = |\xi| = 1, \zeta = \xi \end{array} \right\}.$$

For $|z| = r$ we have

$$\begin{aligned} \log \left| \frac{f(z)}{z} \right| &= \operatorname{Re} \left(\int_{\partial U_x \partial U} K(z, \zeta, \xi) d\mu(\zeta) d\nu(\xi) \right) \\ &\geq \min_{\mu, \nu} \left[\min_{|z|=r} \operatorname{Re} \left(\int_{\partial U_x \partial U} K(z, \zeta, \xi) d\mu(\zeta) d\nu(\xi) \right) \right] \\ &\geq \log \frac{1}{|1+r|^{2\alpha}} + \min_{\mu, \nu} \left[\min_{|z|=r} \left(\int_{\partial U_x \partial U} -2(1-\alpha) \operatorname{Im} \left(\frac{\zeta + \xi}{\zeta - \xi} \right) \right. \right. \\ &\quad \left. \left. \times \arg \left(\frac{1 - \xi z}{1 - \zeta z} \right) d\mu(\zeta) d\nu(\xi) \right) \right] \\ &= \log \frac{1}{|1+r|^{2\alpha}} + \min_{0 < |l| \leq \frac{\pi}{2}} \left[\min_{|z|=r} \left(-2(1-\alpha) \operatorname{Im} \left(\frac{1 + e^{2il}}{1 - e^{2il}} \right) \right. \right. \\ &\quad \left. \left. \times \arg \left(\frac{1 - e^{2il} z}{1 - z} \right); (1-\alpha) \frac{-4r}{1+r} \right) \right], \end{aligned}$$

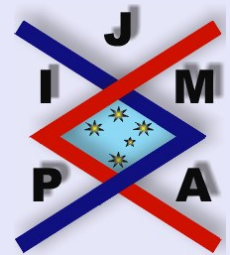
where $e^{2il} = \bar{\zeta}\xi$.

Let

$$\Phi_r(l) = \left\{ \begin{array}{ll} \min_{|z|=r} \left(-2(1-\alpha) \operatorname{Im} \left(\frac{1+e^{2il}}{1-e^{2il}} \right) \arg \left(\frac{1-e^{2il}z}{1-z} \right) \right), & \text{if } 0 < |l| < \frac{\pi}{2} \\ (1-\alpha) \frac{-4r}{1+r} & \text{if } l = 0 \end{array} \right\}.$$

Then $\Phi_r(l)$ is a continuous and even function on $|l| < \frac{\pi}{2}$. Hence

$$\log \left| \frac{f(z)}{z} \right| \geq \log \frac{1}{|1+r|^{2\alpha}} + \min_{0 < |l| \leq \frac{\pi}{2}} \Phi_r(l) = \log \frac{1}{|1+r|^{2\alpha}} + \inf_{0 < l < \frac{\pi}{2}} \Phi_r(l).$$



**Starlike log-harmonic Mappings
of Order α**

Z. AbdulHadi and Y. Abu Muhanna

Title Page

Contents



Go Back

Close

Quit

Page 11 of 13

Since

$$\max_{|z|=r} \arg \left(\frac{1 - e^{2il}z}{1 - z} \right) = 2 \arctan \left(\frac{r \sin(l)}{1 + r \cos(l)} \right),$$

we get

$$\log \left| \frac{f(z)}{z} \right| \geq \log \frac{1}{|1+r|^{2\alpha}} + \inf_{0 < l < \frac{\pi}{2}} \left[-4(1-\alpha) \cot(l) \arctan \left(\frac{r \sin(l)}{1 + r \cos(l)} \right) \right],$$

and using the fact that $|\arctan(x)| \leq |x|$, we have

$$\begin{aligned} \log \left| \frac{f(z)}{z} \right| &\geq \log \frac{1}{|1+r|^{2\alpha}} + \inf_{0 < l < \frac{\pi}{2}} \left[-4(1-\alpha) \frac{r \cos(l)}{1 + r \cos(l)} \right] \\ &\geq \log \frac{1}{|1+r|^{2\alpha}} + \inf_{0 < l < \frac{\pi}{2}} \left[-4(1-\alpha) \frac{r}{1+r} \right]. \end{aligned}$$

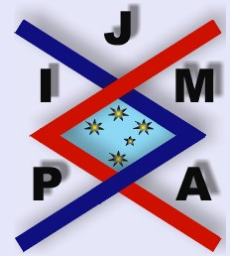
The case of equality is attained by the functions $f(z) = \bar{\zeta} f_0(\zeta z)$, $|\zeta| = 1$. \square

The next application is a consequence of Theorem 2.1.

Theorem 3.2. Let $f(z) = zh(z)\overline{g(z)} \in ST_{Lh}(\alpha)$. Then

$$\left| \arg \frac{f(z)}{z} \right| \leq 2(1-\alpha) \arcsin(|z|).$$

Proof. Let $f(z) = zh(z)\overline{g(z)} \in ST_{Lh}(\alpha)$. Then $\varphi(z) = \frac{zh(z)}{g(z)} \in ST(\alpha)$ by Theorem 2.1. The result follows immediately from $\arg \frac{f(z)}{z} = \arg \frac{\varphi(z)}{z}$ and from [6, p. 142]. \square



Title Page

Contents



Go Back

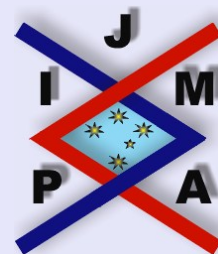
Close

Quit

Page 12 of 13

References

- [1] Z. ABDULHADI, Close-to-starlike logharmonic mappings, *Internat. J. Math. & Math. Sci.*, **19**(3) (1996), 563–574.
- [2] Z. ABDULHADI, Typically real logharmonic mappings, *Internat. J. Math. & Math. Sci.*, **31**(1) (2002), 1–9.
- [3] Z. ABDULHADI AND D. BSHOUTY, Univalent functions in $H\overline{H}$, *Tran. Amer. Math. Soc.*, **305**(2) (1988), 841–849.
- [4] Z. ABDULHADI AND W. HENGARTNER, Spirallike logharmonic mappings, *Complex Variables Theory Appl.*, **9**(2-3) (1987), 121–130.
- [5] Z. ABDULHADI AND W. HENGARTNER, One pointed univalent logharmonic mappings, *J. Math. Anal. Appl.*, **203**(2) (1996), 333–351.
- [6] A.W. GOODMAN, *Univalent functions, Vol. I*, Mariner Publishing Company, Inc., Washington, New Jersey, 1983.
- [7] J.C.C. NITSCHKE, *Lectures on Minimal Surfaces*, Vol. I, New York, 1989.



Starlike log-harmonic Mappings
of Order α

Z. AbdulHadi and Y. Abu Muhanna

Title Page

Contents



Go Back

Close

Quit

Page 13 of 13