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MAXIMAL OPERATORS OF FEJÉR MEANS OF VILENKIN-FOURIER SERIES

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ABSTRACT. The main aim of this paper is to prove that the maximal operator $\sigma^* := \sup_n |\sigma_n|$ of the Fejér means of the Vilenkin-Fourier series is not bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$.

Key words and phrases: Vilenkin system, Hardy space, Maximal operator.

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Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \dots, m_k - 1\}$ the additive group of integers modulo m_k.

Define the group G_m as the complete direct product of the groups Z_{m_j}, with the product of the discrete topologies of Z_{m_j}'s.

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The direct product μ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If the sequence m is bounded, then G_m is called a bounded Vilenkin group, else it is called an unbounded one. The elements of G_m can be represented by sequences $x := (x_0, x_1, \dots, x_j, \dots)$ $(x_j \in Z_{m_j})$. It is easy to give a base for the neighborhoods of G_m :

$$I_0(x) := G_m,$$
$$I_n(x) := \{y \in G_m | y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$$

for $x \in G_m,\ n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{N}_+$.

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \qquad M_{k+1} := m_k M_k (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of n_j's differ from zero. We use the following notations. Let (for $n > 0$) $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$), $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$ and $n_{(k)} := n - n^{(k)}$.

Denote by $L_p(G_m)$ the usual (one dimensional) Lebesgue spaces ($\| \cdot \|_p$ the corresponding norms) $(1 \leq p \leq \infty)$.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first define the complex valued functions $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions as

$$r_k(x) := \exp \frac{2\pi \imath x_k}{m_k} \qquad (\imath^2 = -1,\ x \in G_m,\ k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin system is orthonormal and complete in $L_1(G_m)$ [9].

Now, we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can establish the following definitions in the usual manner:

(Fourier coefficients) $$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu \qquad (k \in \mathbb{N}),$$

(Partial sums) $$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k \qquad (n \in \mathbb{N}_+,\ S_0 f := 0),$$

(Fejér means) $$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_n f \qquad (n \in \mathbb{N}_+),$$

(Dirichlet kernels) $$D_n := \sum_{k=0}^{n-1} \psi_k \qquad (n \in \mathbb{N}_+).$$

Recall that

$$D_{M_n}(x)=\begin{cases} M_n, & \text{if } x\in I_n,\\ 0, & \text{if } x\in G_m\backslash I_n.\end{cases} \tag{1}$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p:=\left(\int_{G_m}|f(x)|^p\mu(x)\right)^{\frac{1}{p}} \qquad (0<p<+\infty).$$

The space weak-$L_p(G_m)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_p(G_m)}:=\sup_{\lambda>0}\lambda\mu(|f|>\lambda)^{\frac{1}{p}}<+\infty.$$

The σ-algebra generated by the intervals $\{I_n(x):(x)\in G_m\}$ will be denoted by $\mathcal{F}_n\ (n\in\mathbb{N})$.

Denote by $f=\left(f^{(n)},n\in\mathbb{N}\right)$ a martingale with respect to $(\mathcal{F}_n,\ n\in\mathbb{N})$ (for details see, e. g. [10, 14]).

The maximal function of a martingale f is defined by

$$f^*=\sup_{n\in\mathbb{N}}\left|f^{(n)}\right|,$$

respectively.

In case $f\in L_1(G_m)$, the maximal functions are also be given by

$$f^*(x)=\sup_{n\in\mathbb{N}}\frac{1}{\mu(I_n(x))}\left|\int_{I_n(x)}f(u)\mu(u)\right|.$$

For $0<p<\infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p}:=\|f^*\|_p<\infty.$$

If $f\in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f):n\in\mathbb{N})$ is a martingale. If f is a martingale, that is $f=(f^{(n)}:n\in\mathbb{N})$, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i)=\lim_{k\to\infty}\int_{G_m}f^{(k)}(x)\overline{\psi}_i(x)\mu(x).$$

The Vilenkin-Fourier coefficients of $f\in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f):n\in\mathbb{N})$ obtained from f.

For a martingale f the maximal operators of the Fejér means are defined by

$$\sigma^*f(x)=\sup_{n\in\mathbb{N}}|\sigma_n(f;x)|.$$

In this one-dimensional case the weak type inequality

$$\mu(\sigma^*f>\lambda)\le\frac{c}{\lambda}\|f\|_1 \quad (\lambda>0)$$

can be found in Zygmund [16] for the trigonometric series, in Schipp [6] for Walsh series and in Pál, Simon [5] for bounded Vilenkin series. Again in one-dimension, Fujji [3] and Simon [8] verified that σ^* is bounded from H_1 to L_1. Weisz [11, 13] generalized this result and proved the boundedness of σ^* from the martingale Hardy space H_p to the space L_p for $p>1/2$. Simon [7] gave a counterexample, which shows that this boundedness does not hold for $0<p<1/2$. In the endpoint case $p=1/2$ Weisz [15] proved that σ^* is bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$. By interpolation it follows that σ^* is not bounded from H_p to the space weak-L_p for all $0<p<1/2$.

Theorem 1. *For any bounded Vilenkin system the maximal operator* σ^* *of the Fejér means is not bounded from the Hardy space* $H_{1/2}$ *to the space* $L_{1/2}$.

The Fejér kernel of order n of the Vilenkin-Fourier series is defined by

$$K_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(x).$$

In order to prove the theorem we need the following lemmas.

Lemma 2 ([4]). *Suppose that* $s,t,n \in \mathbb{N}$ *and* $x \in I_t \backslash I_{t+1}$. *If* $t \leq s \leq |n|$, *then*

$$\begin{aligned}(n^{(s+1)} + M_s)K_{n^{(s+1)}+M_s}(x) &- n^{(s+1)}K_{n^{(s+1)}} \\ &= \begin{cases} M_t M_s \psi_{n^{(s+1)}}(x)\frac{1}{1-r_t(x)}, & \textit{if } x - x_t e_t \in I_s, \\ 0, & \textit{otherwise.}\end{cases}\end{aligned}$$

Lemma 3 ([2]). *Let* $2 < A \in \mathbb{N}_+$, $k \leq s < A$ *and* $n_A^* := M_{2A} + M_{2A-2} + \cdots + M_2 + M_0$. *Then*

$$n_{A-1}^*\left|K_{n_{A-1}^*}(x)\right| \geq \frac{M_{2k}M_{2s}}{4}$$

for

$$\begin{gathered} x \in I_{2A}(0,\ldots,0,x_{2k}\neq 0,0,\ldots,0,x_{2s}\neq 0,x_{2s+1},\ldots,x_{2A-1}), \\ k = 0,1,\ldots,A-3, \qquad s = k+2,k+3,\ldots,A-1. \end{gathered}$$

Proof of Theorem 1. Let $A \in \mathbb{N}_+$ and

$$f_A(x) := D_{M_{2A+1}}(x) - D_{M_{2A}}(x).$$

In the sequel we are going to prove for the function f_A that

$$\frac{\|\sigma^* f_A\|_{1/2}}{\|f_A\|_{H_{1/2}}} \geq c\log_q^2 A,$$

where $q = \sup\{m_0, m_1, \ldots\}$ and constant c depends only on q. This inequality obviously would show the unboundedness of σ^*.

It is evident that

$$\widehat{f}_A(i) = \begin{cases} 1, & \text{if } i = M_{2A},\ldots,M_{2A+1}-1, \\ 0, & \text{otherwise.}\end{cases}$$

Then we can write

$$S_i(f_A;x) = \begin{cases} D_i(x) - D_{M_{2A}}(x), & \text{if } i = M_{2A}+1,\ldots,M_{2A+1}-1, \\ f_A(x), & \text{if } i \geq M_{2A+1}, \\ 0, & \text{otherwise.}\end{cases} \tag{2}$$

Since

$$f_A^*(x) = \sup_{n\in\mathbb{N}}|S_{M_n}(f_A;x)| = |f_A(x)|,$$

from (1) we get

(3)
$$\begin{aligned}\|f_A\|_{H_{1/2}} &= \|f_A^*\|_{1/2} = \left\|D_{M_{2A+1}} - D_{M_{2A}}\right\|_{1/2}\\ &= \left(\int_{I_{2A}\setminus I_{2A+1}} M_{2A}^{\frac{1}{2}} + \int_{I_{2A+1}} |M_{2A+1} - M_{2A}|^{\frac{1}{2}}\right)^2\\ &= \left(\frac{m_{2A}-1}{M_{2A+1}} M_{2A}^{\frac{1}{2}} + \frac{(m_{2A}-1)^{\frac{1}{2}}}{M_{2A+1}} M_{2A}^{\frac{1}{2}}\right)^2\\ &\le 2^2 m_{2A} M_{2A}^{-1}\\ &\le c M_{2A}^{-1}.\end{aligned}$$

Since

$$D_{k+M_{2A}} - D_{M_{2A}} = \psi_{M_{2A}} D_k, \qquad k = 1, 2, \ldots, M_{2A},$$

from (2) we obtain

(4)
$$\begin{aligned}\sigma^* f_A(x) &= \sup_{n\in\mathbb{N}} |\sigma_n(f_A; x)| \ge \left|\sigma_{n_A^*}(f_A; x)\right|\\ &= \frac{1}{n_A^*}\left|\sum_{i=0}^{n_A^*-1} S_i(f_A; x)\right|\\ &= \frac{1}{n_A^*}\left|\sum_{i=M_{2A}+1}^{n_A^*-1} (D_i(x) - D_{M_{2A}}(x))\right|\\ &= \frac{1}{n_A^*}\left|\sum_{i=1}^{n_{A-1}^*-1} (D_{i+M_{2A}}(x) - D_{M_{2A}}(x))\right|\\ &= \frac{n_{A-1}^*}{n_A^*}\left|K_{n_{A-1}^*}(x)\right|.\end{aligned}$$

Let $q := \sup\{m_i : i \in\}$. For every $l = 1, \ldots, \left[\frac{1}{4}\log_q \sqrt{A}\right] - 1$ (A is supposed to be large enough) let k_l be the smallest natural numbers, for which

$$M_{2A}\sqrt{A}\frac{1}{q^{4l}} \le M_{2k_l}^2 < M_{2A}\sqrt{A}\frac{1}{q^{4l-4}}$$

hold.

Denote

$$I_{2A}^{k,s}(x) := I_{2A}(0, \ldots, 0, x_{2k} \ne 0, 0, \ldots, 0, x_{2s} \ne 0, x_{2s+1}, \ldots, x_{2A-1})$$

and let

$$x \in I_{2A}^{k_l, k_l+1}(z)$$

Then from Lemma 3 and (4) we obtain that

$$\sigma^* f_A(x) \ge c\frac{M_{2k_l}^2}{M_{2A}} \ge c\frac{\sqrt{A}}{q^{4l}}$$

On the other hand,

$$\begin{aligned}
\sqrt{\|\sigma^* f_A\|_{1/2}} &\geq c \sum_{l=1}^{\left[\frac{1}{4}\log_q \sqrt{A}\right]} \sum_{x_{2k_l+3}=0}^{m_{2k_l+3}-1} \cdots \sum_{x_{2A-1}=0}^{m_{2A-1}-1} \frac{\sqrt[4]{A}}{q^{2l}} \mu\left(I_{2A}^{k_l,k_l+1}(x)\right) \\
&\geq c\sqrt[4]{A} \sum_{l=1}^{\left[\frac{1}{4}\log_q \sqrt{A}\right]} \frac{m_{2k_l+3}\cdots m_{2A-1}}{q^{2l} M_{2A}} \\
&= c\sqrt[4]{A} \sum_{l=1}^{\left[\frac{1}{4}\log_q \sqrt{A}\right]} \frac{1}{q^{2l} M_{2k_l+2}} \\
&\geq c\sqrt[4]{A} \sum_{l=1}^{\left[\frac{1}{4}\log_q \sqrt{A}\right]} \frac{1}{q^{2l} M_{2k_l}} \\
&\geq c\sqrt[4]{A} \sum_{l=1}^{\left[\frac{1}{4}\log_q \sqrt{A}\right]} \frac{1}{q^{2l}\sqrt{M_{2A}\sqrt{A}q^{-4l+4}}} \\
&\geq c\frac{\log_q A}{\sqrt{M_{2A}}}.
\end{aligned}$$

Combining this with (3) we obtain

$$\frac{\|\sigma^* f_A\|_{1/2}}{\|f_A\|_{H_{1/2}}} \geq \frac{c\log_q^2 A}{M_{2A}} M_{2A} = c\log_q^2 A \to \infty \qquad \text{as} \quad A \to \infty.$$

Thus, the theorem is proved. □

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