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INTEGRABILITY CONDITIONS PERTAINING TO ORLICZ SPACE

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ABSTRACT. Recently S. Tikhonov proved two theorems on the integrability of sine and cosine series with coefficients from the R_0^+BVS class. These results are extended such that the R_0^+BVS class is replaced by the $MRBVS$ class.

Key words and phrases: Trigonometric series, Integrability, Orlicz space.

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1. INTRODUCTION

There are many known and classical theorems pertaining to the integrability of formal sine and cosine series

$$g(x) := \sum_{n=1}^{\infty} \lambda_n \sin nx, \tag{1.1}$$

and

$$f(x) := \sum_{n=1}^{\infty} \lambda_n \cos nx. \tag{1.2}$$

We do not recall such theorems because a nice short survey of these results with references can be found in a recent paper of S. Tikhonov [3], where he proved two theorems providing sufficient conditions of belonging of $f(x)$ and $g(x)$ to Orlicz spaces. In his theorems the sequence of the coefficients λ_n belongs to the class of sequences of rest bounded variation. For notions and notations, please, consult the third section.

In the present paper we shall verify analogous results assuming only that the sequence $\lambda := \{\lambda_n\}$ is a sequence of mean rest bounded variation. We emphasize that the latter sequences may have many zero terms, while the previous ones have no zero term.

Tikhonov's theorems read as follows:

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Theorem 1.1. *Let* $\Phi(x) \in \Delta(p,0)$ $(0 \le p)$. *If* $\{\lambda_n\} \in R_0^+BVS$, *and the sequence* $\{\gamma_n\}$ *is such that* $\{\gamma_n\, n^{-1+\varepsilon}\}$ *is almost decreasing for some* $\varepsilon > 0$, *then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^2}\Phi(n\,\lambda_n) < \infty \Rightarrow \psi(x) \in L(\Phi,\gamma), \tag{1.3}$$

where a function $\psi(x)$ *is either a sine or cosine series.*

Theorem 1.2. *Let* $\Phi(x) \in \Delta(p,q)$ $(0 \le q \le p)$. *If* $\{\lambda_n\} \in R_0^+BVS$, *and the sequence* $\{\gamma_n\}$ *is such that* $\{\gamma_n\, n^{-(1+q)+\varepsilon}\}$ *is almost decreasing for some* $\varepsilon > 0$, *then*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2+q}}\Phi(n^2\,\lambda_n) < \infty \Rightarrow g(x) \in L(\Phi,\gamma). \tag{1.4}$$

2. New Result

Now, we formulate our result in a terse form.

Theorem 2.1. *Theorems 1.1 and 1.2 can be improved when the condition* $\{\lambda_n\} \in R_0^+BVS$ *is replaced by the assumption* $\{\lambda_n\} \in MRBVS$. *Furthermore the conditions of (1.3) and (1.4) may be modified as follows:*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^2}\Phi\left(\sum_{\nu=n}^{2n-1}\lambda_\nu\right) < \infty \Rightarrow \psi(x) \in L(\Phi,\gamma), \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n^{2+q}}\Phi\left(n\sum_{\nu=n}^{2n-1}\lambda_\nu\right) < \infty \Rightarrow g(x) \in L(\Phi,\gamma), \tag{2.2}$$

respectively.

Remark 2.2. It is easy to see that if $\{\lambda_n\} \in R_0^+BVS$ also holds, then

$$\sum_{\nu=n}^{2n-1}\lambda_\nu \ll n\,\lambda_n,$$

that is, our assumptions are not worse than (1.3) and (1.4).

3. Notions and Notations

A null-sequence $\mathbf{c} := \{c_n\}$ $(c_n \to 0)$ of *positive* numbers satisfying the inequalities

$$\sum_{n=m}^{\infty} |\Delta\, c_n| \le K(\mathbf{c})c_m, \quad (\Delta\, c_n := c_n - c_{n+1}),\ m = 1, 2, \ldots$$

with a constant $K(\mathbf{c}) > 0$ is said to be a *sequence of rest bounded variation*, in brief, $\mathbf{c} \in R_0^+BVS$.

A null-sequence $\mathbf{c}$ of *nonnegative* numbers possessing the property

$$\sum_{n=2m}^{\infty} |\Delta\, c_n| \le K(\mathbf{c})m^{-1}\sum_{\nu=m}^{2m-1} c_\nu$$

is called a *sequence of mean rest bounded variation*, in symbols, $\mathbf{c} \in MRBVS$.

It is clear that the class $MRBVS$ includes the class R_0^+BVS.

The author is grateful to the referee for calling his attention to an inaccurancy in the previous definition of the class $MRBVS$ and to some typos.

A sequence γ of positive terms will be called almost increasing (decreasing) if

$$K(\gamma)\gamma_n \geq \gamma_m \quad (\gamma_n \leq K(\gamma)\gamma_m)$$

holds for any $n \geq m$.

Denote by $\Delta(p,q)$ $(0 \leq q \leq p)$ the set of all nonnegative functions $\Phi(x)$ defined on $[0,\infty)$ such that $\Phi(0)=0$ and $\Phi(x)/x^p$ is nonincreasing and $\Phi(x)/x^q$ is nondecreasing.

In this paper a sequence $\gamma := \{\gamma_n\}$ is associated to a function $\gamma(x)$ being defined in the following way: $\gamma\left(\frac{\pi}{n}\right) := \gamma_n,\ n \in \mathbb{N}$ and $K_1(\gamma)\gamma_{n+1} \leq \gamma(x) \leq K_2(\gamma)\gamma_n$ holds for all $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right)$.

A locally integrable almost everywhere positive function $\gamma(x) : [0,\pi] \to [0,\infty)$ is said to be a weight function.

Let $\Phi(t)$ be a nondecreasing continuous function defined on $[0,\infty)$ such that $\Phi(0)=0$ and $\lim\limits_{t\to\infty} \Phi(t) = +\infty$. For a weight function $\gamma(x)$ the weighted Orlicz space $L(\Phi,\gamma)$ is defined by

$$L(\Phi,\gamma) := \left\{ h : \int_0^\pi \gamma(x)\Phi(\varepsilon|h(x)|)dx < \infty \text{ for some } \varepsilon > 0 \right\}.$$

Later on $D_k(x)$ and $\tilde{D}_k(x)$ shall denote the Dirichlet and the conjugate Dirichlet kernels. It is known that, if $x>0,\ |D_k(x)| = O(x^{-1})$ and $|\tilde{D}_k(x)| = O(x^{-1})$ hold.

We shall also use the notation $L \ll R$ if there exists a positive constant K such that $L \leq KR$.

4. LEMMAS

Lemma 4.1 ([1]). *If $a_n \geq 0,\ \rho_n > 0$, and if $p \geq 1$, then*

$$\sum_{n=1}^\infty \rho_n \left(\sum_{\nu=1}^n a_\nu\right)^p \ll \sum_{n=1}^\infty \rho_n^{1-p} a_n^p \left(\sum_{\nu=n}^\infty \rho_\nu\right)^p.$$

Lemma 4.2 ([2]). *Let $\Phi \in \Delta(p,q)$ $(0 \leq q \leq p)$ and $t_j \geq 0,\ j=1,2,\dots,n,\ n \in \mathbb{N}$. Then*

(1) $Q^p\Phi(t) \leq \Phi(Qt) \leq Q^q\Phi(t), \quad 0 \leq Q \leq 1,\ t \geq 0,$

(2) $\Phi\left(\sum\limits_{j=1}^n t_j\right) \leq \left(\sum\limits_{j=1}^n \Phi^{1/p^*}(t_j)\right)^{p^*}, \quad p^* := \max(1,p).$

Lemma 4.3. *Let $\Phi \in \Delta(p,q)$ $(0 \leq q \leq p)$. If $\rho_n > 0,\ a_n \geq 0$, and if*

$$\sum_{\nu=2^m}^{2^{m+1}-1} a_\nu \ll \sum_{\nu=1}^{2^m-1} a_\nu \tag{4.1}$$

holds for all $m \in \mathbb{N}$, then

$$\sum_{k=1}^\infty \rho_k \Phi\left(\sum_{\nu=1}^k a_\nu\right) \ll \sum_{k=1}^\infty \Phi\left(\sum_{\nu=k}^{2k-1} a_\nu\right) \rho_k \left(\frac{1}{k\rho_k}\sum_{\nu=k}^\infty \rho_\nu\right)^{p^*},$$

where $p^* := \max(1,p)$.

Proof. Denote by $A_n := n^{-1}\sum_{\nu=n}^{2n-1} a_\nu$. Let ξ be an integer such that $2^\xi \leq k < 2^{\xi+1}$. Then

$$\sum_{\nu=1}^k a_\nu \leq \sum_{m=0}^\xi \sum_{\nu=2^m}^{2^{m+1}-1} a_\nu = \sum_{m=0}^\xi 2^m A_{2^m}. \tag{4.2}$$

Utilizing the properties of Φ, furthermore (4.1), (4.2) and Lemma 4.2, we obtain that

$$\begin{aligned}
\Phi\left(\sum_{\nu=1}^{k} a_\nu\right) &\ll \Phi\left(\sum_{m=0}^{\xi} 2^m A_{2^m}\right)\\
&\ll \Phi\left(\sum_{m=0}^{\xi-1} 2^m A_{2^m}\right)\\
&\ll \left(\sum_{m=0}^{\xi-1} \Phi^{1/p^*}(2^m A_{2^m})\right)^{p^*}\\
&\ll \left(\sum_{\nu=1}^{k} \nu^{-1}\Phi^{1/p^*}(\nu\, A_\nu)\right)^{p^*}.
\end{aligned}$$

Hence, by Lemma 4.1, we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\rho_k\Phi\left(\sum_{\nu=1}^{k} a_\nu\right) &\ll \sum_{k=1}^{\infty}\rho_k\left(\sum_{\nu=1}^{k} \nu^{-1}\Phi^{1/p^*}(\nu\, A_\nu)\right)^{p^*}\\
&\ll \sum_{k=1}^{\infty}\rho_k^{1-p^*}(k^{-1}\Phi^{1/p^*}(k\, A_k))^{p^*}\left(\sum_{\nu=k}^{\infty}\rho_\nu\right)^{p^*}\\
&\ll \sum_{k=1}^{\infty}\rho_k\Phi(k\, A_k)\left((k\,\rho_k)^{-1}\sum_{\nu=k}^{\infty}\rho_\nu\right)^{p^*}.
\end{aligned}$$

Herewith the proof is complete. □

Lemma 4.4. *If* $\lambda := \{\lambda_n\} \in MRBVS$ *and* $\Lambda_n := n^{-1}\sum_{\nu=n}^{2n-1}\lambda_\nu$, *then*

$$\Lambda_k \ll \Lambda_\ell$$

holds for all $k \geq 2\ell$.

Proof. It is clear that if $m \geq 2\ell$, then

$$\ell^{-1}\sum_{\nu=\ell}^{2\ell-1}\lambda_\nu \gg \sum_{\nu=2\ell}^{\infty}|\Delta\,\lambda_\nu| \geq \sum_{\nu=m}^{\infty}|\Delta\,\lambda_\nu| \geq \lambda_m,$$

whence

$$\Lambda_\ell = \ell^{-1}\sum_{\nu=\ell}^{2\ell-1}\lambda_\nu \gg k^{-1}\sum_{m=k}^{2k-1}\lambda_m = \Lambda_k$$

obviously follows. □

5. PROOF OF THEOREM 2.1

Proof of Theorem 2.1. Let $x \in \left(\frac{\pi}{n+1}, \frac{\pi}{n}\right)$. Using Abel's rearrangement, the known estimate of $D_k(x)$ and the fact that $\lambda \in MRBVS$, we obtain that

$$\begin{aligned} |f(x)| &\leq \sum_{k=1}^{n} \lambda_k + \left| \sum_{k=n+1}^{\infty} \lambda_k \cos kx \right| \\ &\leq \sum_{k=1}^{n} \lambda_k + \sum_{k=n}^{\infty} |\Delta \lambda_k D_k(x)| + \lambda_n |D_n(x)| \\ &\ll \sum_{k=1}^{n} \lambda_k + \sum_{k \geq n/2}^{n} \lambda_k + n\,\lambda_n. \end{aligned}$$

Hence, $\lambda \in MRBVS$, and we obtain that

$$|f(x)| \ll \sum_{k=1}^{n} \lambda_k$$

also holds.

A similar argument yields

$$|g(x)| \ll \sum_{k=1}^{n} \lambda_k,$$

thus we have

$$|\psi(x)| \ll \sum_{k=1}^{n} \lambda_k, \tag{5.1}$$

where $\psi(x)$ is either $f(x)$ or $g(x)$.

By Lemma 4.4, the condition (4.1) with λ_ν in place of a_ν is satisfied, thus we can apply Lemma 4.3, consequently (5.1) and some elementary calculations give that

$$\begin{aligned} \int_0^{\pi} \gamma(x)\Phi(|\psi(x)|)dx &\ll \sum_{n=1}^{\infty} \Phi\left(\sum_{k=1}^{n} \lambda_k\right) \int_{\pi/(n+1)}^{\pi/n} \gamma(x)dx \\ &\ll \sum_{n=1}^{\infty} \gamma_n\, n^{-2} \Phi\left(\sum_{k=1}^{n} \lambda_k\right) \\ &\ll \sum_{k=1}^{\infty} \Phi\left(\sum_{\nu=k}^{2k-1} \lambda_\nu\right) \gamma_k\, k^{-2} \left(k\,\gamma_k^{-1} \sum_{\nu=k}^{\infty} \gamma_\nu\, \nu^{-2}\right)^{p^*}. \end{aligned} \tag{5.2}$$

Since the sequence $\{\gamma_n\, n^{-1+\varepsilon}\}$ is almost decreasing, then

$$k\,\gamma_k^{-1} \sum_{\nu=k}^{\infty} \gamma_\nu\, \nu^{-2} \ll 1,$$

therefore (5.2) proves (2.1).

To prove (2.2) we follow a similar procedure as above. Then

$$
\begin{aligned}
|g(x)| &\leq \sum_{k=1}^{n} kx\lambda_k + \left| \sum_{k=n+1}^{\infty} \lambda_k \sin kx \right| \\
&\ll x \sum_{k=1}^{n} k\lambda_k + \sum_{k=n}^{\infty} |\Delta\lambda_k \tilde{D}_k(x)| + \lambda_n |\tilde{D}_n(x)| \\
&\ll n^{-1} \sum_{k=1}^{n} k\lambda_k + \sum_{k \geq n/2}^{n} \lambda_k + n\,\lambda_n \\
&\ll n^{-1} \sum_{k=1}^{n} k\lambda_k. \qquad (5.3)
\end{aligned}
$$

Using Lemmas 4.2, 4.3, 4.4 and the estimate (5.3), we obtain that

$$
\begin{aligned}
\int_0^{\pi} \gamma(x)\Phi(|g(x)|)dx &\ll \sum_{n=1}^{\infty} \Phi\left(n^{-1} \sum_{k=1}^{n} k\lambda_k \right) \int_{\pi/(n+1)}^{\pi/n} \gamma(x)dx \\
&\ll \sum_{n=1}^{\infty} \gamma_n\, n^{-2-q} \Phi\left(\sum_{k=1}^{n} k\lambda_k \right) \\
&\ll \sum_{k=1}^{\infty} \Phi\left(k \sum_{\nu=k}^{2k-1} \lambda_\nu \right) \gamma_k\, k^{-2-q} \left(k^{1+q}\gamma_k^{-1} \sum_{\nu=k}^{\infty} \gamma_\nu \nu^{-2-q} \right)^{p^*}. \qquad (5.4)
\end{aligned}
$$

By the assumption on $\{\gamma_n\}$,

$$
k^{1+q}\gamma_k^{-1} \sum_{\nu=k}^{\infty} \gamma_\nu \nu^{-2-q} \ll 1,
$$

and thus (5.4) yields that

$$
\int_0^{\pi} \gamma(x)\Phi(|g(x)|)dx \ll \sum_{k=1}^{\infty} \gamma_k k^{-2-q} \Phi\left(k \sum_{\nu=k}^{2k-1} \lambda_\nu \right)
$$

holds, which proves (2.2).

Herewith the proof of Theorem 2.1 is complete. □

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