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ON THE SUBCLASS OF SALAGEAN-TYPE HARMONIC UNIVALENT FUNCTIONS

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ABSTRACT. Using the Salagean derivative, we introduce and study a class of Goodman- Ronning type harmonic univalent functions. We obtain coefficient conditions, extreme points, distortion bounds, convolution conditions, and convex combination for the above class of harmonic functions.

Key words and phrases: Harmonic, Meromorphic, Starlike, Symmetric, Conjugate, Convex functions.

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1. INTRODUCTION

A continuous complex valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain D we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$.

Denote by S_H the class of functions $f = h+\bar{g}$ that are harmonic univalent and sense preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0)-1 = 0$. Then for $f = h+\bar{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

In 1984 Clunie and Sheil-Small [1] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses. Jahangiri et al. [3] make use of the Alexander integral transforms of certain analytic functions (which are starlike or convex of positive order) with a view to investigating the construction of sense-preserving, univalent, and close to convex harmonic functions.

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Definition 1.1. Recently, Rosy et al. [4], defined the subclass $G_H(\gamma) \subset S_H$ consisting of harmonic univalent functions $f(z)$ satisfying the following condition

$$\operatorname{Re}\left\{(1+e^{i\alpha})\frac{zf'(z)}{f(z)} - e^{i\alpha}\right\} \geq \gamma, \quad 0 \leq \gamma < 1, \ \alpha \in \mathbb{R}. \tag{1.2}$$

They proved that if $f = h + \bar{g}$ is given by (1.1) and if

$$\sum_{n=1}^{\infty}\left[\frac{2n-1-\gamma}{1-\gamma}|a_n| + \frac{2n+1+\gamma}{1-\gamma}|b_n|\right] \leq 2, \quad 0 \leq \gamma < 1, \tag{1.3}$$

then f is a Goodman-Ronning type harmonic univalent function in U. This condition is proved to be also necessary if h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty}|a_n|z^n, \qquad g(z) = \sum_{n=1}^{\infty}|b_n|z^n. \tag{1.4}$$

Jahangiri et al. [2] has introduced the modified Salagean operator of harmonic univalent function f as

$$D^k f(z) = D^k h(z) + (-1)^k\overline{D^k g(z)}, \qquad k \in \mathbb{N}, \tag{1.5}$$

where

$$D^k h(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n \quad \text{and} \quad D^k g(z) = \sum_{n=1}^{\infty} n^k b_n z^n.$$

We let $RS_H(k,\gamma)$ denote the family of harmonic functions f of the form (1.1) such that

$$\operatorname{Re}\left\{(1+e^{i\alpha})\frac{D^{k+1}f(z)}{D^k f(z)} - e^{i\alpha}\right\} \geq \gamma, \quad 0 \leq \gamma < 1, \ \alpha \in \mathbb{R}, \tag{1.6}$$

where $D^k f$ is defined by (1.5).

Also, we let the subclass $\overline{RS}_H(k,\gamma)$ consist of harmonic functions $f_k = h + \bar{g}_k$ in $RS_H(k,\gamma)$ so that h and g_k are of the form

$$h(z) = z - \sum_{n=2}^{\infty}|a_n|z^n, \qquad g_k(z) = (-1)^k\sum_{n=1}^{\infty}|b_n|z^n. \tag{1.7}$$

In this paper, the coefficient condition given in [4] for the class $G_H(\gamma)$ is extended to the class $RS_H(k,\gamma)$ of the form (1.6). Furthermore, we determine extreme points, a distortion theorem, convolution conditions and convex combinations for the functions in $\overline{RS}_H(k,\gamma)$.

2. MAIN RESULTS

In our first theorem, we introduce a sufficient coefficient bound for harmonic functions in $RS_H(k,\gamma)$.

Theorem 2.1. *Let $f = h + \bar{g}$ be given by (1.1). If*

$$\sum_{n=1}^{\infty} n^k[(2n-1-\gamma)|a_n| + (2n+1+\gamma)|b_n|] \leq 2(1-\gamma), \tag{2.1}$$

where $a_1 = 1$ and $0 \leq \gamma < 1$, then f is sense preserving, harmonic univalent in U, and $f \in RS_H(k,\gamma)$.

Proof. If the inequality (2.1) holds for the coefficients of $f = h + \bar{g}$, then by (1.3), f is sense preserving and harmonic univalent in U. According to the condition (1.5) we only need to show that if (2.1) holds then

$$\operatorname{Re}\left\{(1+e^{i\alpha})\frac{D^{k+1}f(z)}{D^kf(z)} - e^{i\alpha}\right\} = \operatorname{Re}\left\{(1+e^{i\alpha})\frac{D^{k+1}h(z)-(-1)^k\overline{D^{k+1}g(z)}}{D^kh(z)+(-1)^k\overline{D^kg(z)}} - e^{i\alpha}\right\} \geq \gamma,$$

where $0 \leq \gamma < 1$.

Using the fact that $\operatorname{Re} w \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it suffices to show that

$$\left|(1-\gamma)D^kf(z) + (1+e^{i\alpha})D^{k+1}f(z) - e^{i\alpha}D^kf(z)\right| - \left|(1+\gamma)D^kf(z) - (1+e^{i\alpha})D^{k+1}f(z) + e^{i\alpha}D^kf(z)\right| \geq 0. \tag{2.2}$$

Substituting for D^kf and $D^{k+1}f$ in (2.2) yields

$$\begin{aligned}
&\left|(1-\gamma)D^kf(z) + (1+e^{i\alpha})D^{k+1}f(z) - e^{i\alpha}D^kf(z)\right| \\
&\qquad - \left|(1+\gamma)D^kf(z) - (1+e^{i\alpha})D^{k+1}f(z) + e^{i\alpha}D^kf(z)\right| \\
&= \Bigg|(2-\gamma)z + \sum_{n=2}^{\infty}(1-\gamma-e^{i\alpha}+n+ne^{i\alpha})n^ka_nz^n \\
&\qquad -(-1)^k\sum_{n=1}^{\infty}(n+ne^{i\alpha}-1+\gamma+e^{i\alpha})\overline{b_nz^n}\Bigg| \\
&\qquad - \Bigg|\gamma z - \sum_{n=2}^{\infty}(n+ne^{i\alpha}-1-\gamma-e^{i\alpha})n^ka_nz^n \\
&\qquad +(-1)^k\sum_{n=1}^{\infty}(n+ne^{i\alpha}+1+\gamma+e^{i\alpha})\overline{b_nz^n}\Bigg| \\
&\geq (2-\gamma)|z| - \sum_{n=2}^{\infty}n^k(2n-\gamma)|a_n||z|^n - \sum_{n=1}^{\infty}n^k(2n+\gamma)|b_n||z|^n \\
&\qquad - \gamma|z| - \sum_{n=2}^{\infty}n^k(2n-\gamma-2)|a_n||z|^n - \sum_{n=1}^{\infty}n^k(2n+\gamma+2)|b_n||z|^n \\
&= 2(1-\gamma)|z| - 2\sum_{n=2}^{\infty}n^k(2n-\gamma-1)|a_n||z|^n - 2\sum_{n=1}^{\infty}n^k(2n+\gamma+1)|b_n||z|^n \\
&= 2(1-\gamma)|z|\left\{1 - \sum_{n=2}^{\infty}\frac{n^k(2n-\gamma-1)}{1-\gamma}|a_n||z|^{n-1} - \sum_{n=1}^{\infty}\frac{n^k(2n+\gamma+1)}{1-\gamma}|b_n||z|^{n-1}\right\} \\
&> 2(1-\gamma)\left\{1 - \left(\sum_{n=2}^{\infty}\frac{n^k(2n-\gamma-1)}{1-\gamma}|a_n| + \sum_{n=1}^{\infty}\frac{n^k(2n+\gamma+1)}{1-\gamma}|b_n|\right)\right\}.
\end{aligned}$$

This last expression is non-negative by (2.1), and so the proof is complete. □

The harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\gamma}{n^k(2n-\gamma-1)} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\gamma}{n^k(2n+\gamma+1)} \bar{y}_n \bar{z}^n, \tag{2.3}$$

where

$$\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$$

shows that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.3) are in $RS_H(k,\gamma)$ because

$$\sum_{n=1}^{\infty} \left(\frac{n^k(2n-\gamma-1)}{1-\gamma}|a_n| + \frac{n^k(2n+\gamma+1)}{1-\gamma}|b_n| \right) = 1 + \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 2.$$

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_k = h + \bar{g}_k$, where h and g_k are of the form (1.7).

Theorem 2.2. *Let $f_k = h + \bar{g}_k$ be given by (1.7). Then $f_k \in \overline{RS}_H(k,\gamma)$ if and only if*

$$\sum_{n=1}^{\infty} n^k[(2n-\gamma-1)|a_n| + (2n+\gamma+1)|b_n|] \le 2(1-\gamma). \tag{2.4}$$

Proof. Since $\overline{RS}_H(k,\gamma) \subset RS_H(k,\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_k of the form (1.7), we notice that the condition (1.6) is equivalent to

$$\begin{aligned}
\operatorname{Re}&\left\{ \frac{(1-\gamma)z - \sum_{n=2}^{\infty} n^k[n-\gamma+(n-1)e^{i\alpha}]|a_n|z^n}{z - \sum_{n=2}^{\infty} n^k|a_n|z^n + (-1)^{2k}\sum_{n=1}^{\infty} n^k|b_n|\bar{z}^n} \right.\\
&\qquad \left. - \frac{(-1)^{2k}\sum_{n=1}^{\infty} n^k[n+\gamma+(n+1)e^{i\alpha}]|b_n|\bar{z}^n}{z - \sum_{n=2}^{\infty} n^k|a_n|z^n + (-1)^{2k}\sum_{n=1}^{\infty} n^k|b_n|\bar{z}^n} \right\}\\
&= \operatorname{Re}\left\{ \frac{1-\gamma - \sum_{n=2}^{\infty} n^k[n-\gamma+(n-1)e^{i\alpha}]|a_n|z^{n-1}}{1 - \sum_{n=2}^{\infty} n^k|a_n|z^{n-1} + \frac{\bar{z}}{z}(-1)^{2k}\sum_{n=1}^{\infty} n^k|b_n|\bar{z}^{n-1}} \right.\\
&\qquad \left. - \frac{\frac{\bar{z}}{z}(-1)^{2k}\sum_{n=1}^{\infty} n^k[n+\gamma+(n+1)e^{i\alpha}]|b_n|\bar{z}^{n-1}}{1 - \sum_{n=2}^{\infty} n^k|a_n|z^{n-1} + \frac{\bar{z}}{z}(-1)^{2k}\sum_{n=1}^{\infty} n^k|b_n|\bar{z}^{n-1}} \right\}\\
&\ge 0.
\end{aligned}$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis, where $0 \le z = r < 1$, we must have

$$\begin{aligned}
\operatorname{Re}&\left\{ \frac{1-\gamma - \sum_{n=2}^{\infty} n^k(n-\gamma)|a_n|r^{n-1} - \sum_{n=1}^{\infty} n^k(n+\gamma)|b_n|r^{n-1}}{1 - \sum_{n=2}^{\infty} n^k|a_n|r^{n-1} + \sum_{n=1}^{\infty} n^k|b_n|r^{n-1}} \right.\\
&\qquad \left. - e^{i\alpha}\frac{\sum_{n=2}^{\infty} n^k(n-1)|a_n|r^{n-1} + \sum_{n=1}^{\infty} n^k(n+1)|b_n|r^{n-1}}{1 - \sum_{n=2}^{\infty} n^k|a_n|r^{n-1} + \sum_{n=1}^{\infty} n^k|b_n|r^{n-1}} \right\} \ge 0.
\end{aligned}$$

Since $\operatorname{Re}(-e^{i\alpha}) \ge -|e^{i\alpha}| = -1$, the above inequality reduces to

$$\frac{1-\gamma - \sum_{n=2}^{\infty} n^k(2n-\gamma-1)|a_n|r^{n-1} - \sum_{n=1}^{\infty} n^k(2n+\gamma+1)|b_n|r^{n-1}}{1 - \sum_{n=2}^{\infty} n^k|a_n|r^{n-1} + \sum_{n=1}^{\infty} n^k|b_n|r^{n-1}} \ge 0. \tag{2.5}$$

If the condition (2.4) does not hold then the numerator in (2.5) is negative for r sufficiently close to 1. Thus there exists a $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.5) is negative. This contradicts the condition for $f \in \overline{RS}_H(k,\gamma)$ and hence the result. □

Next we determine a representation theorem for functions in $\overline{RS}_H(k,\gamma)$.

Theorem 2.3. *Let f_k be given by (1.7). Then $f_k \in \overline{RS}_H(k,\gamma)$ if and only if*

$$f_k(z) = \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_{k_n}(z)) \tag{2.6}$$

where $h_1(z) = z$,

$$h_n(z) = z - \frac{1-\gamma}{n^k(2n-\gamma-1)} z^n \quad (n = 2, 3, \dots),$$

$$g_{k_n}(z) = z + (-1)^k \frac{1-\gamma}{n^k(2n+\gamma+1)} \bar{z}^n \quad (n = 1, 2, \dots),$$

$$\sum_{n=1}^{\infty} (X_n + Y_n) = 1,$$

$X_n \geq 0,\ Y_n \geq 0$. *In particular, the extreme points of $\overline{RS}_H(k,\gamma)$ are $\{h_n\}$ and $\{g_{k_n}\}$.*

Proof. For functions f_k of the form (2.6) we have

$$\begin{aligned} f_k(z) &= \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_{k_n}(z)) \\ &= \sum_{n=1}^{\infty} (X_n + Y_n) z - \sum_{n=2}^{\infty} \frac{1-\gamma}{n^k(2n-\gamma-1)} X_n\, z^n \\ &\qquad + (-1)^k \sum_{n=1}^{\infty} \frac{1-\gamma}{n^k(2n+\gamma+1)} Y_n \bar{z}^n. \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{n^k(2n-\gamma-1)}{1-\gamma} |a_n| + \sum_{n=1}^{\infty} \frac{n^k(2n+\gamma+1)}{1-\gamma} |b_n| &= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n \\ &= 1 - X_1 \leq 1 \end{aligned}$$

and so $f_k \in \overline{RS}_H(k,\gamma)$.

Conversely, suppose that $f_k \in \overline{RS}_H(k,\gamma)$. Setting

$$X_n = \frac{n^k(2n-\gamma-1)}{1-\gamma} a_n, \qquad (n = 2, 3, \dots),$$

$$Y_n = \frac{n^k(2n+\gamma+1)}{1-\gamma} b_n, \qquad (n = 1, 2, \dots),$$

where $\sum_{n=1}^{\infty} (X_n + Y_n) = 1$, we obtain

$$f_k(z) = \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_{k_n}(z))$$

as required. □

The following theorem gives the distortion bounds for functions in $\overline{RS}_H(k,\gamma)$ which yields a covering result for this class.

Theorem 2.4. *Let $f_k \in \overline{RS}_H(k,\gamma)$. Then for $|z| = r < 1$ we have*

$$|f_k(z)| \leq (1 + |b_1|) r + \frac{1}{2^k} \left(\frac{1-\gamma}{3-\gamma} - \frac{3+\gamma}{3-\gamma} |b_1| \right) r^2$$

and

$$|f_k(z)| \geq (1 - |b_1|)r - \frac{1}{2^k}\left(\frac{1-\gamma}{3-\gamma} - \frac{3+\gamma}{3-\gamma}|b_1|\right) r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_k \in \overline{RS}_H(k,\gamma)$. Taking the absolute value of f_k we obtain

$$\begin{aligned}
|f_k(z)| &\leq (1 + |b_1|)r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n \\
&\leq (1 + |b_1|)r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^2 \\
&\leq (1 + |b_1|)r + \frac{1-\gamma}{2^k(3-\gamma)} \sum_{n=2}^{\infty} \frac{2^k(3-\gamma)}{1-\gamma}(|a_n| + |b_n|)r^2 \\
&\leq (1 + |b_1|)r + \frac{1-\gamma}{2^k(3-\gamma)} \sum_{n=2}^{\infty} \left(\frac{n^k(2n-\gamma-1)}{1-\gamma}|a_n| + \frac{n^k(2n+\gamma+1)}{1-\gamma}|b_n|\right) r^2 \\
&\leq (1 + |b_1|)r + \frac{1-\gamma}{2^k(3-\gamma)}\left(1 - \frac{3+\gamma}{1-\gamma}|b_1|\right) r^2 \\
&\leq (1 + |b_1|)r + \frac{1}{2^k}\left(\frac{1-\gamma}{3-\gamma} - \frac{3+\gamma}{3-\gamma}|b_1|\right) r^2.
\end{aligned}$$

□

The following covering result follows from the left hand inequality in Theorem 2.4.

Corollary 2.5. *Let f_k of the form (1.7) be so that $f_k \in \overline{RS}_H(k,\gamma)$. Then*

$$\left\{ w : |w| < \frac{3.2^k - 1 - (2^k - 1)\gamma}{2^k(3-\gamma)} - \frac{3(2^k-1) - (2^k+1)\gamma}{2^k(3-\gamma)}|b_1| \right\} \subset f_k(U).$$

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form

$$f_k(z) = z - \sum_{n=2}^{\infty}|a_n|z^n + (-1)^k \sum_{n=1}^{\infty}|b_n|\bar{z}^n$$

and

$$F_k(z) = z - \sum_{n=2}^{\infty}|A_n|z^n + (-1)^k \sum_{n=1}^{\infty}|B_n|\bar{z}^n$$

we define the convolution of f_k and F_k as

$$\begin{aligned}
(f_k * F_k)(z) &= f_k(z) * F_k(z) \\
&= z - \sum_{n=2}^{\infty}|a_n||A_n|z^n + (-1)^k \sum_{n=1}^{\infty}|b_n||B_n|\bar{z}^n.
\end{aligned} \tag{2.7}$$

Theorem 2.6. *For $0 \leq \beta \leq \gamma < 1$, let $f_k \in \overline{RS}_H(k,\gamma)$ and $F_k \in \overline{RS}_H(k,\beta)$. Then the convolution*

$$f_k * F_k \in \overline{RS}_H(k,\gamma) \subset \overline{RS}_H(k,\beta).$$

Proof. Then the convolution $f_k * F_k$ is given by (2.7). We wish to show that the coefficients of $f_k * F_k$ satisfy the required condition given in Theorem 2.2. For $F_k \in \overline{RS}_H(k,\beta)$ we note that $|A_n| \le 1$ and $|B_n| \le 1$. Now, for the convolution function $f_k * F_k$, we obtain

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{n^k(2n-\beta-1)}{1-\beta}|a_n||A_n| + \sum_{n=1}^{\infty} \frac{n^k(2n+\beta+1)}{1-\beta}|b_n||B_n| \\ \le \sum_{n=2}^{\infty} \frac{n^k(2n-\beta-1)}{1-\beta}|a_n| + \sum_{n=1}^{\infty} \frac{n^k(2n+\beta+1)}{1-\beta}|b_n| \\ \le \sum_{n=2}^{\infty} \frac{n^k(2n-\gamma-1)}{1-\gamma}|a_n| + \sum_{n=1}^{\infty} \frac{n^k(2n+\gamma+1)}{1-\gamma}|b_n| \le 1\end{aligned}$$

since $0 \le \beta \le \gamma < 1$ and $f_k \in \overline{RS}_H(k,\gamma)$. Therefore $f_k * F_k \in \overline{RS}_H(k,\gamma) \subset \overline{RS}_H(k,\beta)$. □

Next we discuss the convex combinations of the class $\overline{RS}_H(k,\gamma)$.

Theorem 2.7. *The family $\overline{RS}_H(k,\gamma)$ is closed under convex combination.*

Proof. For $i = 1, 2, \dots,$ suppose that $f_{k_i} \in \overline{RS}_H(k,\gamma)$, where

$$f_{k_i}(z) = z - \sum_{n=2}^{\infty} |a_{i_n}| z^n + (-1)^k \sum_{n=1}^{\infty} |b_{i_n}| \bar{z}^n.$$

Then by Theorem 2.2,

$$\sum_{n=2}^{\infty} \frac{n^k(2n-\gamma-1)}{1-\gamma}|a_{i_n}| + \sum_{n=1}^{\infty} \frac{n^k(2n+\gamma+1)}{1-\gamma}|b_{i_n}| \le 1. \tag{2.8}$$

For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1$, the convex combination of f_{k_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{k_i}(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i_n}| \right) z^n + (-1)^k \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i_n}| \right) \bar{z}^n.$$

Then by (2.8),

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{n^k(2n-\gamma-1)}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i |a_{i_n}| \right) + \sum_{n=1}^{\infty} \frac{n^k(2n+\gamma+1)}{1-\gamma} \left(\sum_{i=1}^{\infty} t_i |b_{i_n}| \right) \\ = \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{n^k(2n-\gamma-1)}{1-\gamma}|a_{i_n}| + \sum_{n=1}^{\infty} \frac{n^k(2n+\gamma+1)}{1-\gamma}|b_{i_n}| \right) \\ \le \sum_{i=1}^{\infty} t_i = 1\end{aligned}$$

and therefore

$$\sum_{i=1}^{\infty} t_i f_{k_i}(z) \in \overline{RS}_H(k,\gamma).$$

□

Following Ruscheweyh [5], we call the $\delta-$neighborhood of f the set

$$N_\delta(f_k) = \Bigg\{ F_k : F_k(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + (-1)^k \sum_{n=1}^{\infty} |B_n| \bar{z}^n \quad \text{and} \quad \sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) + |b_1 - B_1| \leq \delta \Bigg\}.$$

Theorem 2.8. *Assume that*

$$f_k(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + (-1)^k \sum_{n=1}^{\infty} |b_n| \bar{z}^n$$

belongs to $\overline{RS}_H(k,\gamma)$. *If*

$$\delta \leq \frac{1}{3}\left[(1-\gamma)\left(1-\frac{1}{2^k}\right) - \left(3+2\gamma - \frac{1}{2^k}(3+\gamma)\right)|b_1|\right],$$

then $N_\delta(f_k) \subset \overline{RS}_H(0,\gamma)$.

Proof. Let

$$F_k(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + (-1)^k \sum_{n=1}^{\infty} |B_n| \bar{z}^n$$

belong to $N_\delta(f_k)$. We have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{2n-\gamma-1}{1-\gamma}|A_n| + \sum_{n=1}^{\infty} \frac{2n+\gamma+1}{1-\gamma}|B_n| \\
&\leq \sum_{n=2}^{\infty} \frac{2n-\gamma-1}{1-\gamma}|a_n - A_n| + \sum_{n=1}^{\infty} \frac{2n+\gamma+1}{1-\gamma}|b_n - B_n| \\
&\qquad + \sum_{n=2}^{\infty} \frac{2n-\gamma-1}{1-\gamma}|a_n| + \sum_{n=1}^{\infty} \frac{2n+\gamma+1}{1-\gamma}|b_n| \\
&\leq \frac{3}{1-\gamma}\sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) + \frac{3}{1-\gamma}|b_1| + \frac{\gamma}{1-\gamma}|b_1| \\
&\qquad + \frac{1}{2^k}\sum_{n=2}^{\infty} \frac{n^k(2n-\gamma-1)}{1-\gamma}|a_n| + \frac{1}{2^k}\sum_{n=2}^{\infty} \frac{n^k(2n-\gamma-1)}{1-\gamma}|b_n| + \frac{3+\gamma}{1-\gamma}|b_1| \\
&\leq \frac{3\delta}{1-\gamma} + \frac{\gamma}{1-\gamma}|b_1| + \frac{1}{2^k}\left(1 - \frac{3+\gamma}{1-\gamma}|b_1|\right) + \frac{3+\gamma}{1-\gamma}|b_1| \leq 1.
\end{aligned}$$

Thus $F_k \in \overline{RS}_H(0,\gamma)$ for

$$\delta \leq \frac{1}{3}\left[(1-\gamma)\left(1-\frac{1}{2^k}\right) - \left(3+2\gamma - \frac{1}{2^k}(3+\gamma)\right)|b_1|\right].$$

□

REFERENCES

[1] J. CLUNIE AND T. SHEIL-SMALL, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, **9** (1984), 3–25.

[2] J.M. JAHANGIRI, G. MURUGUSUNDARAMOORTHY AND K. VIJAYA, Salagean-type harmonic univalent functions, *South. J. Pure and Appl. Math.*, Issue 2 (2002), 77–82.

[3] J.M. JAHANGIRI, CHAN KIM YONG AND H.M. SRIVASTAVA, Construction of a certain class of harmonic close to convex functions associated with the Alexander integral transform, *Integral Transform. Spec. Funct.*, **14**(3) (2003), 237–242.

[4] T. ROSY, B. ADOLPH STEPHEN AND K.G. SUBRAMANIAN, Goodman Ronning type harmonic univalent functions, *Kyungpook Math. J.*, **41** (2001), 45–54.

[5] St. RUSCHEWEYH, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **81** (1981), 521–527.