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ON CERTAIN CLASSES OF MULTIVALENT ANALYTIC FUNCTIONS

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ABSTRACT. In this paper we introduce the class $\mathcal{B}(p, n, \mu, \alpha)$ of analytic and p-valent functions to obtain some sufficient conditions and some angular properties for functions belonging to this class.

Key words and phrases: Analytic and p-valent functions, p-valent starlike functions and p-valent convex functions, Strongly starlike and strongly convex functions.

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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}(p, n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \qquad (p, n \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. In particular, we set $\mathcal{A}(1, 1) =: \mathcal{A}$. A function $f(z) \in \mathcal{A}(p, n)$ is said to be in the class $\mathcal{S}^*(p, n, \alpha)$ of p-valently starlike of order α in $\mathcal{U}$ if and only if it satisfies the inequality

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad (z \in \mathcal{U}; 0 \le \alpha < p). \tag{1.2}$$

On the other hand, a function $f(z) \in \mathcal{A}(p, n)$ is said to be in the class $\mathcal{K}(p, n, \alpha)$ of p-valently convex of order α in $\mathcal{U}$ if and only if it satisfies the inequality

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \qquad (z \in \mathcal{U}; 0 \le \alpha < p). \tag{1.3}$$

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Furthermore, a function $f(z) \in \mathcal{A}(p,n)$ is said to be in the class $\mathcal{C}(p,n,\alpha)$ of p-valently close-to-convex of order α in $\mathcal{U}$ if and only if it satisfies the inequality

$$\operatorname{Re}\left(\frac{f'(z)}{z^{p-1}}\right) > \alpha \qquad (z \in \mathcal{U}; 0 \leq \alpha < p). \tag{1.4}$$

In particular, we write $\mathcal{S}^*(1,1,0) =: \mathcal{S}^*$, $\mathcal{K}(1,1,0) =: \mathcal{K}$ and $\mathcal{C}(1,1,0) =: \mathcal{C}$, where $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ are the usual subclasses of $\mathcal{A}$ consisting of functions which are starlike, convex and close-to-convex, respectively.

Let $\overline{\mathcal{S}}^*(p,n,\alpha_1,\alpha_2)$ be the subclass of $\mathcal{A}(p,n)$ which satisfies

$$-\frac{\pi\alpha_1}{2} < \arg\frac{zf'(z)}{f(z)} < \frac{\pi\alpha_2}{2}, \qquad (z \in \mathcal{U}; 0 < \alpha_1, \alpha_2 \leq p), \tag{1.5}$$

and let $\overline{\mathcal{K}}(p,n,\alpha_1,\alpha_2)$ be the subclass of $\mathcal{A}(p,n)$ which satisfies

$$-\frac{\pi\alpha_1}{2} < \arg\left(1+\frac{zf''(z)}{f'(z)}\right) < \frac{\pi\alpha_2}{2}, \qquad (z \in \mathcal{U}; 0 < \alpha_1, \alpha_2 \leq p). \tag{1.6}$$

We note that $\overline{\mathcal{S}}^*(1,1,\alpha_1,\alpha_2) =: \mathcal{S}^*(\alpha_1,\alpha_2)$, $\overline{\mathcal{K}}(1,1,\alpha_1,\alpha_2) =: \mathcal{K}(\alpha_1,\alpha_2)$, where $\mathcal{S}^*(\alpha_1,\alpha_2)$ and $\mathcal{K}(\alpha_1,\alpha_2)$ are the subclasses of $\mathcal{A}$ introduced and studied by Takahashi and Nunokawa [7]. Also, we note that $\overline{\mathcal{S}}^*(1,1,\alpha,\alpha) =: \mathcal{S}^*_{st}(\alpha)$ and $\overline{\mathcal{K}}(1,1,\alpha,\alpha) =: \mathcal{K}_{st}(\alpha)$ where $\mathcal{S}^*_{st}(\alpha)$ and $\mathcal{K}_{st}(\alpha)$ are the familiar classes of strongly starlike functions of order α and strongly convex functions of order α, respectively.

The object of the present paper is to investigate various properties of the following classes of analytic and p-valent functions defined as follows.

Definition 1.1. A function $f(z) \in \mathcal{A}(p,n)$ is said to be a member of the class $\mathcal{B}(p,n,\mu,\alpha)$ if and *only if*

$$\left|\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p}f'(z) - p\right| < p-\alpha, \qquad (p \in \mathbb{N}), \tag{1.7}$$

for some α $(0 \leq \alpha < p)$, $\mu \geq 0$ and for all $z \in \mathcal{U}$.

Note that condition (1.7) implies that

$$\operatorname{Re}\left(\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p}f'(z)\right) > \alpha. \tag{1.8}$$

We note that $\mathcal{B}(p,n,2,\alpha) \equiv \mathcal{S}^*(p,n,\alpha)$, $\mathcal{B}(p,n,1,\alpha) \equiv \mathcal{C}(p,n,\alpha)$. The class $\mathcal{B}(1,1,3,\alpha) \equiv \mathcal{B}(\alpha)$ is the class which has been introduced and studied by Frasin and Darus [3] (see also [1, 2]).

In order to derive our main results, we have to recall the following lemmas.

Lemma 1.1 ([4]). *Let $w(z)$ be analytic in $\mathcal{U}$ and such that $w(0)=0$. Then if $|w(z)|$ attains its maximum value on circle $|z|=r<1$ at a point $z_o \in \mathcal{U}$, we have*

$$z_o w'(z) = kw(z_o), \tag{1.9}$$

where $k \geq 1$ *is a real number.*

Lemma 1.2 ([6]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that $\Phi(z)$ is a mapping from $\mathbb{C}^2 \times \mathcal{U}$ to $\mathbb{C}$ which satisfies $\Phi(ix,y;z) \notin \Omega$ for $z \in \mathcal{U}$, and for all real x,y such that $y \leq -n(1+u_2^2)/2$. If the function $q(z) = 1 + q_n z^n + q_{n+1}z^{n+1} + \cdots$ is analytic in $\mathcal{U}$ such that $\Phi(q(z), zq'(z); z) \in \Omega$ for all $z \in \mathcal{U}$, then $\operatorname{Re} q(z) > 0$.*

Lemma 1.3 ([5]). *Let $q(z)$ be analytic in $\mathcal{U}$ with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in \mathcal{U}$. If there exist two points $z_1, z_2 \in \mathcal{U}$ such that*

$$-\frac{\pi\alpha_1}{2} = \arg q(z_1) < \arg q(z) < \arg q(z_2) = \frac{\pi\alpha_2}{2} \tag{1.10}$$

for $\alpha_1 > 0$, $\alpha_2 > 0$, and for $|z| < |z_1| = |z_2|$, then we have

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{\alpha_1+\alpha_2}{2}\right)m \quad \textit{and} \quad \frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{\alpha_1+\alpha_2}{2}\right)m \tag{1.11}$$

where

$$m \geq \frac{1-|a|}{1+|a|} \quad \textit{and} \quad a = i\tan\frac{\pi}{4}\left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\right). \tag{1.12}$$

2. SUFFICIENT CONDITIONS FOR STARLIKENESS AND CLOSE-TO-CONVEXITY

Making use of Lemma 1.1, we first prove

Theorem 2.1. *If $f(z) \in \mathcal{A}(p,n)$ satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)} + (\mu-1)\left(p - \frac{zf'(z)}{f(z)}\right) + \gamma\left(\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p}f'(z) - p\right)\right| < \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha}, \qquad (z \in \mathcal{U}), \tag{2.1}$$

for some α $(0 \leq \alpha < p)$ and $\mu, \gamma \geq 0$, then $f(z) \in \mathcal{B}(p,n,\mu,\alpha)$.

Proof. Define the function $w(z)$ by

$$\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p}f'(z) = p + (p-\alpha)w(z). \tag{2.2}$$

Then $w(z)$ is analytic in $\mathcal{U}$ and $w(0) = 0$. It follows from (2.2) that

$$1 + \frac{zf''(z)}{f'(z)} - p + (\mu-1)\left(p - \frac{zf'(z)}{f(z)}\right) + \gamma\left(\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p}f'(z) - p\right) = \gamma(p-\alpha)w(z) + \frac{(p-\alpha)zw'(z)}{p+(p-\alpha)w(z)}.$$

Suppose that there exists $z_0 \in \mathcal{U}$ such that

$$\max_{|z|<z_0} |w(z)| = |w(z_0)| = 1. \tag{2.3}$$

Then from Lemma 1.1, we have (1.9). Therefore, letting $w(z_0) = e^{i\theta}$, with $k \geq 1$, we obtain that

$$\begin{aligned} &\left|1 + \frac{z_0 f''(z_0)}{f'(z_0)} + (\mu - 1)\left(p - \frac{z_0 f'(z_0)}{f(z_0)}\right) + \gamma\left(\left(\frac{z_0^p}{f(z_0)}\right)^{\mu-1} z_0^{1-p} f'(z_0) - p\right)\right| \\ &= \left|\gamma(p-\alpha)w(z_0) + \frac{(p-\alpha)zw'(z_0)}{p + (p-\alpha)w(z_0)}\right| \\ &\geq \operatorname{Re}\left\{\gamma(p-\alpha) + \frac{(p-\alpha)k}{p + (p-\alpha)w(z_0)}\right\} \\ &> \gamma(p-\alpha) + \frac{p-\alpha}{2p-\alpha} \\ &= \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha}, \end{aligned}$$

which contradicts our assumption (2.1). Therefore we have $|w(z)| < 1$ in $\mathcal{U}$. Finally, we have

$$\left|\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z) - p\right| = (p-\alpha)|w(z)| < p - \alpha \quad (z \in \mathcal{U}), \tag{2.4}$$

that is, $f(z) \in \mathcal{B}(p, n, \mu, \alpha)$. □

Letting $\mu = 1$ in Theorem 2.1, we obtain

Corollary 2.2. *If* $f(z) \in \mathcal{A}(p, n)$ *satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)} + \gamma(z^{1-p}f'(z) - p)\right| < \frac{(p-\alpha)(\gamma(2p-\alpha)+1)}{2p-\alpha}, \ (z \in \mathcal{U}), \tag{2.5}$$

for some α $(0 \leq \alpha < p)$ *and* $\gamma \geq 0$,*then* $f(z) \in \mathcal{C}(p, n, \alpha)$.

Letting $p = n = 1$ and $\gamma = \alpha = 0$ in Corollary 2.2, we easily obtain

Corollary 2.3. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}, \qquad (z \in \mathcal{U}), \tag{2.6}$$

then $f(z) \in \mathcal{C}$.

Letting $\mu = 2$ and $\gamma = 1$ in Theorem 2.1, we obtain

Corollary 2.4. *If* $f(z) \in \mathcal{A}(p, n)$ *satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < \frac{(p-\alpha)(2p-\alpha+1)}{2p-\alpha} \qquad (0 \leq \alpha < p;\ z \in \mathcal{U}), \tag{2.7}$$

then $f(z) \in \mathcal{S}^*(p, n, \alpha)$.

Letting $p = n = 1$ and $\alpha = 0$ in Corollary 2.4, we easily obtain

Corollary 2.5. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < \frac{3}{2} \qquad (z \in \mathcal{U}), \tag{2.8}$$

then $f(z) \in \mathcal{S}^*$.

Next we prove

Theorem 2.6. *If* $f(z) \in \mathcal{A}(p,n)$ *satisfies*

$$(2.9)\quad \operatorname{Re}\left[\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\right] \times \left\{\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z) + 1 + \frac{zf''(z)}{f'(z)} - (\mu-1)\frac{zf'(z)}{f(z)}\right\} > \delta\left(\delta + \frac{n}{2}\right) + p\left(\delta(2-\mu) - \frac{n}{2}\right),$$

then $f(z) \in \mathcal{B}(p,n,\mu,\delta)$, *where* $0 \le \delta < p$.

Proof. Define the function $q(z)$ by

$$(2.10)\quad \left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z) = \delta + (p-\delta)q(z).$$

Then, we see that $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + \cdots$ is analytic in $\mathcal{U}$. A computation shows that

$$\begin{aligned}
&\left[\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\right]^2 + \left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\left(1 + \frac{zf''(z)}{f'(z)} - (\mu-1)\frac{zf'(z)}{f(z)}\right)\\
&= (p-\delta)zq'(z) + (p-\delta)^2 q^2(z) + (p-\delta)[p(2-\mu) + 2\delta]q(z) + \delta p(2-\mu) + \delta^2\\
&= \Phi(q(z), zq'(z); z),
\end{aligned}$$

where

$$\Phi(r,s;t) = (p-\delta)s + (p-\delta)^2 r^2 + (p-\delta)[p(2-\mu) + 2\delta]r + \delta p(2-\mu) + \delta^2.$$

For all real x, y satisfying $y \le -n(1+x_2^2)/2$, we have

$$\begin{aligned}
\operatorname{Re}\Phi(ix, y; z) &= (p-\delta)y - (p-\delta)^2 x^2 + \delta p(2-\mu) + \delta^2\\
&\le -\frac{n}{2}(p-\delta) - (p-\delta)\left[\frac{n}{2} + p - \delta\right]x^2 + \delta p(2-\mu) + \delta^2\\
&\le \delta p(2-\mu) + \delta^2 - \frac{n}{2}(p-\delta)\\
&= \delta\left(\delta + \frac{n}{2}\right) + p\left(\delta(2-\mu) - \frac{n}{2}\right).
\end{aligned}$$

Let

$$\Omega = \left\{w : \operatorname{Re} w > \delta\left(\delta + \frac{n}{2}\right) + p\left(\delta(2-\mu) - \frac{n}{2}\right)\right\}.$$

Then $\Phi(q(z), zq'(z); z) \in \Omega$ and $\Phi(ix, y; z) \notin \Omega$ for all real x and $y \le -n(1+x_2^2)/2$, $z \in \mathcal{U}$. By using Lemma 1.2, we have $\operatorname{Re} q(z) > 0$, that is, $f(z) \in \mathcal{B}(p,n,\mu,\delta)$. □

Letting $\mu = 1$ in Theorem 2.6, we have the following:

Corollary 2.7. *If* $f(z) \in \mathcal{A}(p,n)$ *satisfies*

$$(2.11)\quad \operatorname{Re}\left\{(z^{1-p}f'(z))^2 + z^{1-p}f'(z) + z^{2-p}f''(z)\right\} > \delta\left(\delta + \frac{n}{2}\right) + p\left(\delta - \frac{n}{2}\right),$$

then $f(z) \in \mathcal{C}(p,n,\delta)$,*where* $0 \le \delta < p$.

Letting $p = n = 1$ and $\delta = 0$ in Corollary 2.7, we easily get

Corollary 2.8. *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left\{(f'(z))^2 + f'(z) + zf''(z)\right\} > -\frac{1}{2}, \tag{2.12}$$

then $f(z) \in \mathcal{C}$.

Letting $\mu = 2$ in Theorem 2.6, we have

Corollary 2.9. *If $f(z) \in \mathcal{A}(p,n)$ satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)}\right) > \delta\left(\delta + \frac{n}{2}\right) - \frac{n}{2}p.$$

then $f(z) \in \mathcal{S}^*(p,n,\delta)$,where $0 \le \delta < p$.

Letting $p = n = 1$ and $\delta = 0$ in Corollary 2.9, we easily get

Corollary 2.10. *If $f(z) \in \mathcal{A}$ satisfies*

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)}\right) > -\frac{1}{2}.$$

then $f(z) \in \mathcal{S}^*$.

3. ARGUMENT PROPERTIES

Theorem 3.1. *Suppose that $\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z) \neq \delta$ for $z \in \mathcal{U}$ and $0 \le \delta < p$. If $f(z) \in \mathcal{A}(p,n)$ satisfies*

$$\begin{aligned}
-\frac{\pi}{2}\alpha_1 - \tan^{-1}&\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right)\\
&< \arg\left\{\left(\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\right)\right.\\
&\qquad \left.\times\left(1 + \frac{zf''(z)}{f'(z)} - p + (\mu-1)\left(p - \frac{zf'(z)}{f(z)}\right) + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta}\right\}\\
&< \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right)
\end{aligned} \tag{3.1}$$

for $\alpha_1, \alpha_2, \gamma > 0$, then

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z) - \delta\right) < \frac{\pi}{2}\alpha_2. \tag{3.2}$$

Proof. Define the function $q(z)$ by

$$q(z) = \frac{1}{p-\delta}\left(\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z) - \delta\right). \tag{3.3}$$

Then we see that $q(z)$ is analytic in $\mathcal{U}$, $q(0) = 1$, and $q(z) \neq 0$ for all $z \in \mathcal{U}$. It follows from (3.3) that

$$\begin{aligned}
\left(\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\right)\left(1 + \frac{zf''(z)}{f'(z)} - p + (\mu-1)\left(p - \frac{zf'(z)}{f(z)}\right) + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta}&\\
= (p-\delta)zq'(z) + \gamma q(z).&
\end{aligned}$$

Suppose that there exists two points $z_1,\, z_2 \in \mathcal{U}$ such that the condition (1.10) is satisfied, then by Lemma 1.3, we obtain (1.11) under the constraint (1.12). Therefore, we have

$$\begin{aligned}\arg(\gamma q(z_1) + (p-\delta) z q'(z_1)) &= \arg q(z_1) + \arg\left(\gamma + (p-\delta)\frac{z_1 q'(z_1)}{q(z_1)}\right)\\ &= -\frac{\pi}{2}\alpha_1 + \arg\left(\gamma - i\frac{(\alpha_1+\alpha_2)(p-\delta)}{2}m\right)\\ &= -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}m\right)\\ &\leq \frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right)\end{aligned}$$

and

$$\arg(\gamma q(z_2) + (p-\delta) z q'(z_2)) \geq \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right),$$

which contradict the assumptions of the theorem. This completes the proof. □

Letting $\mu = 1$ in Theorem 3.1, we have

Corollary 3.2. *Suppose that* $z^{1-p} f'(z) \neq \delta$ *for* $z \in \mathcal{U}$ *and* $0 \leq \delta < p$. *If* $f(z) \in \mathcal{A}(p,n)$ *satisfies*

$$\begin{aligned}-\frac{\pi}{2}\alpha_1 &- \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right)\\ &< \arg\left\{z^{1-p} f'(z)\left(1 + \frac{zf''(z)}{f'(z)} - p + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta}\right\}\\ &< \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right)\end{aligned} \tag{3.4}$$

for $\alpha_1, \alpha_2, \gamma > 0$, *then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(z^{1-p} f'(z) - \delta\right) < \frac{\pi}{2}\alpha_2. \tag{3.5}$$

Letting $\alpha_1 = \alpha_2 = 1$ in Corollary 3.2, we have

Corollary 3.3. *Suppose that* $z^{1-p} f'(z) \neq \delta$ *for* $z \in \mathcal{U}$ *and* $0 \leq \delta < p$. *If* $f(z) \in \mathcal{A}(p,n)$ *satisfies*

$$\left|\arg\left\{z^{1-p} f'(z)\left(1 + \frac{zf''(z)}{f'(z)} - p + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta}\right\}\right| < \frac{\pi}{2} + \tan^{-1}\left(\frac{p-\delta}{\gamma}\right) \tag{3.6}$$

for $\gamma > 0$, *then* $f(z) \in \mathcal{C}(p,n,\delta)$.

Letting $\mu = 2$ in Theorem 3.1, we have

Corollary 3.4. *Suppose that* $zf'(z)/f(z) \neq \delta$ *for* $z \in \mathcal{U}$ *and* $0 \leq \delta < p$. *If* $f(z) \in \mathcal{A}(p,n)$ *satisfies*

$$\begin{aligned}-\frac{\pi}{2}\alpha_1 &- \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right)\\ &< \arg\left\{\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta}\right\}\\ &< \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{1-|a|}{1+|a|}\frac{(\alpha_1+\alpha_2)(p-\delta)}{2\gamma}\right)\end{aligned} \tag{3.7}$$

for $\alpha_1, \alpha_2, \gamma > 0$, *then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(\frac{zf'(z)}{f(z)} - \delta\right) < \frac{\pi}{2}\alpha_2 \tag{3.8}$$

Letting $\alpha_1 = \alpha_2 = 1$ in Corollary 3.4, we have

Corollary 3.5. *Suppose that* $zf'(z)/f(z) \neq \delta$ *for* $z \in \mathcal{U}$ *and* $0 \leq \delta < p$. *If* $f(z) \in \mathcal{A}(p,n)$ *satisfies*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{\gamma}{p-\delta}\right) - \frac{\gamma\delta}{p-\delta}\right\}\right| < \frac{\pi}{2} + \tan^{-1}\left(\frac{p-\delta}{\gamma}\right) \tag{3.9}$$

for $\gamma > 0$, *then* $f(z) \in \mathcal{S}^*(p,n,\delta)$.

Letting $\alpha_1 = \alpha_2$, $\mu = p = n = 1$ and $\delta = 0$ in Theorem 3.1, we have

Corollary 3.6. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\left|\arg\left(zf''(z) + f'(z)(\gamma+1) - \frac{z(f'(z))^2}{f(z)}\right)\right| < \frac{\pi}{2}\alpha + \tan^{-1}\frac{\alpha}{\gamma} \tag{3.10}$$

for $\gamma > 0$, *then*

$$|\arg f'(z)| < \frac{\pi}{2}\alpha, \qquad (0 < \alpha \leq 1). \tag{3.11}$$

Taking $\alpha_1 = \alpha_2$, $p = n = 1, \mu = 2$ and $\delta = 0$ in Theorem 3.1, we obtain

Corollary 3.7. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\left|\arg\left(\frac{z^2f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2 + \frac{zf'(z)}{f(z)}(\gamma+1)\right)\right| < \frac{\pi}{2}\alpha + \tan^{-1}\frac{\alpha}{\gamma} \tag{3.12}$$

for $\gamma > 0$, *then* $f(z) \in \mathcal{S}^*_{st}(\alpha)$.

Finally, we prove

Theorem 3.8. *Let* $q(z)$ *analytic in* $\mathcal{U}$ *with* $q(0) = 1$, *and* $q(z) \neq 0$. *If*

$$-\frac{\pi}{2}\eta_1 < \arg\left\{q(z) + \left[\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p}f'(z)\right]^{-1} zq'(z)\right\} < \frac{\pi}{2}\eta_2 \tag{3.13}$$

for some $f(z) \in \mathcal{B}(p,n,\mu,\alpha)$ *and* $(0 < \eta_1, \eta_2 \leq 1)$ *then*

$$-\frac{\pi}{2}\alpha_1 < \arg q(z) < \frac{\pi}{2}\alpha_2 \tag{3.14}$$

where α_1 *and* α_2 $(0 < \alpha_1, \alpha_2 \leq 1)$ *are the solutions of the following equations:*

$$\eta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)m\sin\left[\frac{\pi}{2}(1-\frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p})\right]}{2(2p-\alpha) + (\alpha_1+\alpha_2)m\sin\left[\frac{\pi}{2}(1-\frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p})\right]}\right) \tag{3.15}$$

and

$$\eta_2 = \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)m\sin\left[\frac{\pi}{2}(1-\frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p})\right]}{2(2p-\alpha) + (\alpha_1+\alpha_2)m\sin\left[\frac{\pi}{2}(1-\frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p})\right]}\right) \tag{3.16}$$

Proof. Suppose that there exists two points $z_1, z_2 \in \mathcal{U}$ such that the condition (1.10) is satisfied, then by Lemma 1.3, we obtain (1.11) under the constraint (1.12). Since $f \in \mathcal{B}(p, n, \mu, \alpha)$, we have

$$\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z) = \rho \exp\left(\frac{i\pi\phi}{2}\right), \tag{3.17}$$

where

$$\begin{cases} \alpha < \rho < 2p - \alpha \\ -\frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p} < \phi < \frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p}. \end{cases} \tag{3.18}$$

Thus, we obtain

$$\begin{aligned}
&\arg\left\{q(z_1) + \left[\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\right]^{-1} z q'(z_1)\right\} \\
&\quad = \arg q(z_1) + \arg\left(1 + \left[\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\right]^{-1} \frac{z_1 q'(z_1)}{q(z_1)}\right) \\
&\quad = -\frac{\pi}{2}\alpha_1 + \arg\left(1 - i\left(\frac{\alpha_1+\alpha_2}{2}\right) m \left[\rho \exp\left(\frac{i\pi\phi}{2}\right)\right]^{-1}\right) \\
&\quad \le -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{(\alpha_1+\alpha_2) m \sin\left[\frac{\pi}{2}(1-\phi)\right]}{2\rho + (\alpha_1+\alpha_2) m \sin\left[\frac{\pi}{2}(1-\phi)\right]}\right) \\
&\quad \le -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{(\alpha_1+\alpha_2) m \sin\left[\frac{\pi}{2}(1-\frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p})\right]}{2(2p-\alpha) + (\alpha_1+\alpha_2) m \sin\left[\frac{\pi}{2}(1-\frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p})\right]}\right) \\
&\quad = -\frac{\pi}{2}\eta_1
\end{aligned}$$

and

$$\begin{aligned}
&\arg(q(z_1) + \left[\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\right]^{-1} z q'(z_1)) \\
&\quad \ge \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{(\alpha_1+\alpha_2) m \sin\left[\frac{\pi}{2}(1-\frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p})\right]}{2(2p-\alpha) + (\alpha_1+\alpha_2) m \sin\left[\frac{\pi}{2}(1-\frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p})\right]}\right) \\
&\quad = \frac{\pi}{2}\eta_2,
\end{aligned}$$

where η_1 and η_2 being given by (3.15) and (3.16), respectively, which contradicts the assumption (3.13). This completes the proof of Theorem 3.8. □

Letting $q(z) = zf'(z)/f(z)$ in Theorem 3.8, we have

Corollary 3.9. *Let* $0 < \eta_1, \eta_2 \le 1$. *If*

$$-\frac{\pi}{2}\eta_1 < \arg\left\{q(z) + \left[\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\right]^{-1} z q'(z)\right\} < \frac{\pi}{2}\eta_2 \tag{3.19}$$

for some $f(z) \in \mathcal{B}(p,n,\mu,\alpha)$ *then* $f(z) \in \overline{\mathcal{S}}^*(p,n,\alpha_1,\alpha_2)$, *where* $0 < \alpha_1, \alpha_2 \leq 1.$

Letting $\eta_1 = \eta_2$ in Corollary 3.9, we have

Corollary 3.10. *Let* $0 < \eta_1 \leq 1$. *If*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)} + \left[\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\right]^{-1} z\left(\frac{zf'(z)}{f(z)}\right)'\right\}\right| < \frac{\pi}{2}\eta_1 \tag{3.20}$$

for some $f(z) \in \mathcal{B}(p,n,\mu,\alpha)$, *then*

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\alpha_1 \qquad (0 < \alpha_1 \leq 1), \tag{3.21}$$

that is, $f(z) \in \mathcal{S}^*_{st}(\alpha_1)$, *where* α_1 *is the solutions of the following equation:*

$$\eta = \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{2\alpha_1 m \sin\left[\frac{\pi}{2}(1 - \frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p})\right]}{2(2p-\alpha) + 2\alpha_1 m \sin\left[\frac{\pi}{2}(1 - \frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p})\right]}\right). \tag{3.22}$$

Letting $q(z) = q(z) = 1 + (zf''(z)/f'(z)$ in Theorem 3.8, we have

Corollary 3.11. *Let* $0 < \eta_1, \eta_2 \leq 1$. *If*

$$-\frac{\pi}{2}\eta_1 < \arg\left\{1 + \frac{zf''(z)}{f'(z)} + \left[\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\right]^{-1} z\left(1 + \frac{zf''(z)}{f'(z)}\right)'\right\} < \frac{\pi}{2}\eta_1 \tag{3.23}$$

for some $f(z) \in \mathcal{B}(p,n,\mu,\alpha)$ *then* $f(z) \in \mathcal{K}(p,n,\alpha_1,\alpha_2)$, *where* $0 < \alpha_1, \alpha_2 \leq 1.$

Letting $\eta_1 = \eta_2$ in Corollary 3.11, we have

Corollary 3.12. *Let* $0 < \eta_1 \leq 1$. *If*

$$\left|\arg\left\{1 + \frac{zf''(z)}{f'(z)} + \left[\left(\frac{z^p}{f(z)}\right)^{\mu-1} z^{1-p} f'(z)\right]^{-1} z\left(1 + \frac{zf''(z)}{f'(z)}\right)'\right\}\right| < \frac{\pi}{2}\eta_1 \tag{3.24}$$

for some $f(z) \in \mathcal{B}(p,n,\mu,\alpha)$ *then*

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\alpha_1 \quad (0 < \alpha_1 \leq 1), \tag{3.25}$$

that is, $f(z) \in \mathcal{K}_{st}(\alpha_1)$, *where* α_1 *is the solution of the following equation:*

$$\eta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{2\alpha_1 m \sin\left[\frac{\pi}{2}\left(1 - \frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p}\right)\right]}{2(2p-\alpha) + 2\alpha_1 m \sin\left[\frac{\pi}{2}(1 - \frac{2}{\pi}\sin^{-1}\frac{p-\alpha}{p})\right]}\right). \tag{3.26}$$

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