

ON THE SUBCLASS OF SALAGEAN-TYPE HARMONIC UNIVALENT FUNCTIONS

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Abstract: Using the Salagean derivative, we introduce and study a class of Goodman- Rønning type harmonic univalent functions. We obtain coefficient conditions, extreme points, distortion bounds, convolution conditions, and convex combination for the above class of harmonic functions.



Salagean-Type Harmonic
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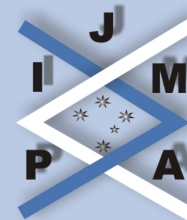
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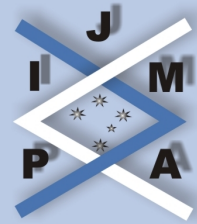
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1. Introduction

A continuous complex valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D . In any simply connected domain D we can write $f = h + \bar{g}$, where h and g are analytic in D . A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$.

Denote by S_H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$ we may express the analytic functions h and g as

$$(1.1) \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.$$

In 1984 Clunie and Sheil-Small [1] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses. Jahangiri et al. [3] make use of the Alexander integral transforms of certain analytic functions (which are starlike or convex of positive order) with a view to investigating the construction of sense-preserving, univalent, and close to convex harmonic functions.

Definition 1.1. Recently, Rosy et al. [4], defined the subclass $G_H(\gamma) \subset S_H$ consisting of harmonic univalent functions $f(z)$ satisfying the following condition

$$(1.2) \quad \operatorname{Re} \left\{ (1 + e^{i\alpha}) \frac{z f'(z)}{f(z)} - e^{i\alpha} \right\} \geq \gamma, \quad 0 \leq \gamma < 1, \quad \alpha \in \mathbb{R}.$$

They proved that if $f = h + \bar{g}$ is given by (1.1) and if

$$(1.3) \quad \sum_{n=1}^{\infty} \left[\frac{2n-1-\gamma}{1-\gamma} |a_n| + \frac{2n+1+\gamma}{1-\gamma} |b_n| \right] \leq 2, \quad 0 \leq \gamma < 1,$$

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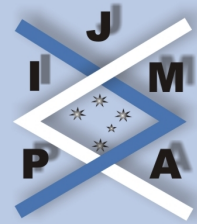
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then f is a Goodman-Ronning type harmonic univalent function in U . This condition is proved to be also necessary if h and g are of the form

$$(1.4) \quad h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n.$$

Jahangiri et al. [2] has introduced the modified Salagean operator of harmonic univalent function f as

$$(1.5) \quad D^k f(z) = D^k h(z) + (-1)^k \overline{D^k g(z)}, \quad k \in \mathbb{N},$$

where

$$D^k h(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n \quad \text{and} \quad D^k g(z) = \sum_{n=1}^{\infty} n^k b_n z^n.$$

We let $RS_H(k, \gamma)$ denote the family of harmonic functions f of the form (1.1) such that

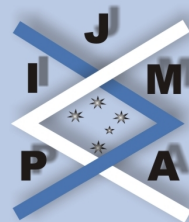
$$(1.6) \quad \operatorname{Re} \left\{ (1 + e^{i\alpha}) \frac{D^{k+1} f(z)}{D^k f(z)} - e^{i\alpha} \right\} \geq \gamma, \quad 0 \leq \gamma < 1, \alpha \in \mathbb{R},$$

where $D^k f$ is defined by (1.5).

Also, we let the subclass $\overline{RS}_H(k, \gamma)$ consist of harmonic functions $f_k = h + \bar{g}_k$ in $RS_H(k, \gamma)$ so that h and g_k are of the form

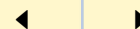
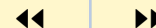
$$(1.7) \quad h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g_k(z) = (-1)^k \sum_{n=1}^{\infty} |b_n| z^n.$$

In this paper, the coefficient condition given in [4] for the class $G_H(\gamma)$ is extended to the class $RS_H(k, \gamma)$ of the form (1.6). Furthermore, we determine extreme points, a distortion theorem, convolution conditions and convex combinations for the functions in $\overline{RS}_H(k, \gamma)$.



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2. Main Results

In our first theorem, we introduce a sufficient coefficient bound for harmonic functions in $RS_H(k, \gamma)$.

Theorem 2.1. *Let $f = h + \bar{g}$ be given by (1.1). If*

$$(2.1) \quad \sum_{n=1}^{\infty} n^k [(2n-1-\gamma)|a_n| + (2n+1+\gamma)|b_n|] \leq 2(1-\gamma),$$

where $a_1 = 1$ and $0 \leq \gamma < 1$, then f is sense preserving, harmonic univalent in U , and $f \in RS_H(k, \gamma)$.

Proof. If the inequality (2.1) holds for the coefficients of $f = h + \bar{g}$, then by (1.3), f is sense preserving and harmonic univalent in U . According to the condition (1.5) we only need to show that if (2.1) holds then

$$\begin{aligned} & \operatorname{Re} \left\{ (1 + e^{i\alpha}) \frac{D^{k+1}f(z)}{D^k f(z)} - e^{i\alpha} \right\} \\ &= \operatorname{Re} \left\{ (1 + e^{i\alpha}) \frac{D^{k+1}h(z) - (-1)^k \overline{D^{k+1}g(z)}}{D^k h(z) + (-1)^k \overline{D^k g(z)}} - e^{i\alpha} \right\} \geq \gamma, \end{aligned}$$

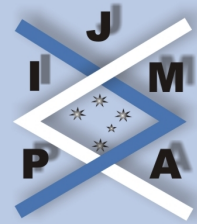
where $0 \leq \gamma < 1$.

Using the fact that $\operatorname{Re} w \geq \gamma$ if and only if $|1 - \gamma + w| \geq |1 + \gamma - w|$, it suffices to show that

$$(2.2) \quad \left| (1 - \gamma)D^k f(z) + (1 + e^{i\alpha})D^{k+1}f(z) - e^{i\alpha}D^k f(z) \right| \\ - \left| (1 + \gamma)D^k f(z) - (1 + e^{i\alpha})D^{k+1}f(z) + e^{i\alpha}D^k f(z) \right| \geq 0.$$

Substituting for $D^k f$ and $D^{k+1}f$ in (2.2) yields

$$\left| (1 - \gamma)D^k f(z) + (1 + e^{i\alpha})D^{k+1}f(z) - e^{i\alpha}D^k f(z) \right|$$



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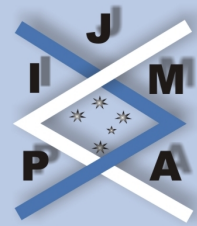
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$$\begin{aligned}
 & - \left| (1 + \gamma)D^k f(z) - (1 + e^{i\alpha})D^{k+1} f(z) + e^{i\alpha}D^k f(z) \right| \\
 = & \left| (2 - \gamma)z + \sum_{n=2}^{\infty} (1 - \gamma - e^{i\alpha} + n + ne^{i\alpha})n^k a_n z^n \right. \\
 & \left. - (-1)^k \sum_{n=1}^{\infty} (n + ne^{i\alpha} - 1 + \gamma + e^{i\alpha})\overline{b_n z^n} \right| \\
 & - \left| \gamma z - \sum_{n=2}^{\infty} (n + ne^{i\alpha} - 1 - \gamma - e^{i\alpha})n^k a_n z^n \right. \\
 & \left. + (-1)^k \sum_{n=1}^{\infty} (n + ne^{i\alpha} + 1 + \gamma + e^{i\alpha})\overline{b_n z^n} \right| \\
 \geq & (2 - \gamma)|z| - \sum_{n=2}^{\infty} n^k (2n - \gamma) |a_n| |z|^n - \sum_{n=1}^{\infty} n^k (2n + \gamma) |b_n| |z|^n \\
 & - \gamma |z| - \sum_{n=2}^{\infty} n^k (2n - \gamma - 2) |a_n| |z|^n - \sum_{n=1}^{\infty} n^k (2n + \gamma + 2) |b_n| |z|^n \\
 = & 2(1 - \gamma)|z| - 2 \sum_{n=2}^{\infty} n^k (2n - \gamma - 1) |a_n| |z|^n - 2 \sum_{n=1}^{\infty} n^k (2n + \gamma + 1) |b_n| |z|^n \\
 = & 2(1 - \gamma)|z| \left\{ 1 - \sum_{n=2}^{\infty} \frac{n^k (2n - \gamma - 1)}{1 - \gamma} |a_n| |z|^{n-1} - \sum_{n=1}^{\infty} \frac{n^k (2n + \gamma + 1)}{1 - \gamma} |b_n| |z|^{n-1} \right\} \\
 > & 2(1 - \gamma) \left\{ 1 - \left(\sum_{n=2}^{\infty} \frac{n^k (2n - \gamma - 1)}{1 - \gamma} |a_n| + \sum_{n=1}^{\infty} \frac{n^k (2n + \gamma + 1)}{1 - \gamma} |b_n| \right) \right\}.
 \end{aligned}$$

This last expression is non-negative by (2.1), and so the proof is complete. \square



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The harmonic function

$$(2.3) \quad f(z) = z + \sum_{n=2}^{\infty} \frac{1-\gamma}{n^k(2n-\gamma-1)} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\gamma}{n^k(2n+\gamma+1)} \bar{y}_n \bar{z}^n,$$

where

$$\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$$

shows that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.3) are in $RS_H(k, \gamma)$ because

$$\sum_{n=1}^{\infty} \left(\frac{n^k(2n-\gamma-1)}{1-\gamma} |a_n| + \frac{n^k(2n+\gamma+1)}{1-\gamma} |b_n| \right) = 1 + \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 2.$$

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_k = h + \bar{g}_k$, where h and g_k are of the form (1.7).

Theorem 2.2. Let $f_k = h + \bar{g}_k$ be given by (1.7). Then $f_k \in \overline{RS}_H(k, \gamma)$ if and only if

$$(2.4) \quad \sum_{n=1}^{\infty} n^k [(2n-\gamma-1)|a_n| + (2n+\gamma+1)|b_n|] \leq 2(1-\gamma).$$

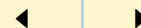
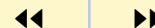
Proof. Since $\overline{RS}_H(k, \gamma) \subset RS_H(k, \gamma)$, we only need to prove the “only if” part of the theorem. To this end, for functions f_k of the form (1.7), we notice that the condition (1.6) is equivalent to

$$\operatorname{Re} \left\{ \frac{(1-\gamma)z - \sum_{n=2}^{\infty} n^k [n-\gamma + (n-1)e^{i\alpha}] |a_n| z^n}{z - \sum_{n=2}^{\infty} n^k |a_n| z^n + (-1)^{2k} \sum_{n=1}^{\infty} n^k |b_n| \bar{z}^n} \right\}$$



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$$= \operatorname{Re} \left\{ \frac{1 - \gamma - \sum_{n=2}^{\infty} n^k [n - \gamma + (n - 1)e^{i\alpha}] |a_n| z^{n-1}}{1 - \sum_{n=2}^{\infty} n^k |a_n| z^{n-1} + \frac{\bar{z}}{z} (-1)^{2k} \sum_{n=1}^{\infty} n^k |b_n| \bar{z}^{n-1}} - \frac{(-1)^{2k} \sum_{n=1}^{\infty} n^k [n + \gamma + (n + 1)e^{i\alpha}] |b_n| \bar{z}^n}{z - \sum_{n=2}^{\infty} n^k |a_n| z^n + (-1)^{2k} \sum_{n=1}^{\infty} n^k |b_n| \bar{z}^n} \right\} \\ \geq 0.$$

The above condition must hold for all values of z , $|z| = r < 1$. Upon choosing the values of z on the positive real axis, where $0 \leq z = r < 1$, we must have

$$\operatorname{Re} \left\{ \frac{1 - \gamma - \sum_{n=2}^{\infty} n^k (n - \gamma) |a_n| r^{n-1} - \sum_{n=1}^{\infty} n^k (n + \gamma) |b_n| r^{n-1}}{1 - \sum_{n=2}^{\infty} n^k |a_n| r^{n-1} + \sum_{n=1}^{\infty} n^k |b_n| r^{n-1}} - e^{i\alpha} \frac{\sum_{n=2}^{\infty} n^k (n - 1) |a_n| r^{n-1} + \sum_{n=1}^{\infty} n^k (n + 1) |b_n| r^{n-1}}{1 - \sum_{n=2}^{\infty} n^k |a_n| r^{n-1} + \sum_{n=1}^{\infty} n^k |b_n| r^{n-1}} \right\} \geq 0.$$

Since $\operatorname{Re}(-e^{i\alpha}) \geq -|e^{i\alpha}| = -1$, the above inequality reduces to

$$(2.5) \quad \frac{1 - \gamma - \sum_{n=2}^{\infty} n^k (2n - \gamma - 1) |a_n| r^{n-1} - \sum_{n=1}^{\infty} n^k (2n + \gamma + 1) |b_n| r^{n-1}}{1 - \sum_{n=2}^{\infty} n^k |a_n| r^{n-1} + \sum_{n=1}^{\infty} n^k |b_n| r^{n-1}} \geq 0.$$

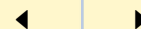
If the condition (2.4) does not hold then the numerator in (2.5) is negative for r sufficiently close to 1. Thus there exists a $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.5) is negative. This contradicts the condition for $f \in \overline{RS}_H(k, \gamma)$ and hence the result. \square

Next we determine a representation theorem for functions in $\overline{RS}_H(k, \gamma)$.



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Theorem 2.3. Let f_k be given by (1.7). Then $f_k \in \overline{RS}_H(k, \gamma)$ if and only if

$$(2.6) \quad f_k(z) = \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_{k_n}(z))$$

where $h_1(z) = z$,

$$h_n(z) = z - \frac{1 - \gamma}{n^k(2n - \gamma - 1)} z^n \quad (n = 2, 3, \dots),$$

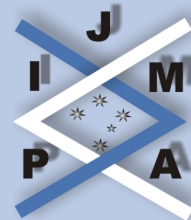
$$g_{k_n}(z) = z + (-1)^k \frac{1 - \gamma}{n^k(2n + \gamma + 1)} \bar{z}^n \quad (n = 1, 2, \dots),$$

$$\sum_{n=1}^{\infty} (X_n + Y_n) = 1,$$

$X_n \geq 0, Y_n \geq 0$. In particular, the extreme points of $\overline{RS}_H(k, \gamma)$ are $\{h_n\}$ and $\{g_{k_n}\}$.

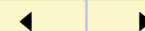
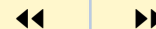
Proof. For functions f_k of the form (2.6) we have

$$\begin{aligned} f_k(z) &= \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_{k_n}(z)) \\ &= \sum_{n=1}^{\infty} (X_n + Y_n) z - \sum_{n=2}^{\infty} \frac{1 - \gamma}{n^k(2n - \gamma - 1)} X_n z^n \\ &\quad + (-1)^k \sum_{n=1}^{\infty} \frac{1 - \gamma}{n^k(2n + \gamma + 1)} Y_n \bar{z}^n. \end{aligned}$$



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Then

$$\sum_{n=2}^{\infty} \frac{n^k(2n - \gamma - 1)}{1 - \gamma} |a_n| + \sum_{n=1}^{\infty} \frac{n^k(2n + \gamma + 1)}{1 - \gamma} |b_n| = \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1$$

and so $f_k \in \overline{RS}_H(k, \gamma)$.

Conversely, suppose that $f_k \in \overline{RS}_H(k, \gamma)$. Setting

$$X_n = \frac{n^k(2n - \gamma - 1)}{1 - \gamma} a_n, \quad (n = 2, 3, \dots),$$

$$Y_n = \frac{n^k(2n + \gamma + 1)}{1 - \gamma} b_n, \quad (n = 1, 2, \dots),$$

where $\sum_{n=1}^{\infty} (X_n + Y_n) = 1$, we obtain

$$f_k(z) = \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_{k_n}(z))$$

as required. □

The following theorem gives the distortion bounds for functions in $\overline{RS}_H(k, \gamma)$ which yields a covering result for this class.

Theorem 2.4. Let $f_k \in \overline{RS}_H(k, \gamma)$. Then for $|z| = r < 1$ we have

$$|f_k(z)| \leq (1 + |b_1|)r + \frac{1}{2^k} \left(\frac{1 - \gamma}{3 - \gamma} - \frac{3 + \gamma}{3 - \gamma} |b_1| \right) r^2$$

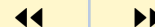
and

$$|f_k(z)| \geq (1 - |b_1|)r - \frac{1}{2^k} \left(\frac{1 - \gamma}{3 - \gamma} - \frac{3 + \gamma}{3 - \gamma} |b_1| \right) r^2.$$



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Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_k \in \overline{RS}_H(k, \gamma)$. Taking the absolute value of f_k we obtain

$$\begin{aligned} |f_k(z)| &\leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \\ &\leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^2 \\ &\leq (1 + |b_1|)r + \frac{1 - \gamma}{2^k(3 - \gamma)} \sum_{n=2}^{\infty} \frac{2^k(3 - \gamma)}{1 - \gamma} (|a_n| + |b_n|)r^2 \\ &\leq (1 + |b_1|)r + \frac{1 - \gamma}{2^k(3 - \gamma)} \sum_{n=2}^{\infty} \left(\frac{n^k(2n - \gamma - 1)}{1 - \gamma} |a_n| + \frac{n^k(2n + \gamma + 1)}{1 - \gamma} |b_n| \right) r^2 \\ &\leq (1 + |b_1|)r + \frac{1 - \gamma}{2^k(3 - \gamma)} \left(1 - \frac{3 + \gamma}{1 - \gamma} |b_1| \right) r^2 \\ &\leq (1 + |b_1|)r + \frac{1}{2^k} \left(\frac{1 - \gamma}{3 - \gamma} - \frac{3 + \gamma}{3 - \gamma} |b_1| \right) r^2. \end{aligned}$$

□

The following covering result follows from the left hand inequality in Theorem 2.4.

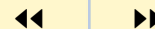
Corollary 2.5. Let f_k of the form (1.7) be so that $f_k \in \overline{RS}_H(k, \gamma)$. Then

$$\left\{ w : |w| < \frac{3 \cdot 2^k - 1 - (2^k - 1)\gamma}{2^k(3 - \gamma)} - \frac{3(2^k - 1) - (2^k + 1)\gamma}{2^k(3 - \gamma)} |b_1| \right\} \subset f_k(U).$$



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For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \bar{z}^n$$

and

$$F_k(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + (-1)^k \sum_{n=1}^{\infty} |B_n| \bar{z}^n$$

we define the convolution of f_k and F_k as

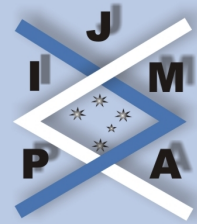
$$(2.7) \quad (f_k * F_k)(z) = f_k(z) * F_k(z) \\ = z - \sum_{n=2}^{\infty} |a_n| |A_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| |B_n| \bar{z}^n.$$

Theorem 2.6. For $0 \leq \beta \leq \gamma < 1$, let $f_k \in \overline{RS}_H(k, \gamma)$ and $F_k \in \overline{RS}_H(k, \beta)$. Then the convolution

$$f_k * F_k \in \overline{RS}_H(k, \gamma) \subset \overline{RS}_H(k, \beta).$$

Proof. Then the convolution $f_k * F_k$ is given by (2.7). We wish to show that the coefficients of $f_k * F_k$ satisfy the required condition given in Theorem 2.2. For $F_k \in \overline{RS}_H(k, \beta)$ we note that $|A_n| \leq 1$ and $|B_n| \leq 1$. Now, for the convolution function $f_k * F_k$, we obtain

$$\sum_{n=2}^{\infty} \frac{n^k(2n - \beta - 1)}{1 - \beta} |a_n| |A_n| + \sum_{n=1}^{\infty} \frac{n^k(2n + \beta + 1)}{1 - \beta} |b_n| |B_n|$$



$$\begin{aligned} &\leq \sum_{n=2}^{\infty} \frac{n^k(2n - \beta - 1)}{1 - \beta} |a_n| + \sum_{n=1}^{\infty} \frac{n^k(2n + \beta + 1)}{1 - \beta} |b_n| \\ &\leq \sum_{n=2}^{\infty} \frac{n^k(2n - \gamma - 1)}{1 - \gamma} |a_n| + \sum_{n=1}^{\infty} \frac{n^k(2n + \gamma + 1)}{1 - \gamma} |b_n| \leq 1 \end{aligned}$$

since $0 \leq \beta \leq \gamma < 1$ and $f_k \in \overline{RS}_H(k, \gamma)$. Therefore $f_k * F_k \in \overline{RS}_H(k, \gamma) \subset \overline{RS}_H(k, \beta)$. \square

Next we discuss the convex combinations of the class $\overline{RS}_H(k, \gamma)$.

Theorem 2.7. *The family $\overline{RS}_H(k, \gamma)$ is closed under convex combination.*

Proof. For $i = 1, 2, \dots$, suppose that $f_{k_i} \in \overline{RS}_H(k, \gamma)$, where

$$f_{k_i}(z) = z - \sum_{n=2}^{\infty} |a_{i_n}| z^n + (-1)^k \sum_{n=1}^{\infty} |b_{i_n}| \bar{z}^n.$$

Then by Theorem 2.2,

$$(2.8) \quad \sum_{n=2}^{\infty} \frac{n^k(2n - \gamma - 1)}{1 - \gamma} |a_{i_n}| + \sum_{n=1}^{\infty} \frac{n^k(2n + \gamma + 1)}{1 - \gamma} |b_{i_n}| \leq 1.$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_{k_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{k_i}(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i_n}| \right) z^n + (-1)^k \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i_n}| \right) \bar{z}^n.$$

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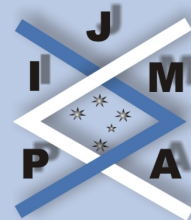
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Then by (2.8),

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{n^k(2n-\gamma-1)}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i |a_{in}| \right) + \sum_{n=1}^{\infty} \frac{n^k(2n+\gamma+1)}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i |b_{in}| \right) \\ &= \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{n^k(2n-\gamma-1)}{1-\gamma} |a_{in}| + \sum_{n=1}^{\infty} \frac{n^k(2n+\gamma+1)}{1-\gamma} |b_{in}| \right) \\ &\leq \sum_{i=1}^{\infty} t_i = 1 \end{aligned}$$

and therefore

$$\sum_{i=1}^{\infty} t_i f_{k_i}(z) \in \overline{RS}_H(k, \gamma).$$

□

Following Ruscheweyh [5], we call the δ -neighborhood of f the set

$$N_{\delta}(f_k) = \left\{ F_k : F_k(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + (-1)^k \sum_{n=1}^{\infty} |B_n| \bar{z}^n \quad \text{and} \right. \\ \left. \sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) + |b_1 - B_1| \leq \delta \right\}.$$

Theorem 2.8. Assume that

$$f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \bar{z}^n$$

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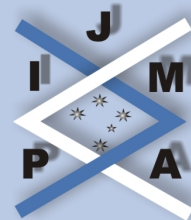
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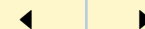
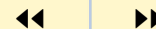
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belongs to $\overline{RS}_H(k, \gamma)$. If

$$\delta \leq \frac{1}{3} \left[(1 - \gamma) \left(1 - \frac{1}{2^k} \right) - \left(3 + 2\gamma - \frac{1}{2^k} (3 + \gamma) \right) |b_1| \right],$$

then $N_\delta(f_k) \subset \overline{RS}_H(0, \gamma)$.

Proof. Let

$$F_k(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + (-1)^k \sum_{n=1}^{\infty} |B_n| \bar{z}^n$$

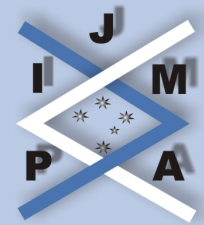
belong to $N_\delta(f_k)$. We have

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{2n - \gamma - 1}{1 - \gamma} |A_n| + \sum_{n=1}^{\infty} \frac{2n + \gamma + 1}{1 - \gamma} |B_n| \\ & \leq \sum_{n=2}^{\infty} \frac{2n - \gamma - 1}{1 - \gamma} |a_n - A_n| + \sum_{n=1}^{\infty} \frac{2n + \gamma + 1}{1 - \gamma} |b_n - B_n| \\ & \quad + \sum_{n=2}^{\infty} \frac{2n - \gamma - 1}{1 - \gamma} |a_n| + \sum_{n=1}^{\infty} \frac{2n + \gamma + 1}{1 - \gamma} |b_n| \\ & \leq \frac{3}{1 - \gamma} \sum_{n=2}^{\infty} n (|a_n - A_n| + |b_n - B_n|) + \frac{3}{1 - \gamma} |b_1| + \frac{\gamma}{1 - \gamma} |b_1| \\ & \quad + \frac{1}{2^k} \sum_{n=2}^{\infty} \frac{n^k (2n - \gamma - 1)}{1 - \gamma} |a_n| + \frac{1}{2^k} \sum_{n=2}^{\infty} \frac{n^k (2n - \gamma - 1)}{1 - \gamma} |b_n| + \frac{3 + \gamma}{1 - \gamma} |b_1| \\ & \leq \frac{3\delta}{1 - \gamma} + \frac{\gamma}{1 - \gamma} |b_1| + \frac{1}{2^k} \left(1 - \frac{3 + \gamma}{1 - \gamma} |b_1| \right) + \frac{3 + \gamma}{1 - \gamma} |b_1| \leq 1. \end{aligned}$$

Thus $F_k \in \overline{RS}_H(0, \gamma)$ for

$$\delta \leq \frac{1}{3} \left[(1 - \gamma) \left(1 - \frac{1}{2^k} \right) - \left(3 + 2\gamma - \frac{1}{2^k} (3 + \gamma) \right) |b_1| \right].$$

□



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