journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 2, Article 59, 8 pp.



# ON THE EXISTENCE OF SOLUTIONS TO A CLASS OF $p$-LAPLACE ELLIPTIC EQUATIONS

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*Received 19 September, 2008; accepted 07 December, 2008*
*Communicated by C. Bandle*

ABSTRACT. We study the equation $-\Delta_p u+|x|^a|u|^{p-2}u = |x|^b|u|^{q-2}u$ with Dirichlet boundary condition on $B(0,R)$ or on $\mathbb{R}^N$. We prove the existence of the radial solution and nonradial solutions of this equation.

*Key words and phrases:* $p$-Laplace elliptic equations, Radial solutions, Nonradial solutions.

2000 *Mathematics Subject Classification.* 35J20.

## 1. INTRODUCTION AND MAIN RESULT

Equations of the form

$$\begin{cases} -\Delta_t u + g(x)|u|^{s-2}u = f(x,u) & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

have attracted much attention. Many papers deal with the problem (1.1) in the case of $t = 2,\ \Omega = \mathbb{R}^N,\ s = 2,\ g$ large at infinity and $f$ superlinear, subcritical and bounded in $x$, see e.g. [1], [2] and [4]. The problem (1.1) with $t = 2,\ \Omega = B(0,1),\ g(x) = 0$ and $f(x,u) = |x|^b u^{l-1}$ was studied in [9]; in particular, it was proved that under some conditions the ground states are not radial symmetric. The case $t = 2,\ \Omega = B(0,1)$ or on $\mathbb{R}^N,\ g(x) = 1$ and $f(x,u) = |x|^b|u|^{l-2}u$ was studied in [7]. The problem (1.1) with $t = s,\ \Omega = \mathbb{R}^N,\ g(x) = V(|x|)$ and $f(x,u) = Q(|x|)|u|^{l-2}u$ was studied by J. Su., Z.-Q. Wang and M. Willem ([11], [12]). They proved embedding results for functions in the weighted $W^{1,p}(\mathbb{R}^N)$ space of radial symmetry. The results were then used to obtain ground state and bound state solutions of equations with unbounded or decaying radial potentials.

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In this paper, we consider the nonlinear elliptic problem

$$\begin{cases} -\Delta_p u + |x|^a |u|^{p-2}u = |x|^b |u|^{q-2}u & \text{in } \Omega \\ u > 0, \ \ u \in W^{1,p}(\Omega), & \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

and prove the existence of the radial and the nonradial solutions of the problem (1.2). Here, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator, and $1 < p < N$, $a \geq 0, \ b \geq 0$.

We denote by $W_r^{1,p}(\mathbb{R}^N)$ the space of radially symmetric functions in

$$W^{1,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : \nabla u \in L^p(\mathbb{R}^N) \right\}.$$

$W_{r,a}^{1,p}(\mathbb{R}^N)$ is denoted by the space of radially symmetric functions in

$$W_a^{1,p}(\mathbb{R}^N) = \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |x|^a |u|^p < \infty \right\}.$$

We also denote by $D_r^{1,p}(\mathbb{R}^N)$ the space of radially symmetric functions in

$$D^{1,p}(\mathbb{R}^N) = \left\{ u \in L^{\frac{Np}{N-p}}\left(\mathbb{R}^N\right) : \nabla u \in L^p\left(\mathbb{R}^N\right) \right\}.$$

Our main results are:

**Theorem 1.1.** *If* $a \geq 0, \ b \geq 0, \ 1 < p < N$ *and*

$$p < q < \tilde{q} = \frac{Np}{N-p} + \frac{bp}{N-p},$$

$$pb - a\left(p + \frac{(p-1)(q-p)}{p}\right) < (q-p)(N-1),$$

*then the problem (1.2) has a radial solution.*

**Remark 1.** In [8], Sirakov proves that the problem (1.2) with $p = 2$ has a solution for

$$2 < q < q^{\#} = \frac{2N}{N-2} - \frac{4b}{a(N-2)}.$$

In [6], P. Sintzoff and M. Willem proved the existence of a solution of the problem (1.2) with

$$p = 2, \ q \leq 2^*, \quad 2b - a\left(1 + \frac{q}{2}\right) < (N-1)(q-2).$$

Theorem 1.1 extends the results of [6] to the general equation with a $p-$Laplacian operator.

**Theorem 1.2.** *Suppose that* $a \geq 0, \ b \geq 0, \ 1 < p < N$ *and*

$$p < q < \frac{Np}{N-p},$$

$$pb - a\left(p + \frac{(p-1)(q-p)}{p}\right) < (q-p)(N-1), \quad aq < pb,$$

*then for every* $R$*, problem (1.2) with* $\Omega = B(0,R)$*,* $R$ *large enough has a radial and a nonradial solution.*

This paper is organized as follows: In Section 2, we study (1.2) in the case of $\Omega = \mathbb{R}^N$. We prove the existence of a radial least energy solution of (1.2) when

$$1 < p < N, \ p < q < \tilde{q}, \quad pb - a\left(p + \frac{(p-1)(q-p)}{p}\right) < (q-p)(N-1).$$

In Section 3, we consider the existence of nonradial solutions of (1.2) with $\Omega = B(0,R),\ R$ large enough. Finally, in Section 4, we consider necessary conditions for the existence of solutions of (1.2).

## 2. RADIAL SOLUTION

In this paper, unless stated otherwise, all integrals are understood to be taken over all of $\mathbb{R}^N$. Also, throughout the paper, we will often denote various constants by the same letter.

**Lemma 2.1.** *Suppose that $1 < p < N$. There exist $A_N > 0$, such that, for every $u \in W^{1,p}_{r,a}(\mathbb{R}^N),\ u \in C(\mathbb{R}^N \backslash \{0\})$, for $a \geq \frac{p}{p-1}(1-N)$, we have that*

$$|x|^{\frac{N-1}{p}+\frac{a(p-1)}{p^2}}|u(x)| \leq A_N \left(\int |x|^a|u|^p\right)^{\frac{p-1}{p^2}} \left(\int |\nabla u|^p\right)^{\frac{1}{p^2}}.$$

*Proof.* Since

$$\begin{aligned}\frac{d}{dr}\left(|u|^p r^{a\cdot\frac{p-1}{p}} r^{N-1}\right) = \frac{p}{2}\left(|u|^2\right)^{\frac{p}{2}-1}\cdot 2u\cdot\frac{du}{dr} r^{a\cdot\frac{p-1}{p}} r^{N-1} \\ + |u|^p\left(a\cdot\frac{p-1}{p}+N-1\right) r^{a\cdot\frac{p-1}{p}-1} r^{N-1},\end{aligned}$$

and

$$a \geq \frac{p}{p-1}(1-N),$$

we get that

$$\frac{d}{dr}\left(|u|^p r^{a\cdot\frac{p-1}{p}} r^{N-1}\right) \geq pu|u|^{p-2}\frac{du}{dr} r^{a\cdot\frac{p-1}{p}} r^{N-1}$$

and obtain

$$\begin{aligned} r^{a\cdot\frac{p-1}{p}} r^{N-1}|u(r)|^p &\leq A_N \int_r^{+\infty} |u|^{p-1}\left|\frac{du}{dr}\right| S^{N-1} S^{a\cdot\frac{p-1}{p}} dS \\ &\leq A_N \int |u|^{p-1}\left|\frac{du}{dr}\right| |x|^{a\cdot\frac{p-1}{p}} dx \\ &\leq A_N \left(\int |x|^a|u|^p\right)^{\frac{p-1}{p}} \left(\int |\nabla u|^p\right)^{\frac{1}{p}}. \end{aligned}$$

It follows that

$$|x|^{N-1+a\cdot\frac{p}{p-1}}|u(x)|^p \leq A_N \left(\int |x|^a|u|^p\right)^{\frac{p-1}{p}} \left(\int |\nabla u|^p\right)^{\frac{1}{p}},$$

and we have

$$|x|^{\frac{N-1}{p}+\frac{a(p-1)}{p^2}}|u(x)| \leq A_N \left(\int |x|^a|u|^p\right)^{\frac{p-1}{p^2}} \left(\int |\nabla u|^p\right)^{\frac{1}{p^2}}.$$

□

**Lemma 2.2.** *If $1 < p < N,\ p \leq r < \frac{pN}{N-p}$, then for any $u \in W^{1,p}(\mathbb{R}^N)$, we have that*

$$\int |u|^r dx \leq C\left(\int |\nabla u|^p\right)^{\frac{N(r-p)}{p^2}} \left(\int |u|^p\right)^{\frac{Np+r(p-N)}{p^2}}.$$

*Proof.* The proof can be adapted directly from the Gagliardo-Nirenberg inequality. □

The following inequality extends the results of [5] to the general equation with the $p-$Laplacian operator.

**Lemma 2.3.** *For*

$$1<p<N, \quad p<q<\frac{pN}{N-p}+\frac{c}{\frac{N-1}{p}+\frac{a(p-1)}{p^2}}, \quad a\geq \frac{p}{p-1}(1-N),$$

*there exist $B_{N,p,c}$ such that for every $u\in D_r^{1,p}(\mathbb{R}^N)$, we have*

$$\int |x|^c|u|^q dx \leq B_{N,p,c}\left(\int |\nabla u|^p\right)^{\frac{c}{p(N-1)+a(p-1)}+\frac{N}{p^2}\left(q-p-\frac{cp^2}{p(N-1)+a(p-1)}\right)}.$$

*Proof.* Using Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\int |x|^c|u|^q dx\\
&=\int \left(|x|^{\frac{N-1}{p}+\frac{a(p-1)}{p^2}}\right)^{\frac{c}{(N-1)/p+a(p-1)p^{-2}}}\left(|u|\right)^{\frac{c}{(N-1)/p+a(p-1)p^{-2}}}\left(|u|\right)^{q-\frac{c}{(N-1)/p+a(p-1)p^{-2}}}dx\\
&\leq \left(\int |x|^a|u|^p\right)^{\frac{p-1}{p^2}\cdot\frac{c}{(N-1)/p+a(p-1)p^{-2}}}\left(\int |\nabla u|^p\right)^{\frac{1}{p^2}\cdot\frac{c}{(N-1)/p+a(p-1)p^{-2}}}\\
&\qquad\cdot\left(\int |\nabla u|^p\right)^{\frac{N}{p^2}(q-p-\frac{c}{(N-1)/p+a(p-1)p^{-2}})}\left(\int |u|^p\right)^{\frac{Np}{p^2}+\frac{p-N}{p^2}\left(q-\frac{c}{(N-1)/p+a(p-1)p^{-2}}\right)}\\
&=\left(\int |x|^a|u|^p dx\right)^{\frac{c(p-1)}{p(N-1)+a(p-1)}}\left(\int |u|^p\right)^{\frac{Np}{p^2}+\frac{p-N}{p^2}\left(q-\frac{cp^2}{p(N-1)+a(p-1)}\right)}\\
&\qquad\cdot\left(\int |\nabla u|^p\right)^{\frac{c}{p(N-1)+a(p-1)}+\frac{N}{p^2}\left(q-p-\frac{cp^2}{p(N-1)+a(p-1)}\right)}\\
&\leq B_{N,p,c}\left(\int |\nabla u|^p\right)^{\frac{c}{p(N-1)+a(p-1)}+\frac{N}{p^2}\left(q-p-\frac{cp^2}{p(N-1)+a(p-1)}\right)}.
\end{aligned}$$

□

Next, to prove Theorem 1.1, we consider the following minimization problem

$$m=m(a,b,p,q)=\inf_{\substack{u\in W_{r,a}^{1,p}(\mathbb{R}^N)\\ \int |x|^b|u|^q dx=1}}\int \left(|\nabla u|^p+|x|^a|u|^p\right)dx.$$

**Theorem 2.4.** *If $a\geq 0,\ b\geq 0,\ 1<p<N$ and*

$$p<q<\tilde{q}=\frac{Np}{N-p}+\frac{bp}{N-p},$$
$$pb-a\left(p+\frac{(p-1)(q-p)}{p}\right)<(q-p)(N-1),$$

*then $m(a,b,p,q)$ is achieved.*

*Proof.* Let $(u_n) \subset W^{1,p}_{r,a}(\mathbb{R}^N)$ be a minimizing sequence for $m = m(a,b,p,q)$ :

$$\int |x|^b |u_n|^q dx = 1,$$

$$\int \left(|\nabla u_n|^p + |x|^a |u_n|^p\right) dx \to m.$$

By going (if necessary) to a subsequence, we can assume that $u_n \rightharpoonup u$ in $W^{1,p}_{r,a}(\mathbb{R}^N)$. Hence, by weak lower semicontinuity, we have

$$\int \left(|\nabla u|^p + |x|^a |u|^p\right) dx \le m,$$

$$\int |x|^b |u|^q dx \le 1.$$

If $c$ is defined by $q = \frac{pN}{N-p} + \frac{pc}{N-p}$, then $c < b$ and it follows from Lemma 2.3 that

$$\int_{|x|\le\varepsilon} |x|^b |u_n|^q dx \le \varepsilon^{b-c} \int |x|^c |u_n|^q dx \le C\varepsilon^{b-c}.$$

Since $(u_n)$ is bounded in $W^{1,p}_{r,a}(\mathbb{R}^N)$. We deduce from Lemma 2.1 that

$$\begin{aligned}\int_{|x|\ge\frac{1}{\varepsilon}} |x|^b |u_n|^q dx &= \int_{|x|\ge\frac{1}{\varepsilon}} |x|^{b-a} |u_n|^{q-p} |x|^a |u_n|^p dx \\ &\le \left(\frac{1}{\varepsilon}\right)^{b-a-(q-p)\left(\frac{N-1}{p}+\frac{a(p-1)}{p^2}\right)} C \int |x|^a |u|^p dx \\ &\le C\varepsilon^{a\left(\frac{q+1}{p}-\frac{q}{p^2}\right)-b+\frac{(q-p)(N-1)}{p}}.\end{aligned}$$

So we get that, for every $t < 1$, there exists $\varepsilon > 0$, such that for every $n$,

$$\int_{\varepsilon\le|x|\le\frac{1}{\varepsilon}} |x|^b |u_n|^q dx \ge t.$$

By the Rellich theorem and Lemma 2.1,

$$1 \ge \int |x|^b |u_n|^q dx \ge \int_{\varepsilon\le|x|\le\frac{1}{\varepsilon}} |x|^b |u_n|^q dx \ge t.$$

Finally $\int |x|^b |u|^q dx = 1$ and $m = m(a,b,p,q)$ is achieved at $u$. □

Now we will prove Theorem 1.1.

*Proof.* By Theorem 2.4, $m$ is achieved. Then by the Lagrange multiplier rule, the symmetric criticality principle (see e.g. [13]) and the maximum principle, we obtain a solution of

$$\begin{cases} -\Delta_p \upsilon + |x|^a |\upsilon|^{p-2}\upsilon = \lambda |x|^b |\upsilon|^{q-2}\upsilon, \\ \upsilon > 0, \quad \upsilon \in W^{1,p}(\mathbb{R}^N). \end{cases}$$

Hence $u = \lambda^{\frac{1}{q-p}}\upsilon$ is a radial solution of (1.2), with $\lambda = \frac{p}{q}m$. □

## 3. NONRADIAL SOLUTIONS

In this section, we will prove Theorem 1.2. We use the preceding results to construct nonradial solutions of problem (1.2) in the case $\Omega = B(0,R)$.

Consider

$$M = M(a,b,p,q) = \inf_{\substack{u\in W_a^{1,p}(\mathbb{R}^N)\\ \int |x|^b|u|^q dx=1}} \int \Big(|\nabla u|^p + |x|^a|u|^p\Big)dx.$$

It is clear that $M \leq m$, and using our previous results, we prove that $M$ is achieved under some conditions.

**Theorem 3.1.** *If $a \geq 0,\ b \geq 0,\ 1 < p < N$ and*

$$p < q < q^{\#} = \frac{pN}{N-p} - \frac{p^2 b}{a(N-p)},$$

*then $M(a,b,p,q)$ is achieved.*

*Proof.* Let $(u_n) \subset W_a^{1,p}(\mathbb{R}^N)$ be a minimizing sequence for $M = M(a,b,p,q)$ :

$$\int |x|^b|u_n|^q dx = 1,$$

$$\int \Big(|\nabla u_n|^p + |x|^a|u_n|^p\Big)dx \to M.$$

By going (if necessary) to a subsequence, we can assume that $u_n \rightharpoonup u$ in $W_a^{1,p}(\mathbb{R}^N)$. Hence, by weak lower semicontinuity, we have

$$\int \Big(|\nabla u|^p + |x|^a|u|^p\Big)dx \leq M,$$

$$\int |x|^b|u|^q dx \leq 1.$$

If $c$ is defined by $q = \frac{pN}{N-p} - \frac{p^2 c}{a(N-p)}$, then $c > b$ and

$$r = \frac{a}{c}, \qquad s = \frac{\frac{aNp}{N-p}}{aq - pc}$$

are conjugate. It follows from the Hölder and Sobolev inequalities that

$$\begin{aligned}\int_{|x|\geq\frac{1}{\varepsilon}} |x|^b|u_n|^q dx &\leq \left(\frac{1}{\varepsilon}\right)^{b-c}\int |x|^c|u_n|^q dx\\ &= \left(\frac{1}{\varepsilon}\right)^{b-c}\int |x|^c|u_n|^{\frac{pc}{a}}|u_n|^{q-\frac{pc}{a}} dx\\ &\leq \varepsilon^{c-b}\left(\int |x|^a|u_n|^p dx\right)^{\frac{1}{r}}\left(\int |u_n|^{\frac{Np}{N-p}} dx\right)^{\frac{1}{s}}\\ &\leq C\varepsilon^{c-b}.\end{aligned}$$

As in Theorem 2.4, for every $t < 1$, there exists $\varepsilon > 0$ such that, for every $n$,

$$\int_{|x|\leq\frac{1}{\varepsilon}} |x|^b|u_n|^q dx \geq t.$$

By the compactness of the Sobolev theorem in the bounded domain, for $1 < p < N,\ p < q < \frac{Np}{N-p}$,

$$1 \geq \int |x|^b |u|^q dx \geq \int_{|x| \leq \frac{1}{\varepsilon}} |x|^b |u_n|^q dx \geq t.$$

Hence $\int |x|^b |u|^q dx = 1$ and $M = M(a, b, p, q)$ is achieved at $u$. □

Now we will prove Theorem 1.2.

*Proof.* By Theorem 2.4, $m(a, b, p, q)$ is positive. Since $pb > aq$, it is easy to verify that $M(a, b, p, q) = 0$. Let us define

$$M(a, b, p, q, R) = \inf_{\substack{u \in W_a^{1,p}(B(0,R)) \\ \int_{B(0,R)} |x|^b |u|^q dx = 1}} \int_{B(0,R)} \left(|\nabla u|^p + |x|^a |u|^p\right) dx,$$

$$m(a, b, p, q, R) = \inf_{\substack{u \in W_{r,a}^{1,p}(B(0,R)) \\ \int_{B(0,R)} |x|^b |u|^q dx = 1}} \int_{B(0,R)} \left(|\nabla u|^p + |x|^a |u|^p\right) dx.$$

It is clear that, for every $R > 0$, $M(a, b, p, q, R)$ and $m(a, b, p, q, R)$ are achieved and

$$\lim_{R \to \infty} M(a, b, p, q, R) = M(a, b, p, q) = 0,$$

$$\lim_{R \to \infty} m(a, b, p, q, R) = m(a, b, p, q) > 0.$$

Then from Theorem 1.1, we know that problem (1.2) with $B(0, R)$ has a radial solution.

On the other hand, by the Lagrange multiplier rule, the symmetric criticality principle (see e.g.[13]) and the maximum principle, we obtain a solution of

$$\begin{cases} -\Delta_p \upsilon + |x|^a |\upsilon|^{p-2} \upsilon = \lambda |x|^b |\upsilon|^{q-2} \upsilon & \text{in } B(0, R) \\ \upsilon > 0, \quad u \in W^{1,p}(B(0, R)), & \\ \upsilon = 0, & \text{on } \partial B(0, R). \end{cases}$$

Hence $u = \lambda^{\frac{1}{q-p}} \upsilon$ is a solution of (1.2), with $\lambda = \frac{p}{q} M(a, b, p, q, R)$. Thus, Problem (1.2) has a nonradial solution. □

## 4. NECESSARY CONDITIONS

In this section we obtain a nonexistence result for the solution of problem (1.2) using a Pohozaev-type identity. The Pohozaev identity has been derived for very general problems by H. Egnell [3].

**Lemma 4.1.** *Let $u \in W^{1,p}(\mathbb{R}^N)$ be a solution of (1.2), then $u$ satisfies*

$$\frac{N-p}{p} \int |\nabla u|^p dx + \frac{N+a}{p} \int |x|^a |u|^p dx - \frac{N+b}{q} \int |x|^b |u|^q dx = 0.$$

**Theorem 4.2.** *Suppose that*

$$\tilde{q} = \frac{Np}{N-p} + \frac{pb}{N-p} \leq q$$

*or*

$$\frac{N+a}{p} \leq \frac{N+b}{q}.$$

*Then there is no solution for problem (1.2).*

*Proof.* Multiplying (1.2) by $u$ and integrating, we see that

$$\int |x|^b |u|^q dx = \int \left(|\nabla u|^p + |x|^a |u|^p\right) dx.$$

On the other hand, using Lemma 4.1, we obtain

$$\left(\frac{N-p}{p} - \frac{N+b}{q}\right) \int |\nabla u|^p dx + \left(\frac{N+a}{p} - \frac{N+b}{q}\right) \int |x|^a |u|^p dx = 0.$$

So, if $u$ is a solution of problem (1.2), we must have

$$\frac{N-p}{p} < \frac{N+b}{q}, \qquad \frac{N+a}{p} > \frac{N+b}{q}.$$

□

**Remark 2.** The second assumption of Theorem 2.4,

$$pb - a\left(p + \frac{(q-p)(p-1)}{p}\right) < (q-p)(N-1)$$

implies that

$$\frac{N+b}{q} < \frac{N+a}{p}.$$

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