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PROVING INEQUALITIES IN ACUTE TRIANGLE WITH DIFFERENCE SUBSTITUTION

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ABSTRACT. In this paper, we prove several inequalities in the acute triangle by means of so-called Difference Substitution. As generalization of the method, we also consider an example that the greatest interior angle is less than or equal to 120° in the triangle.

Key words and phrases: Inequalities, Acute Triangle, Difference Substitution, Linear transformation.

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1. INTRODUCTION

In [1, 2], L. Yang suggested the use of Difference Substitution to prove asymmetric polynomial inequalities, as it had been used previously to deal with symmetric ones.

If $x_1 \le x_2 \le x_3 \le \cdots \le x_n$ with $n \in N^*$, then we set

$$\begin{cases} x_1 = t_1, \\ x_2 = t_1 + t_2, \\ x_3 = t_1 + t_2 + t_3, \\ \dots\dots\dots\dots\dots\dots \\ x_n = t_1 + t_2 + t_3 + \cdots + t_n, \end{cases} \tag{1.1}$$

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where $t_i \geq 0$ for $2 \leq i \leq n$ and $i \in N^*$.

The expansion (1.1) is so-called a "splitting" transformation, and $\{t_1, t_2, \ldots, t_n\}$ is simply the difference sequence of $\{x_1, x_2, \ldots, x_n\}$.

In general, for the n-variant polynomials, there are $n!$ different orders of $\{x_1, x_2, \ldots, x_n\}$, sorting by size. In the instance of $n = 3$, we let $x \leq y \leq z$, and take

$$\begin{cases} x = u, \\ y = u + v, \\ z = u + v + w, \end{cases} \tag{1.2}$$

where $v \geq 0$, $w \geq 0$.

Analogously, if $y \leq x \leq z$, then its "splitting" transformation is

$$\begin{cases} y = u, \\ x = u + v, \\ z = u + v + w, \end{cases} \tag{1.3}$$

where $v \geq 0$, $w \geq 0$.

Sequentially, for $y \leq z \leq x$ or $z \leq x \leq y$ or $z \leq y \leq x$ or $x \leq z \leq y$, we set four similar linear transformations.

For a 3-variant polynomial $F(x, y, z)$, by using the six linear transformations above, we obtain 6 members $P_i(u, v, w)$ with $1 \leq i \leq 6$, and call the set $\{P_1, P_2, \ldots, P_6\}$ the Difference Substitution of $F(x, y, z)$ and denote this by $DS(F)$. If all the coefficients of these members $DS(F)$ are nonnegative, then $F \geq 0$ whenever x, y, z all are nonnegative. In other words, F is positive semi-definite on $\mathbb{R}^3_+$. Difference substitution is a very valid method for proving inequalities. For more information on Difference Substitution, please refer to [3] and [4].

In this paper, by using Difference Substitution, the authors prove several inequalities in acute triangles.

Throughout the paper we denote A, B, C as the interior angles, a, b, c as the side-lengths, S as the area, s as the semi-perimeter, R as the circumradius, r as the inradius, h_a, h_b, h_c as the altitudes, m_a, m_b, m_c as the medians, and r_a, r_b, r_c as the radii of the described circles of triangle ABC respectively. Moreover, we will customarily use the cyclic sum symbol, that is: $\sum f(a) = f(a) + f(b) + f(c)$, and $\sum f(a, b) = f(a, b) + f(b, c) + f(c, a)$, etc.

Let us begin with the well-known Walker's inequality [5]. In the acute triangle, show that

$$s^2 \geq 2R^2 + 8Rr + 3r^2, \tag{1.4}$$

or

$$\begin{aligned} &-2\,a^3b^3 + a^4b^2 - a^4bc + a^5b + ab^5 + b^5c + b^4c^2 \\ &\qquad - 2\,b^3c^3 + b^2c^4 - 2\,c^3a^3 + c^4a^2 + c^5a + c^5b + c^2a^4 \\ &\qquad\qquad + a^5c - ab^4c + a^2b^4 - b^6 - c^6 - a^6 - abc^4 \geq 0. \end{aligned} \tag{1.5}$$

Let

$$\begin{cases} x = \frac{b+c-a}{2} > 0, \\ y = \frac{c+a-b}{2} > 0, \\ z = \frac{a+b-c}{2} > 0. \end{cases} \tag{1.6}$$

Then inequality (1.4) or (1.5) is equivalent to

$$(1.7)\quad \begin{aligned}F(x,y,z) &= 6\,xyz^4+2\,xy^2z^3+2\,xy^3z^2+6\,xy^4z+2\,x^2yz^3+2\,x^2y^3z\\ &+2\,x^3yz^2+2\,x^3y^2z+6\,x^4yz-x^4y^2-x^2z^4-2\,x^3z^3-x^4z^2\\ &-2\,x^3y^3-y^4z^2-y^4x^2-2\,y^3z^3-y^2z^4-18\,x^2y^2z^2\geq 0.\end{aligned}$$

There is no harm in supposing $x \leq y \leq z$ since inequality (1.7) is symmetric for x, y, z. Then, by using (1.2), for the acute triangle, it follows that

$$(1.8)\quad \begin{aligned}b^2+c^2-a^2 &= (z+x)^2+(x+y)^2-(y+z)^2 = 2[x^2+(y+z)x-yz]\\ &= 2\{u^2+[(u+v)+(u+v+w)]u-(u+v)(u+v+w)\}\\ &= 2(2u^2-v^2-vw)>0,\end{aligned}$$

and $F(x,y,z)$ in (1.7) is transformed into

$$(1.9)\quad \begin{aligned}F(x,y,z) &= P(u,v,w)\\ &= \left(2\,u^2-v^2-vw\right)\left[\left(4\,v^2+4\,w^2+4\,vw\right)u^2\right.\\ &\quad+\left(8\,v^3+20\,vw^2+12\,v^2w+8\,w^3\right)u+4\,v^4+8\,v^3w+2\,w^4\\ &\quad\left.+18\,vw^3+22\,v^2w^2\right]+\left(24\,v^3w^2+36\,v^2w^3+12\,vw^4\right)u\\ &\quad+34\,v^3w^3+19\,v^2w^4+2\,vw^5+17\,v^4w^2.\end{aligned}$$

Obviously $F(x,y,z) = P(u,v,w) \geq 0$ from (1.8) and $u > 0, v \geq 0, w \geq 0$, i.e., inequality (1.4) or (1.5) is true.

Now, let us consider another semi-symmetric inequality [6] in the acute triangle

$$(1.10)\quad \cos(B-C) \leq \frac{h_a}{m_a}.$$

It is equivalent to

$$(1.11)\quad -a^4+\left(3\,b^2+3\,c^2\right)a^2-2\,(b-c)^2\,(b+c)^2 \geq 0,$$

and from (1.6), this equals

$$(1.12)\quad F(x,y,z) = \left(-y^2-z^2+14\,yz\right)x^2-(y+z)\left(z^2-14\,yz+y^2\right)x+yz\,(y+z)^2 \geq 0.$$

Calculating $DS(F)$, it consists of 3 polynomials with $u > 0, v \geq 0, w \geq 0$ as follows

$$(1.13)\quad \begin{aligned}&P_1(u,v,w)\\ &= 40\,u^4+112\,u^3v+108\,u^2v^2+56\,u^3w+14\,u^2w^2+40\,uv^3+20\,uvw^2\\ &\quad+60\,uv^2w+108\,u^2vw+8\,v^3w+5\,v^2w^2+vw^3+4\,v^4,\end{aligned}$$

$$(1.14)\quad \begin{aligned}&P_2(u,v,w)\\ &= \left(2\,u^2-v^2-vw\right)\left(20\,u^2+(24\,w+52\,v)\,u+53\,v^2+6\,w^2+52\,vw\right)\\ &\quad+\left(72\,v^3+36\,vw^2+108\,v^2w\right)u+57\,v^2w^2+51\,v^4+6\,vw^3+102\,v^3w,\end{aligned}$$

and

$$(1.15)\quad \begin{aligned}&P_3(u,v,w)\\ &= \left(2\,u^2-v^2-vw\right)\left(20\,u^2+(52\,v+28\,w)\,u+53\,v^2+54\,vw+7\,w^2\right)\\ &\quad+\left(72\,v^3+36\,vw^2+108\,v^2w\right)u+57\,v^2w^2+51\,v^4+6\,vw^3+102\,v^3w.\end{aligned}$$

By (1.8), we immediately obtain $P_i(u,v,w) \geq 0$ for $1 \leq i \leq 3$. Hence, inequality (1.10) is proved.

2. Some Problems and their Proofs

2.1. **The Problems.** In 2004-2005, J. Liu [7, 8] posed the following conjectures for the inequality in the acute triangle.

Problem 2.1. Let $\triangle ABC$ be an acute triangle. Prove the following inequalities

$$\sum \left(\frac{\sin 2A}{\sin B + \sin C}\right)^2 \leq \frac{3}{4}, \tag{2.1}$$

and

$$\sin \frac{A}{2} \leq \frac{\sqrt{m_b m_c}}{2m_a}. \tag{2.2}$$

2.2. **The Proof of Inequality (2.1).**

Proof. Using $\sin 2\alpha = 2\sin\alpha\cos\alpha$, we find that inequality (2.1) is equivalent to

$$\begin{aligned}
&4\,a^{10}b^2 - 10\,a^5b^5c^2 - 24\,b^6c^5a + 16\,a^9b^3 + 4\,a^8b^4 - 5\,b^4c^6a^2\\
&- 8\,a^{11}b - 5\,a^4b^6c^2 + 8\,a^9c^2b + 4\,a^8b^2c^2 - 16\,a^{10}cb + 8\,a^9cb^2 - 5\,a^6b^4c^2\\
&+ 32\,a^8b^3c + 8\,a^7b^4c + 8\,a^7c^4b - 5\,a^6c^4b^2 + 32\,a^8c^3b + 4\,a^{10}c^2 - 8\,a^{11}c\\
&- 4\,a^{12} - 4\,b^{12} - 4\,c^{12} + 4\,a^8c^4 + 16\,a^9c^3 - 8\,a^7b^5 - 8\,a^6b^6 - 8\,a^6c^6\\
&- 8\,a^7c^5 - 8\,a^5b^7 + 4\,a^4b^8 - 24\,a^6b^5c - 24\,a^5b^6c + 2\,a^5b^3c^4 + 6\,a^4b^4c^4\\
&+ 4\,c^{10}b^2 - 26\,a^6b^3c^3 + 2\,a^5b^4c^3 - 24\,a^5c^6b - 5\,a^4c^6b^2 - 24\,a^6c^5b\\
&- 10\,a^5c^5b^2 + 8\,a^4b^7c - 8\,c^{11}b + 32\,b^8a^3c + 8\,b^9a^2c + 4\,b^8a^2c^2\\
&+ 2\,b^5c^3a^4 - 26\,b^6c^3a^3 + 2\,b^5c^4a^3 - 5\,b^6c^4a^2 - 16\,b^{10}ca + 8\,b^9c^2a\\
&+ 32\,b^8c^3a + 8\,b^7c^4a - 10\,b^5c^5a^2 + 16\,b^9a^3 + 4\,b^{10}a^2 - 8\,b^{11}a\\
&- 8\,b^{11}c + 4\,b^{10}c^2 + 16\,b^9c^3 + 4\,b^8c^4 - 8\,b^7c^5 - 8\,b^6c^6 + 4\,a^4c^8\\
&- 8\,a^5c^7 - 24\,b^5c^6a + 8\,a^4c^7b + 2\,c^5b^4a^3 - 26\,c^6b^3a^3 + 2\,c^5b^3a^4\\
&+ 4\,c^8b^2a^2 + 8\,c^9a^2b + 32\,c^8a^3b + 8\,b^4c^7a - 8\,c^{11}a + 32\,c^8b^3a + 8\,c^9b^2a\\
&- 16\,c^{10}ab + 4\,c^{10}a^2 + 16\,c^9a^3 - 8\,b^5c^7 + 4\,b^4c^8 + 16\,c^9b^3 \geq 0.
\end{aligned} \tag{2.3}$$

From (1.6), inequality (2.3) equals

$$\begin{aligned}
F(x,y,z) = &- 4576\,x^7z^5 - 5590\,x^6z^6 - 116\,x^{10}z^2 - 2453\,x^4z^8 - 2453\,x^8z^4\\
&- 4576\,x^5z^7 - 788\,x^3z^9 - 788\,x^9z^3 - 2453\,x^8y^4 - 4576\,y^7z^5\\
&- 2453\,y^8z^4 - 2453\,x^4y^8 - 788\,x^9y^3 - 788\,x^3y^9 - 5590\,x^6y^6\\
&- 4576\,x^7y^5 - 4576\,x^5y^7 - 2453\,y^4z^8 - 4576\,y^5z^7 - 5590\,y^6z^6\\
&- 116\,y^{10}z^2 - 788\,y^9z^3 - 116\,y^{10}x^2 - 788\,y^3z^9 - 116\,y^2z^{10}\\
&+ 13448\,x^6y^5z + 8176\,x^7yz^4 + 13448\,x^6yz^5 + 13448\,x^5yz^6\\
&+ 8176\,x^4yz^7 + 6448\,x^2y^3z^7 + 1220\,xy^9z^2 + 6448\,x^2y^7z^3\\
&+ 6448\,x^3y^7z^2 + 1220\,x^9yz^2 + 10862\,x^4y^6z^2 + 14288\,x^2y^5z^5
\end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&+ 10862\,x^2y^4z^6 + 14288\,x^5y^2z^5 + 10862\,x^6y^2z^4 + 6448\,x^7y^2z^3\\
&- 28248\,x^3y^5z^4 - 28248\,x^4y^5z^3 + 14288\,x^5y^5z^2 - 57474\,x^4y^4z^4\\
&- 28248\,x^3y^4z^5 - 8672\,x^3y^3z^6 + 6448\,x^3y^2z^7 + 10862\,x^2y^6z^4\\
&- 8672\,x^3y^6z^3 - 28248\,x^5y^4z^3 + 10862\,x^6y^4z^2 - 28248\,x^4y^3z^5\\
&- 28248\,x^5y^3z^4 - 8672\,x^6y^3z^3 + 6448\,x^7y^3z^2 + 10862\,x^4y^2z^6\\
&+ 3420\,x^8yz^3 + 3420\,x^8y^3z + 1220\,x^2yz^9 + 280\,xy^{10}z + 4066\,x^2y^2z^8\\
&+ 3420\,x^3yz^8 + 3420\,x^3y^8z + 4066\,x^8y^2z^2 + 4066\,x^2y^8z^2\\
&+ 1220\,xy^2z^9 + 8176\,x^7y^4z + 3420\,xy^3z^8 + 3420\,xy^8z^3\\
&+ 1220\,x^2y^9z + 280\,xyz^{10} + 280\,x^{10}yz + 1220\,x^9y^2z + 8176\,xy^4z^7\\
&+ 13448\,xy^5z^6 + 13448\,xy^6z^5 + 8176\,x^4y^7z + 8176\,xy^7z^4\\
&+ 13448\,x^5y^6z - 116\,x^{10}y^2 - 116\,x^2z^{10} \geq 0.
\end{aligned}
$$

Since (2.4) is symmetric for x, y, z, there is no harm in supposing that $x \leq y \leq z$. Using the transformation (1.2), then $F(x, y, z)$ in (2.4) becomes

$$
\begin{aligned}
F(x,y,z) =& P(u,v,w)\\
=&(2u^2 - v^2 - vw)[(180224\,w^2 + 180224\,v^2 + 180224\,vw)u^8\\
&+ (1794048\,v^2w + 1810432\,vw^2 + 606208\,w^3 + 1196032\,v^3)u^7\\
&+ (4360192\,vw^3 + 7030784\,v^3w + 771072\,w^4 + 7875584\,v^2w^2\\
&+ 3515392\,v^4)u^6 + (6049280\,v^5 + 520704\,w^5 + 19394048\,v^3w^2\\
&+ 13967872\,v^2w^3 + 4689152\,vw^4 + 15123200\,v^4w)u^5\\
&+ (2838144\,vw^5 + 6838400\,v^6 + 12647648\,v^2w^4 + 30324704\,v^4w^2\\
&+ 210048\,w^6 + 26457408\,v^3w^3 + 20515200\,v^5w)u^4 + (19291776\,v^6w\\
&+ 52480\,w^7 + 1074176\,vw^6 + 5511936\,v^7 + 32787968\,v^5w^2\\
&+ 33740480\,v^4w^3 + 20662912\,v^3w^4 + 6899776\,v^2w^5)u^3\\
&+ (32727200\,v^5w^3 + 7968\,w^8 + 2528912\,w^6v^2 + 24395856\,v^4w^4\\
&+ 27385760\,v^6w^2 + 14122880\,v^7w + 268096\,w^7v + 3530720\,v^8\\
&+ 10723072\,v^3w^5)u^2 + (9558576\,v^8w + 4185944\,v^3w^6 + 13383144\,v^4w^5\\
&+ 676240\,v^2w^7 + 2124128\,v^9 + 672\,w^9 + 24737624\,v^5w^4 + 45200\,vw^8\\
&+ 20832112\,v^7w^2 + 28305704\,v^6w^3)u + 15686836\,v^5w^5 + 6092840\,v^4w^6\\
&+ 1326664\,v^3w^7 + 139150\,v^2w^8 + 24651416\,v^6w^4 + 24921352\,v^7w^3\\
&+ 1378920\,v^{10} + 6894600\,v^9w + 16572238\,v^8w^2 + 5112\,vw^9 + 24\,w^{10}]\\
&+ (27659640\,v^9w^2 + 10558592\,v^{10}w + 689380\,v^3w^8 + 4642800\,v^4w^7\\
&+ 36001700\,v^6w^5 + 45278940\,v^8w^3 + 16715660\,v^5w^6 + 720\,vw^{10}\\
&+ 49540936\,v^7w^4 + 1919744\,v^{11} + 44048\,v^2w^9)u + 5020\,v^2w^{10}\\
&+ 49008067\,v^8w^4 + 142314\,v^3w^9 + 23121662\,v^{10}w^2 + 8144784\,v^{11}w\\
&+ 1451049\,v^4w^8 + 40947790\,v^9w^3 + 24\,vw^{11} + 7353016\,v^5w^7\\
&+ 1357464\,v^{12} + 39938152\,v^7w^5 + 21582818\,v^6w^6.
\end{aligned}
\tag{2.5}
$$

This implies $F(x,y,z) = P(u,v,w) \geq 0$ from (1.8). Hence, inequality (2.1) holds. The proof is completed. □

2.3. **The Proof of Inequality** (2.2).

Proof. Inequality (2.2) is equivalent to

$$\sin^4 \frac{A}{2} \leq \frac{m_b^2 m_c^2}{16 m_a^4}. \tag{2.6}$$

By using the formula $\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2}$, the law of cosines and the formulas of the medians, we find that (2.6) is simply the following inequality

$$\begin{aligned} &- a^8 - 4\,b^8 + 6\,a^6c^2 - 34\,b^2c^6 + 20\,b^3a^4c + 12\,a^2c^6 - 32\,b^5a^2c - 32\,c^5a^2b \\ &- 34\,b^6c^2 - 51\,b^4c^4 - 4\,c^8 - 4\,a^6bc + 20\,c^3a^4b - 26\,a^4b^2c^2 + 54\,a^2b^4c^2 \\ &+ 54\,a^2b^2c^4 - 13\,a^4b^4 - 13\,a^4c^4 + 12\,a^2b^6 + 6\,a^6b^2 + 16\,b^7c + 16\,c^7b \\ &- 64\,b^3c^3a^2 + 48\,b^5c^3 + 48\,b^3c^5 \geq 0. \end{aligned} \tag{2.7}$$

Considering (1.6), inequality (2.7) is transformed into

$$\begin{aligned} F(x,y,z) =& x^8 + (4\,z + 4\,y)x^7 + (2\,z^2 + 40\,yz + 2\,y^2)x^6 \\ &+ (-8\,z^3 + 84\,yz^2 - 8\,y^3 + 84\,y^2z)x^5 \\ &+ (-20\,y^2z^2 + 76\,yz^3 + 76\,y^3z - 7\,z^4 - 7\,y^4)x^4 \\ &+ (4\,z^5 + 48\,y^4z - 248\,y^3z^2 - 248\,y^2z^3 + 4\,y^5 + 48\,yz^4)x^3 \\ &+ (4\,y^6 - 234\,y^4z^2 - 234\,y^2z^4 + 28\,y^5z + 4\,z^6 + 28\,yz^5 + 32\,y^3z^3)x^2 \\ &+ (-84\,z^5y^2 + 8\,z^6y - 84\,y^5z^2 + 256\,y^3z^4 + 256\,y^4z^3 + 8\,y^6z)x \\ &+ 84\,z^5y^3 - 63\,z^4y^4 - 12\,y^6z^2 - 12\,z^6y^2 + 84\,y^5z^3 \geq 0. \end{aligned} \tag{2.8}$$

It it easy to see that inequality (2.8) is symmetric for y, z. Therefore, we only need to prove that inequality (2.8) holds when $x \leq y \leq z$, $y \leq x \leq z$ and $y \leq z \leq x$.

Calculating $DS(F)$, it consists of 3 polynomials with $u > 0, v \geq 0, w \geq 0$ as follows

$$\begin{aligned} &P_1(u,v,w) \\ &=(2u^2 - v^2 - vw)[(192\,w^2 + 768\,v^2 + 768\,vw)u^4 \\ &\quad + (256\,w^3 + 2112\,vw^2 + 4800\,v^2w + 3200\,v^3)u^3 \\ &\quad + (5808\,v^4 + 80\,w^4 + 7376\,v^2w^2 + 11616\,v^3w + 1568\,vw^3)u^2 \\ &\quad + (6336\,v^5 + 15840\,v^4w + 13440\,v^3w^2 + 16\,w^5 + 4320\,v^2w^3 + 416\,vw^4)u \\ &\quad + 5112\,v^6 + 15336\,v^5w + 48\,vw^5 + 16560\,v^4w^2 + 7560\,v^3w^3 + 1272\,v^2w^4] \\ &\quad + (7344\,v^7 + 432\,w^5v^2 + 25704\,v^6w + 33912\,v^5w^2 + 20520\,v^4w^3 \\ &\quad + 5400\,v^3w^4)u + 20772\,v^7w + 5193\,v^8 + 36\,w^6v^2 \\ &\quad + 1332\,w^5v^3 + 32418\,v^6w^2 + 24552\,v^5w^3 + 9009\,v^4w^4 \end{aligned} \tag{2.9}$$

for $x \le y \le z$,

$$P_2(u,v,w) =(2u^2 - v^2 - vw)[(-384\,vw + 192\,v^2 + 192\,w^2)u^4 + (-192\,vw^2 + 896\,v^3 - 960\,v^2w + 256\,w^3)u^3 + (-976\,v^2w^2 + 1776\,v^4 + 80\,w^4 + 224\,vw^3 - 288\,v^3w)u^2 + (2032\,v^5 - 480\,v^3w^2 + 16\,w^5 + 1328\,v^4w + 128\,v^2w^3 + 240\,vw^4)u + 1640\,v^6 + 2128\,v^5w + 544\,v^4w^2 + 328\,v^3w^3 + 416\,v^2w^4 + 80\,vw^5] + (2064\,v^5w^2 + 32\,w^6v + 4176\,v^6w + 776\,v^4w^3 + 2320\,v^7 + 416\,v^2w^5 + 968\,v^3w^4)u + 1640\,v^8 + 2708\,w^2v^6 + 817\,w^4v^4 + 524\,w^5v^3 + 956\,w^3v^5 + 84\,w^6v^2 + 3768\,wv^7 \tag{2.10}$$

for $y \le x \le z$, and

$$P_3(u,v,w) =384\,u^6v^2 + 11072\,w^2u^2v^4 + 20992\,w^2u^3v^3 + 19552\,w^2u^4v^2 + 8832\,w^2u^5v + 2008\,w^4uv^3 + 5296\,w^4u^2v^2 + 5376\,w^4u^3v + 36\,w^2v^6 + 1536\,w^2u^6 + 2792\,w^3uv^4 + 10400\,w^3u^2v^3 + 15744\,w^3u^3v^2 + 10816\,w^3u^4v + 2368\,w^2uv^5 + 840\,w^5uv^2 + 1344\,w^5u^2v + 2816\,w^3u^5 + 132\,w^3v^5 + 1888\,w^4u^4 + 193\,w^4v^4 + 184\,w^6uv + 144\,w^5v^3 + 640\,w^5u^3 + 1200\,wuv^6 + 13120\,wu^3v^4 + 6256\,wu^2v^5 + 13824\,wu^4v^3 + 58\,w^6v^2 + 128\,w^6u^2 + 7296\,wu^5v^2 + 3360\,u^4v^4 + 1792\,u^5v^3 + 288\,uv^7 + 1504\,u^2v^6 + 3168\,u^3v^5 + 1536\,wu^6v + 12\,w^7v + w^8 + 16\,w^7u \tag{2.11}$$

for $y \le z \le x$.

It is not difficult to see that $P_1(u,v,w) \ge 0$ and $P_3(u,v,w) \ge 0$ because $u > 0, v \ge 0, w \ge 0$ and $2u^2 - v^2 - vw > 0$.

In order to prove $P_2(u,v,w) \ge 0$, we only need prove the following inequality

$$p(u,v,w) =(-384\,vw + 192\,v^2 + 192\,w^2)u^4 + (-192\,vw^2 + 896\,v^3 - 960\,v^2w + 256\,w^3)u^3 + (-976\,v^2w^2 + 1776\,v^4 + 80\,w^4 + 224\,vw^3 - 288\,v^3w)u^2 + (2032\,v^5 - 480\,v^3w^2 + 16\,w^5 + 1328\,v^4w + 128\,v^2w^3 + 240\,vw^4)u + 1640\,v^6 + 2128\,v^5w + 544\,v^4w^2 + 328\,v^3w^3 + 416\,v^2w^4 + 80\,vw^5 \ge 0, \tag{2.12}$$

where $u > 0$, $v \ge 0$ and $w \ge 0$.

(i) For $u > 0, v \ge w \ge 0$, taking $v = w + t$ with $t \ge 0$, then we have

$$p(u,v,w) =192\,t^2u^4 + (576\,tw^2 + 1728\,t^2w + 896\,t^3)u^3 + (816\,w^4 + 4512\,w^3t + 8816\,w^2t^2 + 6816\,wt^3 + 1776\,t^4)u^2 + (2032\,t^5 + 11488\,wt^4 + 3264\,w^5 + 14528\,w^4t + 26976\,w^3t^2 + 25152\,w^2t^3)u + 50544\,w^4t^2 + 56584\,w^3t^3 + 5136\,w^6 + 24552\,w^5t + 1640\,t^6 + 35784\,w^2t^4 + 11968\,wt^5.$$

It obviously follows that $p(u,v,w) \ge 0$, i.e., inequality (2.12) holds.

(ii) When $u > 0, w \geq v \geq 0$, setting $w = v + t$ for $t \geq 0$, we get

$$\begin{aligned} p(u,v,w) =& (2\,u + 10\,v)t^5 + (10\,u^2 + 40\,uv + 102\,v^2)t^4 \\ &+ (156\,uv^2 + 32\,u^3 + 349\,v^3 + 68\,u^2v)t^3 \\ &+ (24\,u^4 + 188\,uv^3 + 22\,u^2v^2 + 72\,u^3v + 603\,v^4)t^2 \\ &+ v^2(783\,v^3 - 72\,u^3 + 224\,uv^2 - 156\,u^2v)t \\ &+ 6\,v^4(17\,u^2 + 68\,uv + 107\,v^2) \\ =& p_1(u,v,t) + p_2(u,v,t), \end{aligned}$$

where

$$p_1(u,v,t) = (2\,u + 10\,v)t^5 + (10\,u^2 + 40\,uv + 102\,v^2)t^4 + (156\,uv^2 + 32\,u^3 + 349\,v^3 + 68\,u^2v)t^3 \geq 0,$$

and

$$\begin{aligned} p_2(u,v,t) = & (24\,u^4 + 188\,uv^3 + 22\,u^2v^2 + 72\,u^3v + 603\,v^4)t^2 \\ &+ v^2(783\,v^3 - 72\,u^3 + 224\,uv^2 - 156\,u^2v)t \\ &+ 6\,v^4(17\,u^2 + 68\,uv + 107\,v^2). \end{aligned} \tag{2.13}$$

It is easy to see that $24\,u^4 + 188\,uv^3 + 22\,u^2v^2 + 72\,u^3v + 603\,v^4 > 0$, and the discriminant of the quadratic function (2.13) with respect to t is

$$\begin{aligned} \Delta(u,v) = & -v^4(935415\,v^6 + 480144\,u^3v^3 + 1116096\,uv^5 \\ &+ 803456\,u^2v^4 + 4608\,u^6 + 196032\,u^4v^2 + 46080\,u^5v) \leq 0. \end{aligned} \tag{2.14}$$

This is to say that $p_2(u,v,t) \geq 0$.

Hence, $P_2(u,v,w) \geq 0$. From the proof above, the required result (2.6) is proved. □

2.4. **Remarks.**

Remark 2.1. By the same argument as above, we also prove the following inequalities conjectures [9, 10, 11] in the acute triangle

$$\sum m_a^2 h_a^2 \geq \sum m_a^2 r_a^2, \tag{2.15}$$

$$\sum \sin^8 A \geq \sum \cos^8 \frac{A}{2}, \tag{2.16}$$

$$\sum (b-c)^2 \geq \sum \left(\frac{a}{b+c}\right)^2 (r_b - r_c)^2, \tag{2.17}$$

and

$$\sum (h_b + h_c - h_a)^3 \geq 3 m_a m_b m_c. \tag{2.18}$$

Remark 2.2. The operations in this paper were performed using mathematical software Maple 9.0.

3. GENERALIZATION OF THE METHOD

In fact, Difference Substitution can go even further. Now, we consider the following inequality [12]. In $\triangle ABC$, if $\max(A,B,C) \le \frac{2\pi}{3}$, then

$$s^2 \ge R^2 + 10Rr + 3r^2. \tag{3.1}$$

Utilizing the known formulas $R = \frac{abc}{4S}$, $r = \frac{S}{s}$ and $S = \sqrt{s(s-a)(s-b)(s-c)}$, from (1.6), inequality (3.1) is equivalent to

$$\begin{aligned} &3\,c^2a^2b^2 - a^2bc^3 - a^3bc^2 - a^3b^2c - a^2b^3c - ab^2c^3 - ab^3c^2 - 2\,b^3c^3 \\ &\quad - 2\,a^3c^3 + c^4b^2 - 2\,a^3b^3 + c^5b + a^4c^2 + b^4c^2 + b^5a + a^5c + c^5a \\ &\quad\quad - a^6 + a^4b^2 + a^5b - b^6 - c^6 + a^2b^4 + b^5c + a^2c^4 \ge 0, \end{aligned} \tag{3.2}$$

or

$$\begin{aligned} F(x,y,z) = &-42\,x^2y^2z^2 + 14\,y^4zx + 14\,xyz^4 + 2\,xy^2z^3 + 2\,x^2y^3z + 2\,xy^3z^2 \\ &+ 14\,x^4yz + 2\,x^3yz^2 + 2\,x^2yz^3 + 2\,x^3y^2z - x^4y^2 - x^2z^4 - 2\,x^3z^3 \\ &- x^4z^2 - 2\,x^3y^3 - y^4z^2 - y^4x^2 - 2\,y^3z^3 - y^2z^4 \ge 0, \end{aligned} \tag{3.3}$$

where $x > 0, y > 0, z > 0$.

Since inequality (3.3) is symmetric with x, y, z, there is no harm in supposing that $x \le y \le z$. From (1.2), $F(x,y,z)$ in (3.3) is transformed into

$$\begin{aligned} F(x,y,z) =& P(u,v,w) \\ =&(8\,u^2 + 4\,uv + 2\,uw - v^2 - vw)(8\,uvw^2 + 12\,uv^2w + 4\,u^2vw \\ &+ 4\,v^4 + 4\,u^2v^2 + 2\,uw^3 + 8\,uv^3 + 8\,v^3w + 7\,v^2w^2 + 3\,vw^3) \\ &+ 2\,w^2(v + 2\,u)^2(v + 2\,u + w)^2 \ge 0, \end{aligned} \tag{3.4}$$

and for $\max(A,B,C) \le \frac{2\pi}{3}$ and $y = \cos x$ decreasing in $x \in (0,\pi)$, we have

$$\begin{aligned} b^2 + c^2 + bc - a^2 &= b^2 + c^2 - \frac{1}{2}bc\cos\frac{2\pi}{3} - a^2 \\ &= 3x^2 + 3(y+z)x - yz \\ &= 8u^2 + 4uv + 2uw - v^2 - vw \\ &\ge b^2 + c^2 - \frac{1}{2}bc\cos A - a^2 = 0. \end{aligned} \tag{3.5}$$

Since $F(x,y,z) = P(u,v,w) \ge 0$ for $u > 0, v \ge 0$ and $w \ge 0$, inequality (3.1) is obtained.

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