

# A SIMULTANEOUS SYSTEM OF FUNCTIONAL INEQUALITIES AND MAPPINGS WHICH ARE WEAKLY OF A CONSTANT SIGN

**DOROTA KRASSOWSKA**

Fac. of Math., Computer Science and Econometrics  
University of Zielona Góra  
Podgórná 50, PL-65-246 Zielona Góra, Poland  
EMail: D.krassowska@wmie.uz.zgora.pl

**JANUSZ MATKOWSKI**

Institute of Mathematics  
Silesian University  
Bankowa 14, PL-40-007 Katowice, Poland  
EMail: j.matkowski@wmie.uz.zgora.pl

*Received:* 17 October, 2006

*Accepted:* 07 June, 2007

*Communicated by:* Zs. Pales

*2000 AMS Sub. Class.:* Primary 39B72; Secondary 26D05.

*Key words:* Linear functional inequalities, Kronecker's theorem, Mappings of weakly constant sign.

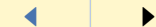
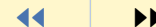
*Abstract:* It is shown that, under some algebraic conditions on fixed reals  $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$  and vectors  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1} \in \mathbb{R}^n$ , every continuous at a point function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfying the simultaneous system of inequalities

$$f(\mathbf{x} + \mathbf{a}_i) \leq \alpha_i + f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, i = 1, 2, \dots, n + 1,$$

has to be of the form  $f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0})$ ,  $\mathbf{x} \in \mathbb{R}^n$ , with uniquely determined  $\mathbf{p} \in \mathbb{R}^n$ . For mappings with values in a Banach space which are weakly of a constant sign, a counterpart of this result is given.

Title Page

Contents



Page 1 of 19

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

# Contents

1	Introduction	3
2	Kronecker's Theorem and a Lemma	4
3	Main Result	7
4	Application for Mappings Which are Weakly of a Constant Sign	16



---

**Functional Inequalities**  
Dorota Krassowska  
and Janusz Matkowski  
vol. 8, iss. 2, art. 35, 2007

---

Title Page

Contents



Page 2 of 19

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



## 1. Introduction

In this paper we consider the simultaneous system of functional inequalities

$$(1.1) \quad f(\mathbf{x} + \mathbf{a}_i) \leq \alpha_i + f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \quad i = 1, 2, \dots, n + 1,$$

where  $n \in \mathbb{N}$ ,  $\alpha_1, \alpha_2, \dots, \alpha_{n+1} \in \mathbb{R}$ ,  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1} \in \mathbb{R}^n$  are fixed and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is an unknown function.

In Section 3, assuming Kronecker's type conditions on the vectors  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1}$  and a inequality involving some determinants depending on these vectors and scalars  $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ , we show (Theorem 3.1) that for every continuous at least at one point function  $f$  satisfying (1.1) there exists a unique vector  $\mathbf{p} \in \mathbb{R}^n$  such that

$$f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0}), \quad \mathbf{x} \in \mathbb{R}^n.$$

This result seems to be a little surprising in the context of an obvious fact that, in general, the accompanying simultaneous system of functional equations has a lot of very regular but nonlinear solutions, even “depending on an arbitrary function” (cf. M. Kuczma [4], [2] or [3], where the full construction of the solution in a special one dimensional case is given). Theorem 3.1 generalizes a suitable result of [2], where the one dimensional case

$$f(x + a) \leq \alpha + f(x), \quad f(x + b) \leq \beta + f(x), \quad x \in \mathbb{R},$$

is considered. Let us mention that in the case when  $\alpha = \beta = 0$ , Montel [5] considered the accompanying simultaneous system of equations.

In Section 4 we define a mapping to be weakly of a constant sign. Using this notion and a total system of linear functionals, we present a counterpart of Theorem 3.1 for functions with values in a Banach space.

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 3 of 19

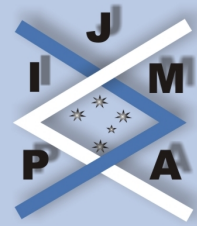
Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 4 of 19

[Go Back](#)

[Full Screen](#)

[Close](#)

## 2. Kronecker's Theorem and a Lemma

The symbols  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$  are reserved, respectively, for the set of natural, integer, rational and real numbers.

We begin this section by recalling the following:

**Theorem 2.1 (Kronecker (cf. [1, p. 382])).** *If the reals  $v_1, v_2, \dots, v_n, 1$  are linearly independent over the field  $\mathbb{Q}$ , the numbers  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$  are arbitrary, and  $N$  and  $\epsilon$  are positive, then there are the integers  $p_1, p_2, \dots, p_n$ , and  $m > N$ , such that  $|mv_i - p_i - \alpha_i| < \epsilon$  for each  $i = 1, 2, \dots, n$ .*

As an immediate consequence of this theorem we obtain the following

**Corollary 2.2.** *Let  $e_1, e_2, \dots, e_n$  be the standard base of the real linear space  $\mathbb{R}^n$ . If reals  $v_1, v_2, \dots, v_n, 1$  are linearly independent over the field  $\mathbb{Q}$ , and  $\mathbf{v} := (v_1, v_2, \dots, v_n)$ , then the set*

$$\{m\mathbf{v} + p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + \dots + p_n\mathbf{e}_n : m \in \mathbb{N}, p_1, p_2, \dots, p_n \in \mathbb{Z}\}$$

is dense in  $\mathbb{R}^n$ .

In sequel we need a more special result which guarantees that the set of all linear combinations of the elements of the set  $B = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{n+1}\}$  with natural coefficients is dense in  $\mathbb{R}^n$ .

**Lemma 2.3.** *Let  $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$  be an arbitrary base of linear space  $\mathbb{R}^n$  over  $\mathbb{R}$ . Suppose that  $v_1, v_2, \dots, v_n \in \mathbb{R}$  are negative and the system of numbers  $v_1, v_2, \dots, v_n, 1$  is linearly independent over the field  $\mathbb{Q}$ . If*

$$\mathbf{a}_{n+1} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \dots + v_n\mathbf{a}_n,$$



then the set

$$A := \left\{ \sum_{i=1}^{n+1} m^{(i)} \mathbf{a}_i : m^{(i)} \in \mathbb{N}, i = 1, 2, \dots, n+1 \right\}$$

is dense in  $\mathbb{R}^n$ .

*Proof.* Let  $N > 0$  be fixed. Take  $\mathbf{x} \in \mathbb{R}^n$ . Then there exists a unique system of numbers  $x_1, x_2, \dots, x_n \in \mathbb{R}$  such that

$$\mathbf{x} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \dots + x_n \mathbf{a}_n.$$

By Kronecker's theorem, for every  $k \in \mathbb{N}$  there exist  $m_k > N$  and  $p_{ik} \in \mathbb{Z}$  such that

$$|m_k v_i + p_{ik} - x_i| = |m_k v_i - (-p_{ik}) - x_i| < \frac{1}{k}, \quad i = 1, 2, \dots, n,$$

whence

$$\lim_{k \rightarrow \infty} (m_k v_i + p_{ik}) = x_i, \quad i = 1, 2, \dots, n.$$

Since  $m_k \in \mathbb{N}$ ,  $v_i < 0$  for  $i = 1, 2, \dots, n$ , it follows that  $p_{ik} \in \mathbb{N}$  for  $k$  large enough (in the opposite case we would have  $\lim_{k \rightarrow \infty} (m_k v_i + p_{ik}) = -\infty$ ). It follows that, for  $k$  large enough, setting  $m_k^{(n+1)} := m_k$  and  $m_k^{(i)} := p_{ik}$ , we get

$$m_k^{(i)} \in \mathbb{N}, \quad i = 1, 2, \dots, n+1.$$

Put

$$\mathbf{t}_k := \sum_{i=1}^n \left( m_k^{(n+1)} v_i + m_k^{(i)} \right) \mathbf{a}_i, \quad k \in \mathbb{N}.$$

By the definition of  $\mathbf{a}_{n+1}$  we hence get

$$\mathbf{t}_k = \sum_{i=1}^{n+1} m_k^{(i)} \mathbf{a}_i, \quad k \in \mathbb{N},$$

[Title Page](#)

[Contents](#)

[◀◀](#) [▶▶](#)

[◀](#) [▶](#)

Page 5 of 19

[Go Back](#)

[Full Screen](#)

[Close](#)

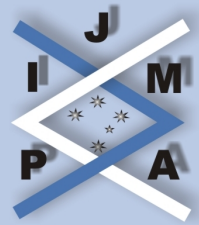
journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

whence  $\mathbf{t}_k \in A$  for  $k$  sufficiently large. Moreover

$$\begin{aligned}\lim_{k \rightarrow \infty} \mathbf{t}_k &= \lim_{k \rightarrow \infty} \sum_{i=1}^n \left( m_k^{(n+1)} v_i + m_k^{(i)} \right) \mathbf{a}_i \\ &= \sum_{i=1}^n \lim_{k \rightarrow \infty} \left( m_k^{(n+1)} v_i + m_k^{(i)} \right) \mathbf{a}_i \\ &= \sum_{i=1}^n x_i \mathbf{a}_i = \mathbf{x}.\end{aligned}$$

This completes the proof of the density of  $A$  in  $\mathbb{R}^n$ . □



**Functional Inequalities**  
Dorota Krassowska  
and Janusz Matkowski  
vol. 8, iss. 2, art. 35, 2007

Title Page

Contents



Page 6 of 19

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756



### 3. Main Result

**Theorem 3.1.** Let  $n \in \mathbb{N}$ ,  $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \dots, a_{i,n}) \in \mathbb{R}^n$ ,  $i = 1, 2, \dots, n$ , negative reals  $v_1, v_2, \dots, v_n$  and  $\alpha_1, \alpha_2, \dots, \alpha_{n+1} \in \mathbb{R}$  be fixed and such that:

- (i)  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$  form a base of the linear space  $\mathbb{R}^n$  over  $\mathbb{R}$ ;
- (ii)  $v_1, v_2, \dots, v_n, 1$  are linearly independent over  $\mathbb{Q}$ ,
- (iii)  $\mathbf{a}_{n+1} = v_1 \mathbf{a}_1 + v_2 \mathbf{a}_2 + \dots + v_n \mathbf{a}_n$ ,
- (iv)

$$(-1)^n (\text{sgn } \mathbf{W}^{(n)}) \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \alpha_{n+1} \\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix} \leq 0,$$

where

$$\mathbf{W}^{(n)} := \begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} \end{vmatrix}.$$

If a continuous at least at one point function  $f : \mathbb{R}^n \longrightarrow \mathbb{R}$  satisfies the simultaneous system of functional inequalities

$$(3.1) \quad f(\mathbf{x} + \mathbf{a}_i) \leq \alpha_i + f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \quad i = 1, 2, \dots, n + 1,$$

then there exists a unique  $\mathbf{p} \in \mathbb{R}^n$  such that

$$f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0}), \quad \mathbf{x} \in \mathbb{R}^n.$$

Title Page

Contents



Page 7 of 19

Go Back

Full Screen

Close



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 8 of 19

Go Back

Full Screen

Close

Moreover,

$$\mathbf{p} = \frac{1}{\mathbf{W}^{(n)}} \left[ p_1^{(n)}, p_2^{(n)}, \dots, p_n^{(n)} \right],$$

where

$$p_i^{(n)} := \begin{vmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,i-1} & a_{2,i-1} & \cdots & a_{n,i-1} \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ a_{1,i+1} & a_{2,i+1} & \cdots & a_{n,i+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} \end{vmatrix}, \quad i = 1, 2, \dots, n.$$

*Proof.* Suppose that the function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies system (3.1). Using an easy induction argument, one can show that  $f$  satisfies the inequality

$$(3.2) \quad f \left( \sum_{i=1}^{n+1} m^{(i)} \mathbf{a}_i + \mathbf{x} \right) \leq \sum_{i=1}^{n+1} m^{(i)} \alpha_i + f(\mathbf{x}),$$

for all  $m^{(i)} \in \mathbb{N}$ ,  $i = 1, 2, \dots, n + 1$  and all  $\mathbf{x} \in \mathbb{R}^n$ . By Lemma 2.3, the set

$$A := \left\{ \sum_{i=1}^{n+1} m^{(i)} \mathbf{a}_i : m^{(i)} \in \mathbb{N}, \quad i = 1, 2, \dots, n + 1 \right\}$$

is dense in  $\mathbb{R}^n$ . Thus there exist the sequences of positive integers  $(m_k^{(i)})_{k \in \mathbb{N}}$  for





Title Page

Contents



Page 9 of 19

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

$i = 1, 2, \dots, n + 1$  such that

$$\begin{aligned} \lim_{k \rightarrow \infty} \sum_{i=1}^{n+1} m_k^{(i)} \mathbf{a}_i &= \lim_{k \rightarrow \infty} \left( \sum_{i=1}^{n+1} m_k^{(i)} a_{i,1}, \sum_{i=1}^{n+1} m_k^{(i)} a_{i,2}, \dots, \sum_{i=1}^{n+1} m_k^{(i)} a_{i,n} \right) \\ &= (0, 0, \dots, 0), \end{aligned}$$

and, obviously,

$$\lim_{k \rightarrow \infty} m_k^{(i)} = \infty, \quad i = 1, 2, \dots, n + 1.$$

It follows that, for each  $i = 1, 2, \dots, n$ , the limit  $\lim_{k \rightarrow \infty} \frac{m_k^{(i)}}{m_k^{(n+1)}}$  exists, and

$$\sum_{i=1}^{n+1} \left( \lim_{k \rightarrow \infty} \frac{m_k^{(i)}}{m_k^{(n+1)}} a_{i,j} \right) = \lim_{k \rightarrow \infty} \frac{\sum_{i=1}^{n+1} m_k^{(i)} a_{i,j}}{m_k^{(n+1)}} = 0, \quad j = 1, 2, \dots, n.$$

Consequently,

$$(3.3) \quad \lim_{k \rightarrow \infty} \frac{m_k^{(i)}}{m_k^{(n+1)}} = \frac{\mathbf{W}_i^{(n)}}{\mathbf{W}^{(n)}}, \quad i = 1, 2, \dots, n,$$

where

$$\mathbf{W}_i^{(n)} := \begin{vmatrix} a_{1,1} & \cdots & a_{i-1,1} & -a_{n+1,1} & a_{i+1,1} & \cdots & a_{n,1} \\ a_{1,2} & \cdots & a_{i-1,2} & -a_{n+1,2} & a_{i+1,2} & \cdots & a_{n,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & \cdots & a_{i-1,n} & -a_{n+1,n} & a_{i+1,n} & \cdots & a_{n,n} \end{vmatrix}, \quad i = 1, 2, \dots, n.$$



Let  $\mathbf{x}_0$  be a point of the continuity of  $f$ . From inequality (3.2), we get

$$\frac{f\left(\sum_{i=1}^{n+1} m_k^{(i)} \mathbf{a}_i + \mathbf{x}_0\right)}{m_k^{(n+1)}} \leq \sum_{i=1}^{n+1} \frac{m_k^{(i)}}{m_k^{(n+1)}} \alpha_i + \frac{f(\mathbf{x}_0)}{m_k^{(n+1)}}.$$

Letting here  $k \rightarrow \infty$  and applying (3.3), we obtain

$$(3.4) \quad 0 \leq \sum_{i=1}^n \frac{\mathbf{W}_i^{(n)}}{\mathbf{W}^{(n)}} \alpha_i + \alpha_{n+1}.$$

Setting

$$\overline{\mathbf{W}_i^{(n)}} := - \begin{vmatrix} a_{1,1} & \cdots & a_{i-1,1} & a_{i+1,1} & \cdots & a_{n,1} & a_{n+1,1} \\ a_{1,2} & \cdots & a_{i-1,2} & a_{i+1,2} & \cdots & a_{n,2} & a_{n+1,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & \cdots & a_{i-1,n} & a_{i+1,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix}, \quad i = 1, 2, \dots, n,$$

we can write inequality (3.4) in the form

$$0 \leq \operatorname{sgn} \mathbf{W}^{(n)} \left( \sum_{i=1}^n (-1)^{n-i+1} \overline{\mathbf{W}_i^{(n)}} \alpha_i + \mathbf{W}^{(n)} \alpha_{n+1} \right),$$

whence, by the Laplace expansion of a determinant,

$$0 \leq (-1)^n \operatorname{sgn} \mathbf{W}^{(n)} \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \alpha_{n+1} \\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix}.$$

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 10 of 19

Go Back

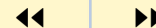
Full Screen

Close



Title Page

Contents



Page 11 of 19

Go Back

Full Screen

Close

Hence, taking into account condition (iv), we infer that

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n & \alpha_{n+1} \\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix} = 0.$$

Now, by Laplace's expansion theorem,

$$\alpha_{n+1} = (-1)^{n+1} \frac{\sum_{i=1}^n (-1)^{1-i} \overline{\mathbf{W}}_i^{(n)} \alpha_i}{\mathbf{W}^{(n)}},$$

whence by (3.2),

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} m^{(i)} \mathbf{a}_i + \mathbf{x}\right) &\leq \sum_{i=1}^n m^{(i)} \alpha_i + m^{(n+1)} (-1)^{n+1} \frac{\sum_{i=1}^n (-1)^{1-i} \overline{\mathbf{W}}_i^{(n)} \alpha_i}{\mathbf{W}^{(n)}} + f(\mathbf{x}) \\ &= \frac{1}{\mathbf{W}^{(n)}} \left[ \sum_{i=1}^n m^{(i)} \mathbf{W}^{(n)} \alpha_i + m^{(n+1)} \sum_{i=1}^n (-1)^{n+2-i} \overline{\mathbf{W}}_i^{(n)} \alpha_i \right] + f(\mathbf{x}), \end{aligned}$$

for all  $m^{(i)} \in \mathbb{N}$ , and  $x \in \mathbb{R}^n$ .

Applying Laplace's theorem for the  $i$ th column of  $\mathbf{W}^{(n)}$  and for the last columnne



of  $\overline{W_i^{(n)}}$ , we have

$$\begin{aligned} & \sum_{i=1}^n m^{(i)} W_i^{(n)} \alpha_i + m^{(n+1)} \sum_{i=1}^n (-1)^{n+2-i} \overline{W_i^{(n)}} \alpha_i \\ &= \sum_{i=1}^n m^{(i)} \alpha_i \sum_{j=1}^n (-1)^{i+j} a_{i,j} W_{ij}^{(n)} \\ & \quad + m^{(n+1)} \left( \sum_{i=1}^n (-1)^{n-i} a_{i,j} \alpha_i \left( \sum_{j=1}^n (-1)^{n+j} a_{n+1,j} \overline{W_{ij}^{(n)}} \right) \right), \end{aligned}$$

where  $W_{ij}^{(n)}$  is obtained from  $W_i^{(n)}$  by deleting the  $j$ th row and  $i$ th column, and  $\overline{W_{ij}^{(n)}}$  is obtained from  $\overline{W_i^{(n)}}$  by deleting the  $j$ th row and the last column. Since

$$W_{ij}^{(n)} = \overline{W_{ij}^{(n)}}, \quad i, j = 1, 2, \dots, n,$$

applying Fubini's theorem for sums, we have

$$\begin{aligned} & \sum_{i=1}^n m^{(i)} W_i^{(n)} \alpha_i + m^{(n+1)} \sum_{i=1}^n (-1)^{n+2-i} \overline{W_i^{(n)}} \alpha_i \\ &= \sum_{j=1}^n \sum_{k=1}^n m^{(k)} a_{k,j} \left( (-1)^{k+j} W_{kj}^{(n)} \alpha_k \right) + m^{(n+1)} a_{n+1,j} \sum_{i=1}^n (-1)^{j-i} W_{ij}^{(n)} \alpha_i. \end{aligned}$$

Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 12 of 19

Go Back

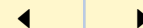
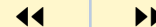
Full Screen

Close



Title Page

Contents



Page 13 of 19

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

Adding and subtracting the term  $\sum_{i=1}^n (-1)^{j-i} \mathbf{W}_{ij}^{(n)} \alpha_i$  in the sum over  $k$  gives

$$\begin{aligned} & \sum_{i=1}^n m^{(i)} \mathbf{W}^{(n)} \alpha_i + m^{(n+1)} \sum_{i=1}^n (-1)^{n+2-i} \overline{\mathbf{W}_i^{(n)}} \alpha_i \\ &= \sum_{j=1}^n \left( \sum_{k=1}^n m^{(k)} a_{k,j} + m^{(n+1)} a_{n+1,j} \right) \left( \sum_{i=1}^n (-1)^{j-i} \mathbf{W}_{ij}^{(n)} \alpha_i \right) \\ & \quad + \sum_{j=1}^n \left( \sum_{k=1}^n m^{(k)} a_{k,j} \left( \sum_{i=1}^n (-1)^{j-i+1} \mathbf{W}_{ij}^{(n)} \alpha_i + (-1)^{k+j} \mathbf{W}_{kj}^{(n)} \alpha_k \right) \right). \end{aligned}$$

Let us note that

$$\sum_{j=1}^n \left( \sum_{k=1}^n m^{(k)} a_{k,j} \left( \sum_{i=1}^n (-1)^{j-i+1} \mathbf{W}_{ij}^{(n)} \alpha_i + (-1)^{k+j} \mathbf{W}_{kj}^{(n)} \alpha_k \right) \right) = 0.$$

Indeed, we have

$$\begin{aligned} & \sum_{j=1}^n \left( \sum_{k=1}^n m^{(k)} a_{k,j} \left( \sum_{i=1}^n (-1)^{j-i+1} \mathbf{W}_{ij}^{(n)} \alpha_i + (-1)^{k+j} \mathbf{W}_{kj}^{(n)} \alpha_k \right) \right) \\ &= \sum_{j=1}^n \sum_{k=1}^n m^{(k)} a_{k,j} \left( \sum_{i \in \{1,2,\dots,n\}-k} (-1)^{j-i+1} \mathbf{W}_{ij}^{(n)} \alpha_i \right) \\ &= \sum_{k=1}^n m^{(k)} \left( \sum_{i \in \{1,2,\dots,n\}-k} \alpha_i \left( \sum_{j=1}^n (-1)^{j-i+1} a_{k,j} \mathbf{W}_{ij}^{(n)} \right) \right) \end{aligned}$$

and

$$\sum_{j=1}^n (-1)^{j-i+1} a_{k,j} \mathbf{W}_{ij}^{(n)}$$

is a determinant with two equal columns. Thus we have shown that  $f$  satisfies the inequality

$$f\left(\sum_{i=1}^{n+1} m^{(i)} \mathbf{a}_i + \mathbf{x}\right) \leq \sum_{j=1}^n \left( \left( \frac{1}{\mathbf{W}^n} \sum_{i=1}^n (-1)^{j-i} \mathbf{W}_{ij}^{(n)} \alpha_i \right) \left( \sum_{i=1}^{n+1} m^{(i)} a_{ij} \right) \right) + f(\mathbf{x}),$$

for all  $m^{(i)} \in \mathbb{N}$ , and  $x \in \mathbb{R}^n$ , which can be written in the form

$$(3.5) \quad f(\mathbf{t} + \mathbf{x}) \leq \mathbf{p} \cdot \mathbf{t} + f(\mathbf{x}), \quad \mathbf{t} \in A, \mathbf{x} \in \mathbb{R}^n.$$

Now take an arbitrary  $\mathbf{x} \in \mathbb{R}^n$ . By the density of  $A$  there is a sequence  $(\mathbf{t}_n)$  such that

$$\mathbf{t}_n \in A \quad (n \in \mathbb{N}), \quad \lim_{n \rightarrow \infty} \mathbf{t}_n = \mathbf{x}_0 - \mathbf{x}.$$

From (3.5) we have

$$f(\mathbf{t}_n + \mathbf{x}) \leq \mathbf{p} \cdot \mathbf{t}_n + f(\mathbf{x}), \quad n \in \mathbb{N}.$$

Letting here  $n \rightarrow \infty$ , and making use of the continuity of  $f$  at  $\mathbf{x}_0$ , we obtain

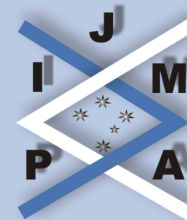
$$f(\mathbf{x}_0) \leq \mathbf{p} \cdot (\mathbf{x}_0 - \mathbf{x}) + f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n.$$

To prove the converse inequality, note that replacing  $\mathbf{x}$  by  $\mathbf{x} - \mathbf{t}$  in (3.5) we get

$$f(\mathbf{x}) \leq \mathbf{p} \cdot \mathbf{t} + f(\mathbf{x} - \mathbf{t}), \quad \mathbf{t} \in A, \mathbf{x} \in \mathbb{R}^n.$$

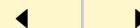
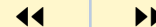
Taking a sequence  $(\mathbf{t}_n)$  such that

$$\mathbf{t}_n \in A \quad (n \in \mathbb{N}), \quad \lim_{n \rightarrow \infty} \mathbf{t}_n = \mathbf{x} - \mathbf{x}_0,$$



Title Page

Contents



Page 14 of 19

Go Back

Full Screen

Close



(which, by the density of  $A$ , exists) we hence get

$$f(\mathbf{x}) \leq \mathbf{p} \cdot \mathbf{t}_n + f(\mathbf{x} - \mathbf{t}_n), \quad n \in \mathbf{N}.$$

Letting here  $n \rightarrow \infty$ , and again making use of the continuity of  $f$  at  $\mathbf{x}_0$ , we obtain the inequality

$$f(\mathbf{x}) \leq \mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_0) + f(\mathbf{x}_0), \quad \mathbf{x} \in \mathbb{R}^n.$$

Thus

$$f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + (f(\mathbf{x}_0) - \mathbf{p} \cdot \mathbf{x}_0), \quad \mathbf{x} \in \mathbb{R}^n.$$

Since  $f(\mathbf{0}) = f(\mathbf{x}_0) - \mathbf{p} \cdot \mathbf{x}_0$ , the proof is completed. □

Title Page

Contents

◀◀      ▶▶

◀      ▶

Page 15 of 19

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756

## 4. Application for Mappings Which are Weakly of a Constant Sign

We start this section with the following.

**Definition 4.1.** Let  $X$  be an arbitrary nonempty set,  $Y$  - an arbitrary real linear topological space,  $Y^*$  - the conjugate space of  $Y$ , and  $T \subset Y^*$ .

- (i) We say that a mapping  $G : X \rightarrow Y$  is  $T$ -weakly of a constant sign if for each functional  $\phi \in T$  either  $\phi \circ G$  is nonpositive or  $\phi \circ G$  is nonnegative.
- (ii) The mappings  $G_1 : X \rightarrow Y$  and  $G_2 : X \rightarrow Y$  are said to be  $T$ -weakly of the same sign if  $(\phi \circ G_1) \cdot (\phi \circ G_2) \geq 0$  for every functional  $\phi \in T$ .

Now applying Theorem 3.1 we prove:

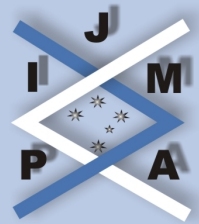
**Theorem 4.1.** Let  $Y$  be a real Banach space and  $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\} \subset \mathbb{R}^n$ ,  $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \dots, a_{i,n})$ ,  $i = 1, 2, \dots, n$  be a base of the real linear space  $\mathbb{R}^n$ . Suppose that reals  $v_1, v_2, \dots, v_n$  are negative and  $v_1, v_2, \dots, v_n, 1$  are linearly independent over the field  $\mathbb{Q}$ . Let  $\mathbf{a}_{n+1} = v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + \dots + v_n\mathbf{a}_n$ .

If a mapping  $F : \mathbb{R}^n \rightarrow Y$  is continuous at least at one point and there exist a total system  $T \subset Y^*$  and the vectors  $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_{n+1} \in Y$  such that

(i) the mappings

$$(*) \quad \mathbb{R}^n \ni \mathbf{x} \rightarrow F(\mathbf{x} + \mathbf{a}_i) - F(\mathbf{x}) - \mathbf{y}_i, \quad i \in \{1, 2, \dots, n + 1\},$$

are  $T$ -weakly of a constant and the same sign;



Title Page

Contents



Page 16 of 19

Go Back

Full Screen

Close





Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 17 of 19

Go Back

Full Screen

Close

(ii) for every  $\phi \in T$ ,

$$(-1)^n (\operatorname{sgn} \mathbf{W}^{(n)}) \begin{vmatrix} \phi(\mathbf{y}_1) & \phi(\mathbf{y}_2) & \cdots & \phi(\mathbf{y}_n) & \phi(\mathbf{y}_{n+1}) \\ a_{1,1} & a_{2,1} & \cdots & a_{n,1} & a_{n+1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n} & a_{2,n} & \cdots & a_{n,n} & a_{n+1,n} \end{vmatrix} \leq 0,$$

then there exists a unique linear continuous mapping  $L : \mathbb{R}^n \rightarrow Y$  such that

$$(4.1) \quad F(\mathbf{x}) = L(\mathbf{x}) + F(\mathbf{0}), \quad \mathbf{x} \in \mathbb{R}^n.$$

*Proof.* Assume that  $F : \mathbb{R}^n \rightarrow Y$  satisfies assumptions (i) and (ii). By Definition 4.1, we have either, for all  $\phi \in T$  and  $i = 1, 2, \dots, n + 1$ ,

$$\phi(F(\mathbf{x} + \mathbf{a}_i) - F(\mathbf{x}) - \mathbf{y}_i) \leq 0, \quad \mathbf{x} \in \mathbb{R}^n,$$

or, for all  $\phi \in T$  and  $i = 1, 2, \dots, n + 1$ ,

$$\phi(F(\mathbf{x} + \mathbf{a}_i) - F(\mathbf{x}) - \mathbf{y}_i) \geq 0, \quad \mathbf{x} \in \mathbb{R}^n.$$

Without any loss of generality we may confine the consideration to the first case. From the linearity of  $\phi$  we obtain the simultaneous system of inequalities

$$\phi(F(\mathbf{x} + \mathbf{a}_i)) \leq \phi(F(\mathbf{x})) + \phi(\mathbf{y}_i), \quad \mathbf{x} \in \mathbb{R}^n; \quad \mathbf{y}_i \in Y, i = 1, 2, \dots, n+1; \quad \phi \in T.$$

Let us fix  $\phi \in T$ . Setting  $f := \phi \circ F$  and  $\alpha_i := \phi(\mathbf{y}_i)$ , we hence get

$$f(\mathbf{x} + \mathbf{a}_i) \leq \alpha_i + f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \quad i = 1, 2, \dots, n + 1,$$

i.e.  $f$  satisfies system (3.1). Since all the required assumptions of Theorem 3.1 are satisfied, there exists a vector  $\mathbf{p}$ , such that

$$f(\mathbf{x}) = \mathbf{p} \cdot \mathbf{x} + f(\mathbf{0}), \quad \mathbf{x} \in \mathbb{R}^n.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 18 of 19

Go Back

Full Screen

Close

Hence, by the definition of  $f$ ,

$$\phi(F(\mathbf{x}) - F(\mathbf{0})) = \mathbf{p} \cdot \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n.$$

It follows that for every  $\phi \in T$ , there exists  $\mathbf{p} := \mathbf{p}_\phi$ , such that

$$\phi(F(\mathbf{x}) - F(\mathbf{0})) = \mathbf{p}_\phi \cdot \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n.$$

Setting  $L := F - F(\mathbf{0})$ , we hence get

$$\begin{aligned} \phi(L(\mathbf{x} + \mathbf{y})) &= \phi(F(\mathbf{x} + \mathbf{y}) - F(\mathbf{0})) \\ &= \mathbf{p}_\phi \cdot (\mathbf{x} + \mathbf{y}) \\ &= \mathbf{p}_\phi \cdot \mathbf{x} + \mathbf{p}_\phi \cdot \mathbf{y} \\ &= \phi(F(\mathbf{x}) - F(\mathbf{0})) + \phi(F(\mathbf{y}) - F(\mathbf{0})) \\ &= \phi(L(\mathbf{x})) + \phi(L(\mathbf{y})), \end{aligned}$$

for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , whence

$$\phi(L(\mathbf{x} + \mathbf{y}) - L(\mathbf{x}) - L(\mathbf{y})) = 0, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

for every functional  $\phi \in T$ . Since  $T$  is total set of functionals (cf. [6]),

$$L(\mathbf{x} + \mathbf{y}) - L(\mathbf{x}) - L(\mathbf{y}) = 0, \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

that is,  $L$  is an additive mapping. Now the continuity of  $F$  at least at one point implies that  $L := F - F(\mathbf{0})$  is continuous and, consequently, a linear map. The proof is completed.  $\square$

## References

- [1] G.H. HARDY AND E.M. WRIGHT, *An Introduction to the Theory of Numbers*, Oxford, 1960.
- [2] D. KRASSOWSKA AND J. MATKOWSKI, A pair of functional inequalities and a characterization of  $L^p$ -norm, *Ann. Polon. Math.*, **85** (2005), 1–11.
- [3] D. KRASSOWSKA AND J. MATKOWSKI, A pair of functional inequalities of iterative type related to a Cauchy functional equation, in *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht-Boston-London, 2003, 73–89.
- [4] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities, Cauchy's Equation and Jensen's Inequality*, PWN - Polish Sci. Publ. and Silesian University, Warszawa-Kraków-Katowice, 1985.
- [5] P. MONTEL, Sur les propriétés périodiques des fonctions, *C. R. Acad. Sci. Paris*, **251** (1960), 2111–2112.
- [6] A. WILANSKY, *Functional Analysis*, Blaisdel Publishing Company, New York-Toronto-London, 1964.



---

Functional Inequalities  
Dorota Krassowska  
and Janusz Matkowski  
vol. 8, iss. 2, art. 35, 2007

---

Title Page

Contents



Page 19 of 19

Go Back

Full Screen

Close

journal of **inequalities**  
in pure and applied  
mathematics

issn: 1443-5756