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ON A UNIFORMLY INTEGRABLE FAMILY OF POLYNOMIALS DEFINED ON THE UNIT INTERVAL

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ABSTRACT. In this short note, we establish the uniform integrability and pointwise convergence of an (unbounded) family of polynomials on the unit interval that arises in work on statistical density estimation using Bernstein polynomials. These results are proved by first establishing/generalizing some combinatorial and probability inequalities that rely on a new family of completely monotonic functions.

Key words and phrases: Uniform integrability, Bernstein polynomials, Probability inequalities, Combinatorial inequalities, Completely monotonic functions.

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1. INTRODUCTION

Let $P_{n,k}(x)\colon [0,1] \to [0,1]$ denote the probability of exactly k successes in n independent Bernoulli trials with success probability x, i.e.

$$P_{n,k}(x) = \Pr\{\text{Bin}(n,x) = k\} = \binom{n}{k} x^k (1-x)^{n-k},$$

and, for integers $r, s \geq 1$, define the family of functions $\{S_{n,r,s}\}_{n=1}^{\infty}$ by

$$S_{n,r,s}(x) := \sqrt{n} \sum_{k=0}^{n} P_{rn,rk}(x) P_{sn,sk}(x). \tag{1.1}$$

This family of polynomials arises in the context of statistical density estimation based on Bernstein polynomials. Specifically, the case $r = s = 1$ has been considered by many authors (for example, Babu *et al.* [3], Kakizawa [5] and Vitale [8]) and the case $r = 1$ and $s = 2$ has been considered by Leblanc [6]. Issues linked to uniform integrability and pointwise convergence of $\{S_{n,1,1}\}$ and $\{S_{n,1,2}\}$ have also been addressed by these authors. However, the generalization to

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any $r, s \geq 1$ has not yet been considered. In the present paper we will establish the following result.

Theorem 1.1. *Let r, s be fixed positive integers. Then*

(i) $0 \leq S_{n,r,s}(x) \leq \sqrt{n}$ *for* $x \in [0,1]$ *and* $S_{n,r,s}(0) = S_{n,r,s}(1) = \sqrt{n}$.
(ii) $\{S_{n,r,s}\}_{n=0}^{\infty}$ *is uniformly integrable (w.r.t. Lebesgue measure) on* $[0,1]$.
(iii) $S_{n,r,s}(x) \to \gcd(r,s)[rs(r+s)2\pi x(1-x)]^{-1/2}$ *for* $x \in (0,1)$ *as* $n \to \infty$.

For the case $r = s = 1$, Babu *et al.* [3, Lemma 3.1] contains a proof of (iii). Leblanc [6, Lemma 3] gives a proof of Theorem 1.1 when $r = 1$ and $s = 2$. The proof herein generalizes (but follows along the same lines as) these previous results. As an application of Theorem 1.1 we have, for any function f that is bounded on $[0,1]$,

$$\lim_{n\to\infty} \int_0^1 S_{n,r,s}(x) f(x)\, dx = \frac{\gcd(r,s)}{\sqrt{rs(r+s)}} \int_0^1 \frac{f(x)}{\sqrt{2\pi x(1-x)}}\, dx, \tag{1.2}$$

the latter integral generally being easier to evaluate (or approximate). This simple consequence of Theorem 1.1 plays an important role in assessing the performance of nonparametric density estimators based on Bernstein polynomials. Kakizawa [5], for example, went to great lengths to establish (1.2) for the case $r = s = 1$.

In establishing Theorem 1.1, we first show that, for all $0 \leq k \leq n$ and $x \in [0,1]$, (see Corollary 2.4)

$$P_{n,k}(x) \geq P_{2n,2k}(x) \geq P_{3n,3k}(x) \geq \cdots . \tag{1.3}$$

The proof of this inequality is based on a class of completely monotonic functions and hence is of general interest. Using completely different methods, Leblanc and Johnson [7] previously showed that $\{P_{2^j n, 2^j k}(x)\}_{j=0}^{\infty}$ is decreasing in j and hence (1.3) is a generalization of this earlier result.

The remainder of this paper is organized as follows. In Section 2 we introduce a new family of completely monotonic functions and obtain some necessary combinatorial and probability inequalities. In Section 3, we prove Theorem 1.1. Finally, in Section 4, we highlight the fact that the results in Section 2 can be used to obtain other interesting inequalities.

2. PRELIMINARY RESULTS

Recall that a real valued function f is said to be completely monotonic on (a,b) if and only if $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in (a,b)$ and integers $n \geq 0$ (cf. Feller [4, XIII.4]). We begin with the following lemma.

Lemma 2.1. *Let $\{a_k\}_{k=1}^m$ and $\{b_k\}_{k=1}^m$ be real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_m \geq 0$ and let ψ denote the digamma function. Define*

$$\phi_\delta(x) := \sum_{k=1}^m a_k \psi(b_k x + \delta), \qquad x > 0, \quad \delta \geq 0.$$

If $\delta \geq 1/2$ and $\sum_{k=1}^m a_k \geq 0$, then ϕ'_δ is completely monotonic on $(0,\infty)$ and hence ϕ_δ is increasing and concave on $(0,\infty)$.

The proof follows along the same lines as that in Alzer and Berg [2], who show that ϕ_0 is completely monotonic (and hence decreasing and convex) if and only if $\sum a_k = 0$ and $\sum a_k \ln b_k \geq 0$.

Proof. Let $x > 0$ and $\delta \geq 1/2$ and recall that the integral representation of $\psi^{(n)}$ is (*cf.* Abramowitz and Stegun [1, pp. 260])

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}\, dt, \qquad n = 1, 2, \ldots .$$

Therefore, for $n = 1, 2, \ldots$,

$$(-1)^{n+1}\phi_\delta^{(n)}(x) = (-1)^{n+1} \sum_{k=1}^m a_k b_k^n \psi^{(n)}(b_k x + \delta) = \sum_{k=1}^m a_k \int_0^\infty \frac{(b_k t)^n e^{-x b_k t}}{e^{\delta t}(1-e^{-t})}\, dt. \tag{2.1}$$

The substitution(s) $u = b_k t$ yield

$$(-1)^{n+1}\phi_\delta^{(n)}(x) = \int_0^\infty u^{n-1} e^{-ux} \sum_{k=1}^m a_k \eta(u/b_k)\, du, \tag{2.2}$$

where $\eta(x) = xe^{-\delta x}(1-e^{-x})^{-1} > 0$. A little calculus shows that, for $\delta \geq 1/2$, η is strictly decreasing on $(0,\infty)$ and hence, for every $u > 0$, $\{\eta(u/b_k)\}_{k=1}^m$ is decreasing [note that, if $b_k = 0$, there is no difficulty in taking $\eta(u/b_k) = \eta(\infty) = \lim_{x\to\infty} \eta(x) = 0$, since these terms vanish in (2.1)]. Since $\{a_k\}_{k=1}^m$ is also decreasing, Chebyshev's inequality for sums yields

$$\sum_{k=1}^m a_k \eta(u/b_k) \geq \frac{1}{m} \left(\sum_{k=1}^m a_k \right) \left(\sum_{k=1}^m \eta(u/b_k) \right).$$

We see that, if $\sum_{k=1}^m a_k \geq 0$, the integrand in (2.2) is non-negative and hence $(-1)^{n+1}\phi_\delta^{(n)} \geq 0$ on $(0,\infty)$. We conclude that ϕ_δ' is completely monotonic on $(0,\infty)$ and, in particular, ϕ_δ is increasing and concave on $(0,\infty)$ whenever $\delta \geq 1/2$ and $\sum a_k \geq 0$. ☐

Lemma 2.2. *Let* n, k, j *be integers such that* $0 \leq k \leq n$ *and* $j \geq 1$ *and define*

$$Q_{n,k}(j) = \frac{\binom{(j-1)n}{(j-1)k}}{\binom{jn}{jk}} = \frac{\Gamma((j-1)n+1)\Gamma(jk+1)\Gamma(j(n-k)+1)}{\Gamma(jn+1)\Gamma((j-1)k+1)\Gamma((j-1)(n-k)+1)}.$$

Then $Q_{n,k}(j)$ *is decreasing in* j *and*

$$\lim_{j\to\infty} Q_{n,k}(j) = \left(\frac{k}{n}\right)^k \left(\frac{n-k}{n}\right)^{n-k}.$$

Proof. The limit is easily verified using Stirling's formula, thus we need only show that $Q_{n,k}(j)$ is decreasing in j. Treating $Q_{n,k}(j)$ as a continuous function in j and differentiating we obtain

$$Q_{n,k}'(j) = Q_{n,k}(j) \left\{ k\Big(q_j(k) - q_j(n)\Big) + (n-k)\Big(q_j(n-k) - q_j(n)\Big) \right\},$$

where $q_j(x) = \psi(jx+1) - \psi(jx - x + 1)$. Now, taking $\delta = 1$, $a_1 = 1$, $a_2 = -1$, $b_1 = j$ and $b_2 = j-1$ in Lemma 2.1, we have that $q_j(x)$ is increasing on $(0,\infty)$ and hence $Q_{n,k}'(j) \leq 0$ for all $j \geq 1$ since $Q_{n,k}(j) > 0$ always. ☐

Remark 2.3. In light of Lemma 2.1, we may define, for $j \geq 1$ and $\delta > 0$,

$$Q_{n,k,\delta}(j) = \frac{\Gamma((j-1)n+\delta)}{\Gamma((j-1)n+\delta)\,\Gamma((j-1)k+\delta)} \Bigg/ \frac{\Gamma(jn+\delta)}{\Gamma(jk+\delta)\,\Gamma(j(n-k)+\delta)}.$$

The same arguments in the proof of Theorem 2.2 show that $Q_{n,k,\delta}(j)$ is decreasing in j for all $\delta \geq 1/2$ and has the same limiting value of $(k/n)^k(1-k/n)^{n-k}$.

Corollary 2.4. *Let* $0 \le k \le n$*. Then* $\{P_{jn,jk}(x)\}_{j=1}^{\infty}$ *is decreasing in* j *for every fixed* $x \in [0,1]$*.*

Proof. $P_{(j-1)n,(j-1)k}(x) \ge P_{jn,jk}(x)$ if and only if $Q_{n,k}(j) \ge x^k(1-x)^{n-k}$ and we have, by Lemma 2.2,

$$Q_{n,k}(j) \ge (k/n)^k(1-k/n)^{n-k} = \sup_{x\in[0,1]} x^k(1-x)^{n-k},$$

which completes the proof. □

3. Proof of Theorem 1.1

We now give a proof of Theorem 1.1. First note that (i) holds since

$$\sum_{k=0}^{n} P_{rn,rk}(x)P_{sn,sk}(x) \le \sum_{k=0}^{n} P_{rn,rk}(x) \le \sum_{k=0}^{rn} P_{rn,k}(x) = 1,$$

with equality if and only if $x = 0, 1$. Similarly, (ii) holds since $\{S_{n,1,1}\}_{n=1}^{\infty}$ is uniformly integrable on $[0,1]$ (cf. [6]) and, by Corollary 2.4, we have $S_{n,r,s}(x) \le S_{n,1,1}(x)$ for all $x \in [0,1]$.

To prove (iii), let $U_1, \dots, U_n$ and $V_1, \dots, V_n$ be two sequences of independent random variables such that U_i is Binomial(r,x) and V_i is Binomial(s,x). Now, define $W_i = r^{-1}U_i - s^{-1}V_i$ so that W_i has a lattice distribution with span $\gcd(r,s)/rs$ (*cf.* Feller [4]). We can write $S_{n,r,s}(x)$ in terms of the W_i as

$$\frac{S_{n,r,s}(x)}{\sqrt{n}} = \sum_{k=0}^{n} P_{rn,rk}(x)P_{sn,sk}(x) = \mathbb{P}\left(\sum_{i=1}^{n}\frac{U_i}{r} = \sum_{i=1}^{n}\frac{V_i}{s}\right) = \mathbb{P}\left(\sum_{i=1}^{n} W_i = 0\right).$$

Now, define the standardized variables $W_i^* = W_i\sqrt{rs}/\sqrt{(r+s)x(1-x)}$ so that $\mathbb{V}\mathrm{ar}(W_i^*) = 1$ and note that these also have a lattice distribution, but with span $\gcd(r,s)/\sqrt{rs(r+s)x(1-x)}$. Theorem 3 of Section XV.5 of Feller [4] now leads to

$$\lim_{n\to\infty}\frac{S_{n,r,s}(x)}{\sqrt{n}} = \lim_{n\to\infty}\mathbb{P}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{m} W_i^* = 0\right) = \frac{\gcd(r,s)\phi(0)}{\sqrt{nrs(r+s)x(1-x)}},$$

where ϕ corresponds to the standard normal probability density function. The result now follows from the fact that $\phi(0) = 1/\sqrt{2\pi}$. □

4. Concluding Comments

We conclude by pointing out the fact that Lemma 2.2 also leads to some other interesting combinatorial and discrete probability inequalities. For example, since $Q_{n,k}(j)$ is decreasing, we immediately obtain

$$\binom{(j-1)n}{(j-1)k}\binom{(j+1)n}{(j+1)k} \ge \binom{jn}{jk}^2.$$

Indeed, since $Q_{n,k}(j-m+1) \ge Q_{n,j}(j+m)$ for $m = 1, \dots, j$, we see that the sequence $\{A_m\}_{m=1}^{j}$ defined by

$$A_m = \binom{(j+m)n}{(j+m)k}\binom{(j-m)n}{(j-m)k} \tag{4.1}$$

is increasing.

Finally, Corollary 2.4 trivially leads to a similar family of inequalities for "number of failure" negative binomial probabilities. Let $H_{n,k}$ be the probability of exactly n failures $(n \ge 0)$ before the kth success $(k \geq 1)$ in a sequence of i.i.d. Bernoulli trials with success probability $p \in [0,1]$ so that, for $j = 1, 2, \ldots$,

$$H_{jn,jk} = \binom{jn+jk-1}{jk-1} p^{jk}(1-p)^{jn} = \frac{k}{n+k} P_{j(n+k),jk}.$$

Hence, as a direct consequence of Corollary 2.4, we have that $\{H_{jn,jk}\}_{j=1}^{\infty}$ is also decreasing.

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