# ON CLASS $wF(p, r, q)$ OPERATORS AND QUASISIMILARITY

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*Abstract:* Let $T$ be a bounded linear operator on a complex Hilbert space $H$. In this paper, we show that if $T$ belongs to class $wF(p, r, q)$ operators, then we have (i) $T^*X = XN^*$ whenever $TX = XN$ for some $X \in B(H)$, where $N$ is normal and $X$ is injective with dense range. (ii) $T$ satisfies the property $(\beta)_\varepsilon$, i.e., $T$ is subscalar, moreover, $T$ is subdecomposable. (iii) Quasisimilar class $wF(p, r, q)$ operators have the same spectra and essential spectra.

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# 1. Introduction

Let $X$ denote a Banach space, $T \in B(X)$ is said to be generalized scalar ([3]) if there exists a continuous algebra homomorphism (called a spectral distribution of $T$) $\Phi : \varepsilon(\mathcal{C}) \to B(X)$ with $\Phi(1) = I$ and $\Phi(z) = T$, where $\varepsilon(\mathcal{C})$ denotes the algebra of all infinitely differentiable functions on the complex plane $\mathcal{C}$ with the topology defined by uniform convergence of such functions and their derivatives ([2]). An operator similar to the restriction of a generalized scalar (decomposable) operator to one of its closed invariant subspaces is said to be subscalar (subdecomposable). Subscalar operators are subdecomposable operators ([3]). Let $H, K$ be complex Hilbert spaces and $B(H), B(K)$ be the algebra of all bounded linear operators in $H$ and $K$ respectively, $B(H, K)$ denotes the algebra of all bounded linear operators from $H$ to $K$. A capital letter (such as $T$) means an element of $B(H)$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for any $x \in H$. An operator $T$ is said to be $p-$hyponormal if $(T^*T)^p \geq (TT^*)^p$, $0 < p \leq 1$.

**Definition 1.1 ([10]).** *For $p > 0, r \geq 0,$ and $q \geq 1$, an operator $T$ belongs to class $wF(p, r, q)$ if*

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} \geq |T^*|^{\frac{2(p+r)}{q}}$$

*and*

$$|T|^{2(p+r)(1-\frac{1}{q})} \geq (|T|^p|T^*|^{2r}|T|^p)^{1-\frac{1}{q}}.$$

Let $T = U|T|$ be the polar decomposition of $T$. We define

$$\widetilde{T}_{p,r} = |T|^pU|T|^r (p + r = 1).$$

The operator $\widetilde{T}_{p,r}$ is known as the generalized Aluthge transform of $T$. We define $(\widetilde{T}_{p,r})^{(1)} = \widetilde{T}_{p,r}$, $(\widetilde{T}_{p,r})^{(n)} = [(\widetilde{T}_{p,r})^{(n-1)}]_{p,r}$, where $n \geq 2$.



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The following Fuglede-Putnam's theorem is famous. We extend this theorem for class $wF(p, r, q)$ operators.

**Theorem 1.2 (Fuglede-Putnam's Theorem [7]).** *Let $A$ and $B$ be normal operators and $X$ be an operator on a Hilbert space. Then the following hold and follow from each other:*

*(i) (Fuglede) If $AX = XA$, then $A^*X = XA^*$.*

*(ii) (Putnam) If $AX = XB$, then $A^*X = XB^*$.*


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## 2. Preliminaries

**Lemma 2.1 ([9]).** *If $N$ is a normal operator on $H$, then we have*

$$\bigcap_{\lambda\in\mathcal{C}}(N-\lambda)\mathcal{H}=\{0\}.$$

**Lemma 2.2 ([5]).** *Let $T=U|T|$ be the polar decomposition of a $p$-hyponormal operator for $p>0$. Then the following assertions hold:*

(i) *$\widetilde{T}_{s,t}=|T|^sU|T|^t$ is $\frac{p+\min(s,t)}{s+t}$-hyponormal for any $s>0$ and $t>0$ such that $\max\{s,t\}\geq p$.*

(ii) *$\widetilde{T}_{s,t}=|T|^sU|T|^t$ is hyponormal for any $s>0$ and $t>0$ such that $\max\{s,t\}\leq p$.*

**Lemma 2.3 ([8]).** *Let $T\in B(H)$, $D\in B(H)$ with $0\leq D\leq M(T-\lambda)(T-\lambda)^*$ for all $\lambda$ in $\mathcal{C}$, where $M$ is a positive real number. Then for every $x\in D^{\frac{1}{2}}H$ there exists a bounded function $f:\mathcal{C}\to H$ such that $(T-\lambda)f(\lambda)\equiv x$.*

**Lemma 2.4 ([10]).** *If $T\in wF(p,r,q)$, then $\left|\widetilde{T}_{p,r}\right|^{2m}\geq|T|^{2m}\geq\left|(\widetilde{T}_{p,r})^*\right|^{2m}$, where $m=\min\left\{\frac{1}{q},\max\left\{\frac{p}{p+r},1-\frac{1}{q}\right\}\right\}$, i.e., $\widetilde{T}_{p,r}=|T|^pU|T|^r$ is m-hyponormal operator.*

**Lemma 2.5 ([11]).** *Let $A,B\geq 0$, $\alpha_0,\beta_0>0$ and $-\beta_0\leq\delta\leq\alpha_0$, $-\beta_0\leq\bar{\delta}\leq\alpha_0$, if $0\leq\delta\leq\alpha_0$ and $\left(B^{\frac{\beta_0}{2}}A^{\alpha_0}B^{\frac{\beta_0}{2}}\right)^{\frac{\beta_0+\delta}{\alpha_0+\beta_0}}\geq B^{\beta_0+\delta}$, then*

$$\left(B^{\frac{\beta}{2}}A^{\alpha}B^{\frac{\beta}{2}}\right)^{\frac{\beta+\delta}{\alpha+\beta}}\geq B^{\beta+\delta},$$


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*and*

$$A^{\alpha-\bar{\delta}} \geq \left(A^{\frac{\alpha}{2}} B^{\beta} A^{\frac{\alpha}{2}}\right)^{\frac{\alpha-\bar{\delta}}{\alpha+\beta}}$$

*hold for each* $\alpha \geq \alpha_0, \beta \geq \beta_0$ *and* $0 \leq \bar{\delta} \leq \alpha$.

**Lemma 2.6 ([6]).** *Let* $A \geq 0$, $B \geq 0$, *if* $B^{\frac{1}{2}} A B^{\frac{1}{2}} \geq B^2$ *and* $A^{\frac{1}{2}} B A^{\frac{1}{2}} \geq A^2$ *then* $A = B$.

**Lemma 2.7.** *Let* $A, B \geq 0$, $s, t \geq 0$, *if* $B^s A^{2t} B^s = B^{2s+2t}$, $A^t B^{2s} A^t = A^{2s+2t}$ *then* $A = B$.

*Proof.* We choose $k > \max\{s, t$. Since $B^s A^{2t} B^s = B^{2s+2t}$, $A^t B^{2s} A^t = A^{2s+2t}$ it follows from Lemma 2.5 that:

$$(B^k A^{2k} B^k)^{\frac{2k+2t}{4k}} \geq B^{2k+2t},$$

$$A^{2k-2t} \geq (A^k B^{2k} A^k)^{\frac{2k-2t}{4k}},$$

and

$$(A^k B^{2k} A^k)^{\frac{2k+2s}{4k}} \geq A^{2k+2s},$$

$$B^{2k-2s} \geq (B^k A^{2k} B^k)^{\frac{2k-2s}{4k}}.$$

So

$$A^k B^{2k} A^k = A^{4k}, \qquad B^k A^{2k} B^k = B^{4k},$$

by Lemma 2.6

$$A = B.$$

□

**Lemma 2.8 ([11]).** *Let* $T$ *be a class* $wF(p, r, q)$ *operator, if* $\widetilde{T}_{p,r} = |T|^p U |T|^r$ *is normal, then* $T$ *is normal.*


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The following theorem have been shown by T. Huruya in [3], here we give a simple proof.

**Theorem 2.9 (Furuta inequality [4]).** *If* $A \geq B \geq 0$, *then for each* $r > 0$,

*(i)* $\left(B^{\frac{r}{2}} A^p B^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(B^{\frac{r}{2}} B^p B^{\frac{r}{2}}\right)^{\frac{1}{q}}$ *and*

*(ii)* $\left(A^{\frac{r}{2}} A^p A^{\frac{r}{2}}\right)^{\frac{1}{q}} \geq \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{1}{q}}$

*hold for* $p \geq 0$ *and* $q \geq 1$ *with* $(1+r)q \geq p+r$.

**Theorem 2.10.** *Let* $T$ *be a* $p-$*hyponormal operator on* $H$ *and let* $T = U|T|$ *be the polar decomposition of* $T$, *if* $\widetilde{T}_{s,t} = |T|^s U |T|^t$ $(s+t=1)$ *is normal, then* $T$ *is normal.*

*Proof.* First, consider the case $\max\{s,t\} \geq p$. Let $A = |T|^{2p}$ and $B = |T^*|^{2p}$, $p$-hyponormality of $T$ ensures $A \geq B \geq 0$. Applying Theorem 2.9 to $A \geq B \geq 0$, since

$$\left(1+\frac{t}{p}\right)\frac{s+t}{p+\min(s,t)} \geq \frac{s}{p}+\frac{t}{p} \quad \text{and} \quad \frac{s+t}{p+\min(s,t)} \geq 1,$$

we have

$$\begin{aligned}
(\widetilde{T}^*_{s,t}\widetilde{T}_{s,t})^{\frac{p+\min(s,t)}{s+t}} &= (|T|^t U^* |T|^{2s} U |T|^t)^{\frac{p+\min(s,t)}{s+t}} \\
&= (U^* U |T|^t U^* |T|^{2s} U |T|^t U^* U)^{\frac{p+\min(s,t)}{s+t}} \\
&= (U^* |T^*|^t |T|^{2s} |T^*|^t U)^{\frac{p+\min(s,t)}{s+t}} \\
&= U^* (|T^*|^t |T|^{2s} |T^*|^t)^{\frac{p+\min(s,t)}{s+t}} U \\
&= U^* (B^{\frac{t}{2p}} A^{\frac{s}{p}} B^{\frac{t}{2p}})^{\frac{p+\min(s,t)}{s+t}} U \\
&\geq U^* B^{\frac{p+\min(s,t)}{p}} U = U^* |T^*|^{2(p+\min(s,t))} U = |T|^{2(p+\min(s,t))}.
\end{aligned}$$

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Similarly, we also have

$$(\widetilde{T}_{s,t}\widetilde{T}^*_{s,t})^{\frac{p+\min(s,t)}{s+t}} \le |T|^{2(p+\min(s,t))}.$$

Therefore, we have

$$(\widetilde{T}^*_{s,t}\widetilde{T}_{s,t})^{\frac{p+\min(s,t)}{s+t}} \ge |T|^{2(p+\min(s,t))} \ge (\widetilde{T}_{s,t}\widetilde{T}^*_{s,t})^{\frac{p+\min(s,t)}{s+t}}.$$

If

$$\widetilde{T}_{s,t} = |T|^s U |T|^t \quad (s+t=1)$$

is normal, then

$$(\widetilde{T}^*_{s,t}\widetilde{T}_{s,t})^{\frac{p+\min(s,t)}{s+t}} = |T|^{2(p+\min(s,t))} = (\widetilde{T}_{s,t}\widetilde{T}^*_{s,t})^{\frac{p+\min(s,t)}{s+t}},$$

which implies

$$|T^*|^t|T|^{2s}|T^*|^t = |T^*|^{2(s+t)} \quad \text{and} \quad |T|^s|T^*|^{2t}|T|^s = |T|^{2(s+t)},$$

then $|T^*| = |T|$ by Lemma 2.7. Next, consider the case $\max\{s,t\} \le p$. Firstly, $p-$hyponormality of $T$ ensures $|T|^{2s} \ge |T^*|^{2s}$ and $|T|^{2t} \ge |T^*|^{2t}$ for $\max\{s,t\} \le p$ by the Löwner-Heinz theorem. Then we have

$$\begin{aligned}
\widetilde{T}^*_{s,t}\widetilde{T}_{s,t} &= |T|^t U^* |T|^{2s} U |T|^t \ge |T|^t U^* |T^*|^{2s} U |T|^t \\
&= |T|^{2(s+t)} \\
\widetilde{T}_{s,t}\widetilde{T}^*_{s,t} &= |T|^s U |T|^{2t} U^* |T|^s \\
&\le |T|^{2(s+t)}.
\end{aligned}$$

If $\widetilde{T}_{s,t} = |T|^s U |T|^t$ $(s+t=1)$ is normal, then

$$\widetilde{T}^*_{s,t}\widetilde{T}_{s,t} = |T|^{2((s+t)} = \widetilde{T}_{s,t}\widetilde{T}^*_{s,t},$$


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which implies

$$|T^*|^t|T|^{2s}|T^*|^t = |T^*|^{2(s+t)} \quad \text{and} \quad |T|^s|T^*|^{2t}|T|^s = |T|^{2(s+t)},$$

then $|T^*| = |T|$ by Lemma 2.7. $\square$



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## 3. Main Theorem

**Theorem 3.1.** *Assume that $T$ is a class $wF(p,r,q)$ operator with $\mathrm{Ker}(T) \subset \mathrm{Ker}(T^*)$, and $N$ is a normal operator on $H$ and $K$ respectively. If $X \in B(K,H)$ is injective with dense range which satisfies $TX = XN$, then $T^*X = XN^*$.*

*Proof.* $\mathrm{Ker}(T) \subset \mathrm{Ker}(T^*)$ implies $\mathrm{Ker}(T)$ reduces $T$. Also $\mathrm{Ker}(N)$ reduces $N$ since $N$ is normal. Using the orthogonal decompositions $H = \overline{\mathrm{Ran}(|T|)} \bigoplus \mathrm{Ker}(T)$ and $H = \overline{\mathrm{Ran}(N)} \bigoplus \mathrm{Ker}(N)$, we can represent $T$ and $N$ as follows.

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $T_1$ is an injective class $wF(p,r,q)$ operator on $\overline{\mathrm{Ran}(|T|)}$ and $N_1$ is injective normal on $\overline{\mathrm{Ran}(N)}$. The assumption $TX = XN$ asserts that $X$ maps $\mathrm{Ran}(N)$ to $\mathrm{Ran}(T) \subset \overline{\mathrm{Ran}(|T|)}$ and $\mathrm{Ker}(N)$ to $\mathrm{Ker}(T)$, hence $X$ is of the form:

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix},$$

where $X_1 \in B(\overline{\mathrm{Ran}(N)}, \overline{\mathrm{Ran}(|T|)})$, $X_2 \in B(\mathrm{Ker}(N), \mathrm{Ker}(T))$. Since $TX = XN$, we have that $T_1X_1 = X_1N_1$. Since $X$ is injective with dense range, $X_1$ is also injective with dense range. Put $W_1 = |T_1|^p X_1$, then $W_1$ is also injective with dense range and satisfies $\widetilde{(T_1)}_{p,r} W_1 = W_1 N$. Put $W_n = \left|(\widetilde{T_1})^{(n)}_{p,r}\right|^p W_{(n-1)}$, then $W_n$ is also injective with dense range and satisfies $(\widetilde{T_1})^{(n)}_{p,r} W_n = W_n N$. From Lemma 2.2 and Lemma 2.4, if there is an integer $\alpha_0$ such that $(\widetilde{T_1})^{(\alpha_0)}_{p,r}$ is a hyponormal operator, then

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$(\widetilde{T}_1)^{(n)}_{p,r}$ is a hyponormal operator for $n \geq \alpha_0$. It follows from Lemma 2.3 that there exists a bounded function $f : \mathcal{C} \to H$ such that

$$\left(\left(\left(\widetilde{T}_1\right)^{(n)}_{p,r}\right)^* - \lambda\right) f(\lambda) \equiv x, \text{ for every}$$

$$x \in \left(\left(\left(\widetilde{T}_1\right)^{(n)}_{p,r}\right)^* \left(\widetilde{T}_1\right)^{(n)}_{p,r} - \left(\widetilde{T}_1\right)^{(n)}_{p,r} \left(\left(\widetilde{T}_1\right)^{(n)}_{p,r}\right)^*\right)^{\frac{1}{2}} H.$$

Hence

$$\begin{aligned} W_n^* x &= W_n^* \left(\left(\left(\widetilde{T}_1\right)^{(n)}_{p,r}\right)^* - \lambda\right) f(\lambda) \\ &= (N_1^* - \lambda) W_n^* f(\lambda) \in \operatorname{Ran}(N_1^* - \lambda) \quad \text{for all } \lambda \in \mathcal{C} \end{aligned}$$

By Lemma 2.1, we have $W_n^* x = 0$, and hence $x = 0$ because $W_n^*$ is injective. This implies that $(\widetilde{T}_1)^{(n)}_{p,r}$ is normal. By Lemma 2.8 and Theorem 2.10, $T_1$ is nomal and therefore $T = T_1 \bigoplus 0$ is also normal. The assertion is immediate from Fuglede-Putnam's theorem. □

Let $X$ be a $Banach$ space, $U$ be an open subset of $\mathcal{C}$. $\varepsilon(U, X)$ denotes the *Fréchet* space of all $X-$valued $\mathcal{C}^\infty-$functions, i.e., infinitely differentiable functions on $U$ ([3]). $T \in B(X)$ is said to satisfy property $(\beta)_\varepsilon$ if for each open subset $U$ of $\mathcal{C}$, the map

$$T_z : \varepsilon(U, X) \to \varepsilon(U, X), \quad f \mapsto (T - z)f$$

is a topological monomorphism, i.e., $T_z f_n \to 0$ $(n \to \infty)$ in $\varepsilon(U, X)$ implies $f_n \to 0$ $(n \to \infty)$ in $\varepsilon(U, X)$ ([3]).

**Lemma 3.2 ([1]).** *Let $T \in B(X)$. $T$ is subscalar if and only if $T$ satisfies property* $(\beta)_\varepsilon$.



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**Lemma 3.3.** *Let $T \in B(X)$. $T$ satisfies property $(\beta)_\varepsilon$ if and only if $\widetilde{T}_{p,r}$ satisfies property $(\beta)_\varepsilon$.*

*Proof.* First, we suppose that $T$ satisfies property $(\beta)_\varepsilon$, $U$ is an open subset of $\mathcal{C}$, $f_n \in \varepsilon(U, X)$ and

$$(\widetilde{T}_{p,r} - z)f_n \to 0 \quad (n \to \infty), \tag{3.1}$$

in $\varepsilon(U, X)$, then

$$(T - z)U|T|^r f_n = U|T|^r(\widetilde{T}_{p,r} - z)f_n \to 0 \quad (n \to \infty).$$

Since $T$ satisfies property $(\beta)_\varepsilon$, we have $U|T|^r f_n \to 0$ $(n \to \infty)$. and therefore

$$\widetilde{T}_{p,r} f_n \to 0 \quad (n \to \infty). \tag{3.2}$$

(3.1) and (3.2) imply that

$$z f_n = \widetilde{T}_{p,r} f_n - (\widetilde{T}_{p,r} - z)f_n \to 0 \quad (n \to \infty) \tag{3.3}$$

in $\varepsilon(U, X)$. Notice that $T = 0$ is a subscalar operator and hence satisfies property $(\beta)_\varepsilon$ by Lemma 3.2. Now we have

$$f_n \to 0 \quad (n \to \infty). \tag{3.4}$$

(3.1) and (3.4) imply that $\widetilde{T}_{p,r}$ satisfies property $(\beta)_\varepsilon$. Next we suppose that $\widetilde{T}_{p,r}$ satisfies property $(\beta)_\varepsilon$, $U$ is an open subset of $\mathcal{C}$, $f_n \in \varepsilon(U, X)$ and

$$(T - z)f_n \to 0 \quad (n \to \infty), \tag{3.5}$$

in $\varepsilon(U, X)$. Then

$$(\widetilde{T}_{p,r} - z)|T|^p f_n = |T|^p(T - z)f_n \to 0 \quad (n \to \infty).$$


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Since $\widetilde{T}_{p,r}$ satisfies property $(\beta)_\varepsilon$, we have $|T|^p f_n \to 0$ $(n \to \infty)$, and therefore

$$Tf_n \to 0 \quad (n \to \infty). \tag{3.6}$$

(3.5) and (3.6) imply

$$zf_n = Tf_n - (T - z)f_n \to 0 \quad (n \to \infty).$$

So $f_n \to 0$ $(n \to \infty)$. Hence $T$ satisfies property $(\beta)_\varepsilon$. □

**Lemma 3.4 ([1]).** *Suppose that $T$ is a $p-$hyponormal operator, then $T$ is subscalar.*

**Theorem 3.5.** *Let $T \in wF(p, r, q)$ and $p + r = 1$, then $T$ is subdecomposable.*

*Proof.* If $T \in wF(p, r, q)$, then $\widetilde{T}_{p,r}$ is a $m$-hyponormal operator by Lemma 2.4, and it follows from Lemma 3.4 that $\widetilde{T}_{p,r}$ is subscalar. So we have $T$ is subscalar by Lemma 3.2 and Lemma 3.3. It is well known that subscalar operators are subdecomposable operators ([3]). Hence $T$ is subdecomposable. □

Recall that an operator $X \in B(H)$ is called a quasiaffinity if $X$ is injective and has dense range. For $T_1, T_2 \in B(H)$, if there exist quasiaffinities $X \in B(H_2, H_1)$ and $Y \in B(H_1, H_2)$ such that $T_1X = XT_2$ and $YT_1 = T_2Y$ then we say that $T_1$ and $T_2$ are quasisimilar.

**Lemma 3.6 ([2]).** *Let $S \in B(H)$ be subdecomposable, $T \in B(H)$. If $X \in B(K, H)$ is injective with dense range which satisfies $XT = SX$, then $\sigma(S) \subset \sigma(T)$; if $T$ and $S$ are quasisimilar, then $\sigma_e(S) \subseteq \sigma_e(T)$.*

**Theorem 3.7.** *Let $T_1, T_2 \in wF(p, r, q)$. If $T_1$ and $T_2$ are quasisimilar then $\sigma(T_1) = \sigma(T_2)$ and $\sigma_e(T_1) = \sigma_e(T_2)$.*

*Proof.* Obvious from Theorem 3.5 and Lemma 3.6. □



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