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TWO MAPPINGS RELATED TO MINKOWSKI'S INEQUALITIES

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ABSTRACT. In this paper, by the Minkowski's inequalities we define two mappings, investigate their properties, obtain some refinements for Minkowski's inequalities and some new inequalities.

Key words and phrases: Minkowski's inequality, Jensen's inequality, convex function, concave function, refinement.

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1. INTRODUCTION

Throughout this paper, for any given positive integer n and two real numbers a, b such that $a < b$, let $a_i > 0, b_i > 0$ $(i = 1, 2, \ldots, n)$ and $f, g : [a, b] \to (0, +\infty)$ be two functions, $0^r = 0$ $(r < 0)$ is assumed.

Let f^p, g^p and $(f + g)^p$ be integrable functions on $[a, b]$. If $p > 1$, then

$$\left(\sum_{i=1}^{n} {a_i}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} {b_i}^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{p}}, \tag{1.1}$$

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} + \left(\int_a^b g^p(x)dx\right)^{\frac{1}{p}} \geq \left(\int_a^b (f(x) + g(x))^p\, dx\right)^{\frac{1}{p}}. \tag{1.2}$$

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The inequalities (1.1) and (1.2) are equivalent to the following:

$$\left[\left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i{}^p\right)^{\frac{1}{p}} - \left(\sum_{i=1}^{n} (a_i+b_i)^p\right)^{\frac{1}{p}}\right]\left(\sum_{i=1}^{n} (a_i+b_i)^p\right)^{\frac{1}{q}} = \left(\left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i{}^p\right)^{\frac{1}{p}}\right)\left(\sum_{i=1}^{n} (a_i+b_i)^p\right)^{\frac{1}{q}} - \sum_{i=1}^{n} (a_i+b_i)^p \geq 0 \tag{1.3}$$

and

$$\begin{aligned}\left[\left(\int_a^b f^p(s)ds\right)^{\frac{1}{p}} + \left(\int_a^b g^p(s)ds\right)^{\frac{1}{p}} - \left(\int_a^b (f(s)+g(s))^p\,ds\right)^{\frac{1}{p}}\right]& \\ \times\left(\int_a^b (f(s)+g(s))^p\,ds\right)^{\frac{1}{q}}& \\ = \left(\left(\int_a^b f^p(s)ds\right)^{\frac{1}{p}} + \left(\int_a^b g^p(s)ds\right)^{\frac{1}{p}}\right)\left(\int_a^b (f(s)+g(s))^p\,ds\right)^{\frac{1}{q}}& \\ - \int_a^b (f(s)+g(s))^p\,ds& \\ \geq 0,&\end{aligned} \tag{1.4}$$

respectively.

If $p < 1$ ($p \neq 0$), then the inequalities in (1.1), (1.2), (1.3) and (1.4) are reversed.

The inequality (1.1) is called the Minkowski inequality, (1.2) is the integral form of inequality (1.1) (see [1] – [5]). For some recent results which generalize, improve, and extend this classic inequality, see [6] and [7].

To go further into (1.1) and (1.2), we define two mappings M and m by

$$M : \{(j,k) \mid 1 \leq j \leq k \leq n; j,k \in \mathbb{N}\} \to \mathbb{R},$$

$$M(j,k) = \left[\left(\sum_{i=j}^{k} a_i{}^p\right)^{\frac{1}{p}} + \left(\sum_{i=j}^{k} b_i{}^p\right)^{\frac{1}{p}}\right]\left(\sum_{i=j}^{k} (a_i+b_i)^p\right)^{\frac{1}{q}} - \sum_{i=j}^{k} (a_i+b_i)^p,$$

$$m : \{(x,y) \mid a \leq x \leq y \leq b\} \to \mathbb{R},$$

$$m(x,y) = \left[\left(\int_x^y f^p(s)ds\right)^{\frac{1}{p}} + \left(\int_x^y g^p(s)ds\right)^{\frac{1}{p}}\right]\left(\int_x^y (f(s)+g(s))^p\,ds\right)^{\frac{1}{q}} - \int_x^y (f(s)+g(s))^p\,ds,$$

where p and q be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$.

M and m are generated by (1.3) and (1.4), respectively.

The aim of this paper is to study the properties of M and m, thus obtaining some new inequalities and refinements of (1.1) and (1.2).

2. MAIN RESULTS

The properties of the mapping M are embodied in the following theorem.

Theorem 2.1. *Let $a_i > 0, b_i > 0$ $(i = 1, 2, \dots, n; n > 1)$, p and q be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$, and M be defined as in the first section. We write*

$$D(j,k) = \left[\left\{ \left(\sum_{i=j}^{k} a_i{}^p \right)^{\frac{1}{p}} + \left(\sum_{i=j}^{k} b_i{}^p \right)^{\frac{1}{p}} \right\} \left(\sum_{i=j}^{k} (a_i + b_i)^p \right)^{\frac{1}{q}} + \sum_{i=k+1}^{n} (a_i + b_i)^p + \sum_{i=1}^{j-1} (a_i + b_i)^p \right] \times \left(\sum_{i=1}^{n} (a_i + b_i)^p \right)^{-\frac{1}{q}}, \qquad (1 \le j \le k \le n),$$

where $\sum_{i=v}^{v-1} (a_i + b_i)^p = 0$ $(v = 1, n + 1)$.

When $p > 1$, we get the following three class results.

(1) *For any three positive integers r, j and k such that $1 \le r \le j < k \le n$, we have*

$$M(r,k) \ge M(r,j) + M(j+1,k). \tag{2.1}$$

(2) *For $l, j = 1, 2, \dots, n-1$, we have*

$$M(1, l+1) \ge M(1, l), \tag{2.2}$$

$$M(j,n) \ge M(j+1,n). \tag{2.3}$$

(3) *For any two real numbers $\alpha \ge 0$ and $\beta \ge 0$ such that $\alpha + \beta = 1$, we get the following refinements of (1.1)*

$$\begin{aligned} \left(\sum_{i=1}^{n} a_i{}^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i{}^p \right)^{\frac{1}{p}} &= D(1,n) \\ &\ge \alpha D(1, n-1) + \beta D(2,n) \\ &\ge \cdots \\ &\ge \alpha D(1,2) + \beta D(n-1,n) \\ &\ge \alpha D(1,1) + \beta D(n,n) \\ &= \left(\sum_{i=1}^{n} (a_i + b_i)^p \right)^{\frac{1}{p}}. \end{aligned} \tag{2.4}$$

When $p < 1$ $(p \ne 0)$, the inequalities in (2.1) – (2.4) are reversed.

The properties of the mapping m are given in the following theorem.

Theorem 2.2. *Let f^p, g^p and $(f+g)^p$ be integrable functions on $[a,b]$, p and q be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$, and m be defined as in the first section. Then we obtain the following four class results.*

(1) *If $p > 1$, for any $x, y, z \in [a,b]$ such that $x < y < z$, then*

$$m(x,z) \ge m(x,y) + m(y,z). \tag{2.5}$$

If $p < 1$ $(p \ne 0)$, then the inequality in (2.5) is reversed.

(2) *The mapping $m(x,b)$ monotonically decreases when $p > 1$, and monotonically increases for $p < 1$ $(p \ne 0)$ on $[a,b]$ with respect to x.*

(3) *The mapping* $m(a, y)$ *monotonically increases when* $p > 1$*, and monotonically decreases for* $p < 1$ $(p \neq 0)$ *on* $[a, b]$ *with respect to* y*.*
(4) *For any* $x \in (a, b)$ *and any two real numbers* $\alpha \geq 0$ *and* $\beta \geq 0$ *such that* $\alpha + \beta = 1$*, when* $p > 1$*, we get the following refinement of (1.2)*

$$\begin{aligned}
(2.6)\quad & \left(\int_a^b f^p(s)ds\right)^{\frac{1}{p}} + \left(\int_a^b g^p(s)ds\right)^{\frac{1}{p}} \\
& \geq \alpha \left[\left(\left(\int_a^x f^p(s)ds\right)^{\frac{1}{p}} + \left(\int_a^x g^p(s)ds\right)^{\frac{1}{p}} \right) \left(\int_a^x (f(s)+g(s))^p\, ds\right)^{\frac{1}{q}} \right. \\
& \qquad \left. + \left(\int_x^b (f(s)+g(s))^p\, ds\right) \right] \left(\int_a^b (f(s)+g(s))^p\, ds\right)^{-\frac{1}{q}} \\
& \qquad + \beta \left[\left(\left(\int_x^b f^p(s)ds\right)^{\frac{1}{p}} + \left(\int_x^b g^p(s)ds\right)^{\frac{1}{p}} \right) \left(\int_x^b (f(s)+g(s))^p\, ds\right)^{\frac{1}{q}} \right. \\
& \qquad \left. + \left(\int_a^x (f(s)+g(s))^p\, ds\right) \right] \left(\int_a^b (f(s)+g(s))^p\, ds\right)^{-\frac{1}{q}} \\
& \geq \left(\int_a^b (f(s)+g(s))^p\, ds\right)^{\frac{1}{p}}.
\end{aligned}$$

If $p < 1$ $(p \neq 0)$*, then the inequalities in (2.6) are reversed.*

3. SEVERAL LEMMAS

In order to prove the above theorems, we need the following two lemmas.

Lemma 3.1. *Let* $c_i > 0, d_i > 0$ $(i = 1, 2, \ldots, n; n > 1)$*,* p *and* q *be two non-zero real numbers such that* $p^{-1} + q^{-1} = 1$*. We write*

$$H(j, k; c_i, d_i) = \left(\sum_{i=j}^k {c_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=j}^k {d_i}^q\right)^{\frac{1}{q}} - \sum_{i=j}^k c_i d_i\,, \quad (1 \leq j \leq k \leq n).$$

For any three positive integers r, j *and* k *such that* $1 \leq r \leq j < k \leq n$*, if* $p > 1$*, we obtain*

$$H(r, k; c_i, d_i) \geq H(r, j; c_i, d_i) + H(j + 1, k; c_i, d_i). \tag{3.1}$$

The inequality in (3.1) is reversed for $p < 1$ $(p \neq 0)$*.*

Proof of Lemma 3.1.
Case 1: $p > 1$. Clearly, $0 < p^{-1} < 1$ and $x^{\frac{1}{p}}$ is a concave function on $(0, +\infty)$ with respect to x. Using Jensen's inequality for concave functions (see [2] – [4] and [8]) and $p^{-1} + q^{-1} = 1$,

for any three positive integers r, j and k such that $1 \le r \le j < k \le n$, we have

$$
\begin{aligned}
(3.2)\qquad & H(r,k;c_i,d_i) \\
&= \left(\sum_{i=r}^{k} c_i{}^p\right)^{\frac{1}{p}} \left(\sum_{i=r}^{k} d_i{}^q\right)^{\frac{1}{q}} - \sum_{i=r}^{k} c_i d_i \\
&= \left(\sum_{i=r}^{k} d_i{}^q\right) \left[\left(\sum_{i=r}^{k} d_i{}^q\right)^{-1} \left(\left(\sum_{i=r}^{j} d_i{}^q\right) \left[\left(\sum_{i=r}^{j} d_i{}^q\right)^{-1} \left(\sum_{i=r}^{j} c_i{}^p\right)\right]\right.\right. \\
&\qquad\qquad \left.\left. + \left(\sum_{i=j+1}^{k} d_i{}^q\right) \left[\left(\sum_{i=j+1}^{k} d_i{}^q\right)^{-1} \left(\sum_{i=j+1}^{k} c_i{}^p\right)\right]\right)\right]^{\frac{1}{p}} - \sum_{i=r}^{k} c_i d_i \\
&\ge \left(\sum_{i=r}^{j} d_i{}^q\right) \left[\left(\sum_{i=r}^{j} d_i{}^q\right)^{-1} \left(\sum_{i=r}^{j} c_i{}^p\right)\right]^{\frac{1}{p}} \\
&\qquad\qquad + \left(\sum_{i=j+1}^{k} d_i{}^q\right) \left[\left(\sum_{i=j+1}^{k} d_i{}^q\right)^{-1} \left(\sum_{i=j+1}^{k} c_i{}^p\right)\right]^{\frac{1}{p}} - \sum_{i=r}^{k} c_i d_i \\
&= \left(\sum_{i=r}^{j} c_i{}^p\right)^{\frac{1}{p}} \left(\sum_{i=r}^{j} d_i{}^q\right)^{\frac{1}{q}} + \left(\sum_{i=j+1}^{k} c_i{}^p\right)^{\frac{1}{p}} \left(\sum_{i=j+1}^{k} d_i{}^q\right)^{\frac{1}{q}} \\
&\qquad\qquad - \sum_{i=r}^{j} c_i d_i - \sum_{i=j+1}^{k} c_i d_i \\
&= H(r,j;c_i,d_i) + H(j+1,k;c_i,d_i),
\end{aligned}
$$

which is (3.1).

Case 2: $p < 1$ ($p \neq 0$). Clearly, $x^{\frac{1}{p}}$ is a convex function on $(0, +\infty)$. Using Jensen's inequality for convex functions (see [2] – [4] and [8]), we obtain the reverse of (3.2), which is the reverse of (3.1).

The proof of Lemma 3.1 is completed. □

Lemma 3.2. *Let p and q be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$, and let u^p, v^p and $(u+v)^p$ be positive integrable functions on $[a,b]$. We write*

$$h(x,y;u,v) = \left(\int_x^y u^p(s)ds\right)^{\frac{1}{p}} \left(\int_x^y v^q(s)ds\right)^{\frac{1}{q}} - \int_x^y u(s)v(s)ds, \quad (a \le x \le y \le b).$$

When $p > 1$, for any $x, y, z \in [a,b]$ such that $x < y < z$, we obtain

$$h(x,z;u,v) \ge h(x,y;u,v) + h(y,z;u,v). \tag{3.3}$$

When $p < 1$ $(p \neq 0)$, the inequality in (3.3) is reversed.

Proof of Lemma 3.2. When $p > 1$, i. e. $0 < p^{-1} < 1$, $x^{\frac{1}{p}}$ is a concave function on $(0, +\infty)$. Using Jensen's integral inequality for concave functions (see [2] – [4] and [8]) and $p^{-1} + q^{-1} =$

1, for any $x, y, z \in [a, b]$ such that $x < y < z$, we obtain

$$
\begin{aligned}
(3.4)\qquad & h(x,z;u,v) \\
&= \int_x^z v^q(s)ds \left[\left(\int_x^z v^q(s)ds\right)^{-1} \left(\int_x^y v^q(s)ds \left(\int_x^y v^q(s)ds\right)^{-1} \int_x^y u^p(s)ds \right.\right. \\
&\qquad\left.\left. + \int_y^z v^q(s)ds \left(\int_y^z v^q(s)ds\right)^{-1} \int_y^z u^p(s)ds \right)\right]^{\frac{1}{p}} - \int_x^z u(s)v(s)ds \\
&\geq \left[\int_x^y v^q(s)ds \left(\left(\int_x^y v^q(s)ds\right)^{-1} \int_x^y u^p(s)ds \right)^{\frac{1}{p}} \right. \\
&\qquad \left. + \int_y^z v^q(s)ds \left(\left(\int_y^z v^q(s)ds\right)^{-1} \int_y^z u^p(s)ds \right)^{\frac{1}{p}} \right] \\
&\qquad - \int_x^y u(s)v(s)ds - \int_y^z u(s)v(s)ds \\
&= \left(\int_x^y u^p(s)ds\right)^{\frac{1}{p}} \left(\int_x^y v^q(s)ds\right)^{\frac{1}{q}} + \left(\int_y^z u^p(s)ds\right)^{\frac{1}{p}} \left(\int_y^z v^q(s)ds\right)^{\frac{1}{q}} \\
&\qquad - \int_x^y u(s)v(s)ds - \int_y^z u(s)v(s)ds \\
&= h(x,y;u,v) + h(y,z;u,v),
\end{aligned}
$$

which is (3.3).

When $p < 1$ ($p \neq 0$), $x^{\frac{1}{p}}$ is a convex function on $(0, +\infty)$. Using Jensen's integral inequality for convex functions (see [2] – [4] and [8]), we obtain the reverse of (3.4), which is the reverse of (3.3).

The proof of Lemma 3.2 is completed. □

4. PROOF OF THE THEOREMS

Proof of Theorem 2.1. From $p^{-1} + q^{-1} = 1$ (i. e. $p = q(p-1)$) and definitions of M and H, we get

$$(4.1)\qquad M(j,k) = H\left(j,k;a_i,(a_i+b_i)^{p-1}\right) + H\left(j,k;b_i,(a_i+b_i)^{p-1}\right).$$

Case 1: $p > 1$.

(1) For any three positive integers r, j and k such that $1 \leq r \leq j < k \leq n$, from (4.1) and (3.1), we obtain

$$
\begin{aligned}
(4.2)\qquad M(r,k) &= H\left(r,k;a_i,(a_i+b_i)^{p-1}\right) + H\left(r,k;b_i,(a_i+b_i)^{p-1}\right) \\
&\geq H\left(r,j;a_i,(a_i+b_i)^{p-1}\right) + H\left(r,j;b_i,(a_i+b_i)^{p-1}\right) \\
&\qquad + H\left(j+1,k;a_i,(a_i+b_i)^{p-1}\right) + H\left(j+1,k;b_i,(a_i+b_i)^{p-1}\right) \\
&= M(r,j) + M(j+1,k),
\end{aligned}
$$

which is (2.1).

(2) For $l = 1, 2, \ldots, n-1$, replacing r, j and k in (2.1) with $1, l$ and $l+1$, respectively, then (2.1) reduces to (2.2) (because $M(l+1, l+1) = 0$). For $j = 1, 2, \ldots, n-1$,

replacing r and k in (2.1) with j and n, respectively, then (2.1) reduces to (2.3) (because $M(j,j) = 0$).

(3) From the definitions of D and M, we have

$$D(j,k) = \left[M(j,k) + \sum_{i=1}^{n}(a_i+b_i)^p\right]\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{-\frac{1}{q}}. \tag{4.3}$$

Using (4.3), from $\alpha \geq 0$, $\beta \geq 0$, (2.2) and (2.3), we get

$$\alpha D(1,n) \geq \alpha D(1,n-1) \geq \cdots \geq \alpha D(1,2) \geq \alpha D(1,1) \tag{4.4}$$

and

$$\beta D(1,n) \geq \beta D(2,n) \geq \cdots \geq \beta D(n-1,n) \geq \beta D(n,n), \tag{4.5}$$

respectively. From $\alpha + \beta = 1$, expression (4.4) combined with (4.5) yields (2.4).

Case 2: $p < 1$ ($p \neq 0$). The reverse of (3.1) implies the reverse of (4.2). Further, the reverse of (4.2) implies the reverse of (2.1), (2.2) and (2.3). The reverse of (2.2) and (2.3) implies the reverse of (4.4) and (4.5), respectively. The reverse of (4.4) combined with the reverse of (4.5) yields the reverse of (2.4).

The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2. From $p^{-1} + q^{-1} = 1$ (i. e. $p = q(p-1)$) and the definitions of m and h, we get

$$m(x,y) = h\left(x,y;f,(f+g)^{p-1}\right) + h\left(x,y;g,(f+g)^{p-1}\right). \tag{4.6}$$

(1) If $p > 1$, for any $x, y, z \in [a,b]$ such that $x < y < z$, from (4.6) and (3.3), we get

$$\begin{aligned} m(x,z) &= h(x,z;f,(f+g)^{p-1}) + h(x,z;g,(f+g)^{p-1}) \\ &\geq h(x,y;f,(f+g)^{p-1}) + h(x,y;g,(f+g)^{p-1}) \\ &\qquad + h(y,z;f,(f+g)^{p-1}) + h(y,z;g,(f+g)^{p-1}) \\ &= m(x,y) + m(y,z), \end{aligned} \tag{4.7}$$

which is (2.5).

If $p < 1$ ($p \neq 0$), then the reverse of (3.3) implies the reverse of (4.7). Further, (2.5) is reversed.

(2) When $p > 1$, for any $x_1, x_2 \in [a,b]$, $x_1 < x_2$, if $x_2 < b$, taking $z = b$, $x = x_1$ and $y = x_2$ in (2.5) and using $m(x_1,x_2) \geq 0$, we obtain

$$m(x_1,b) \geq m(x_1,x_2) + m(x_2,b) \geq m(x_2,b). \tag{4.8}$$

If $x_2 = b$, by the definition of m we have

$$m(x_1,b) \geq 0 = m(b,b) = m(x_2,b). \tag{4.9}$$

Then (4.8) and (4.9) imply that $m(x,b)$ is monotonically decreasing on $[a,b]$.

When $p < 1$ ($p \neq 0$), then the inequality in (2.5) is reversed, $m(x,y) \leq 0$ and $m(x,b) \leq 0$. Further, the inequalities in (4.8) and (4.9) are reversed, which implies that $m(x,b)$ is monotonically increasing on $[a,b]$.

(3) Using the same method as that for the proof of the monotonicity of $m(x,b)$, we can prove the monotonicity of $m(a,y)$ on $[a,b]$ with respect to y.

(4) **Case 1:** $p > 1$. For any $x \in (a, b)$, from the increasing property of $m(a, y)$ on $[a, b]$ with respect to y, $m(a, a) = 0$ and $\alpha \geq 0$, we get

$$\begin{aligned} &\alpha \left[m(a,b) + \int_a^b (f(s)+g(s))^p \, ds \right] \left(\int_a^b (f(s)+g(s))^p \, ds \right)^{-\frac{1}{q}} \\ &\geq \alpha \left[m(a,x) + \int_a^b (f(s)+g(s))^p \, ds \right] \left(\int_a^b (f(s)+g(s))^p \, ds \right)^{-\frac{1}{q}} \\ &\geq \alpha \left[m(a,a) + \int_a^b (f(s)+g(s))^p \, ds \right] \left(\int_a^b (f(s)+g(s))^p \, ds \right)^{-\frac{1}{q}} . \end{aligned} \tag{4.10}$$

From the decreasing property of $m(x, b)$ on $[a, b]$ with respect to x, $m(b, b) = 0$ and $\beta \geq 0$, we get

$$\begin{aligned} &\beta \left[m(a,b) + \int_a^b (f(s)+g(s))^p \, ds \right] \left(\int_a^b (f(s)+g(s))^p \, ds \right)^{-\frac{1}{q}} \\ &\geq \beta \left[m(x,b) + \int_a^b (f(s)+g(s))^p \, ds \right] \left(\int_a^b (f(s)+g(s))^p \, ds \right)^{-\frac{1}{q}} \\ &\geq \beta \left[m(b,b) + \int_a^b (f(s)+g(s))^p \, ds \right] \left(\int_a^b (f(s)+g(s))^p \, ds \right)^{-\frac{1}{q}} . \end{aligned} \tag{4.11}$$

From $\alpha + \beta = 1$, expression (4.10) plus (4.11), with a simple manipulation, we obtain (2.6).

Case 2: $p < 1$ ($p \neq 0$). The decreasing property of $m(a, y)$ on $[a, b]$ with respect to y and the increasing property of $m(x, b)$ on $[a, b]$ with respect to x imply the reverse of (4.10) and (4.11), respectively. The reverse of (4.10) and (4.11) yields the reverse of (2.6).

The proof of Theorem 2.2 is completed. □

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