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ON HADAMARD TYPE INEQUALITIES FOR GENERALIZED WEIGHTED QUASI-ARITHMETIC MEANS

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ABSTRACT. In the present paper we establish some integral inequalities analogous to the well-known Hadamard inequality for a class of generalized weighted quasi-arithmetic means in integral form.

Key words and phrases: Integral mean, Hadamard's inequality, generalized weighted quasi-arithmetic mean, convex function.

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1. INTRODUCTION

In papers [6] and [7] we have investigated some basic properties of a class of generalized weighted quasi-arithmetic means in integral form and we have presented some inequalities involving such a class of means.

In this paper we extend our considerations to inequalities of Hadamard type. Recall that the inequality, cf. [5],

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

which holds for all convex functions $f:[a,b]\to\mathbb{R}$, is known in the literature as the *Hadamard inequality* (sometimes denoted as the Hermite-Hadamard inequality). This inequality has became an important cornerstone in mathematical analysis and optimization and has found uses in a variety of settings. There is a growing literature providing new proofs, extensions and considering its refinements, generalizations, numerous interpolations and applications, for example,

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in the theory of special means and information theory. For some results on generalization, extensions and applications of the Hadamard inequality, see [1], [2], [4], [8], [9] and [10].

In general, the inequality (1.1) is a special case of a result of Fejér, [3],

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \le \int_a^b p(x)f(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx, \tag{1.2}$$

which holds when f is convex and p is a nonnegative function whose graph is symmetric with respect to the center of the interval $[a,b]$. It is interesting to investigate the role of symmetry in this result and whether such an inequality holds also for other functions (instead of convex ones). In this paper we consider the weight function p as a positive Lebesgue integrable function defined on the closed interval $[a,b]\subset\mathbb{R}$, $a<b$ with a finite norm, i.e. p belongs to the vector space $L_1^+([a,b])$ (see Section 2). We also give an elementary proof of the Jensen inequality and state a result which corresponds to some of its conversions. In the last section of this article we give a result involving two convex (concave) functions and (not necessarily symmetric) a weight function p on the interval $[a,b]$ which is a generalization of a result given in [9].

The main aim of this paper is to establish some integral inequalities analogous to that of the weighted Hadamard inequality (1.2) for a class of generalized weighted quasi-arithmetic means in integral form.

2. PRELIMINARIES

The notion of a convex function plays a fundamental role in modern mathematics. As usual, a function $f: I\to\mathbb{R}$ is called *convex* if

$$f((1-\lambda)x+\lambda y)\le(1-\lambda)f(x)+\lambda f(y)$$

for all $x,y\in I$ and all $\lambda\in[0,1]$. Note that if $(-f)$ is convex, then f is called *concave*. In this paper we will use the following simple characterization of convex functions. For the proof, see [10].

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$. Then the following statements are equivalent:*

(i) *f is convex on $[a,b]$;*
(ii) *for all $x,y\in[a,b]$ the function $g:[0,1]\to\mathbb{R}$, defined by $g(t)=f((1-t)x+ty)$ is convex on $[0,1]$.*

For the convenience of the reader we continue by recalling the definition of a class of generalized weighted quasi-arithmetic means in integral form, cf. [6].

Let $L_1([a,b])$ be the vector space of all real Lebesgue integrable functions defined on the interval $[a,b]\subset\mathbb{R}$, $a<b$, with respect to the classical Lebesgue measure. Let us denote by $L_1^+([a,b])$ the positive cone of $L_1([a,b])$, i.e. the vector space of all real positive Lebesgue integrable functions on $[a,b]$. In what follows $\|p\|_{[a,b]}$ denotes the finite L_1-norm of a function $p\in L_1^+([a,b])$. For the purpose of integrability of the product of two functions we also consider the space $L_\infty([a,b])$ as the dual to $L_1([a,b])$ (and $L_\infty^+([a,b])$ as the dual to $L_1^+([a,b])$, respectively).

Definition 2.1. Let $(p,f)\in L_1^+([a,b])\times L_\infty^+([a,b])$ and $g:[0,\infty]\to\mathbb{R}$ be a real continuous monotone function. The generalized weighted quasi-arithmetic mean of function f with respect to weight function p is a real number $M_{[a,b],g}(p,f)$ given by

$$M_{[a,b],g}(p,f)=g^{-1}\left(\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)g(f(x))\,dx\right), \tag{2.1}$$

where g^{-1} denotes the inverse function to the function g.

Many known means in the integral form of two variables p, f are a special case of $M_{[a,b],g}(p,f)$ when taking the suitable functions p, f, g. For instance, putting $p \equiv 1$ on $[a,b]$ we obtain classical quasi-arithmetic integral means of a function f. Means $M_{[a,b],g}(p,f)$ generalize also other types of means, cf. [7], e.g. generalized weighted arithmetic, geometric and harmonic means, logarithmic means, intrinsic means, power means, one-parameter means, extended logarithmic means, extended mean values, generalized weighted mean values, and others. Hence, from $M_{[a,b],g}(p,f)$ we can deduce most of the two variable means.

Some basic properties of means $M_{[a,b],g}(p,f)$ related to properties of input functions f, g were studied in [6] and [7] in connection with the weighted integral Jensen inequality for convex functions. The following lemma states Jensen's inequality in the case of means $M_{[a,b],g}(p,f)$ and we give its elementary proof.

Lemma 2.2 (Jensen's Inequality). *Let* $(p,f) \in L_1^+([a,b]) \times L_\infty^+([a,b])$ *such that* $c < f(x) < d$ *for all* $x \in [a,b]$, *where* $-\infty < c < d < \infty$.

(i) *If* g *is a convex function on* (c,d), *then*

$$g\big(A_{[a,b]}(p,f)\big) \leq A_{[a,b]}(p, g \circ f). \tag{2.2}$$

(ii) *If* g *is a concave function on* (c,d), *then*

$$g\big(A_{[a,b]}(p,f)\big) \geq A_{[a,b]}(p, g \circ f),$$

where $A_{[a,b]}(p,f)$ *denotes the weighted arithmetic mean of the function* f *on* $[a,b]$.

Proof. Let g be a convex function. Put

$$\xi = A_{[a,b]}(p,f). \tag{2.3}$$

From the mean value theorem, it follows that $c < \xi < d$. Put

$$\eta = \sup_{\tau \in (c,d)} \frac{g(\xi) - g(\tau)}{\xi - \tau},$$

i.e., the supremum of slopes of secant lines. From the convexity of g it follows that

$$\eta \leq \frac{g(\theta) - g(\xi)}{\theta - \xi}, \quad \text{for any} \quad \theta \in (\xi, d).$$

Therefore, we have that

$$g(\tau) \geq g(\xi) + \eta(\tau - \xi), \quad \text{for any} \quad \tau \in (c,d),$$

which is equivalent to

$$g(\xi) - g(\tau) \leq \eta(\xi - \tau), \tag{2.4}$$

for any $\tau \in (c,d)$. Choosing, in particular, $\tau = f(x)$, multiplying both sides of (2.4) by $p(x)/\|p\|_{[a,b]}$ and integrating over the interval $[a,b]$ with respect to x, we get

$$g(\xi) - A_{[a,b]}(p, g \circ f) \leq \eta \cdot A_{[a,b]}(p, \xi - f). \tag{2.5}$$

The integral at the right side of the inequality (2.5) is equal to 0. Indeed,

$$\eta \cdot A_{[a,b]}(p, \xi - f) = \eta\left(\xi - A_{[a,b]}(p,f)\right) = 0.$$

Replacing ξ by (2.3), we have

$$g\Big(A_{[a,b]}(p,f)\Big) - A_{[a,b]}(p, g \circ f) \leq 0.$$

Hence the result (2.2). □

As a direct consequence of Jensen's inequality we obtain the following

Corollary 2.3. *Let* $(p, f) \in L_1^+([a,b]) \times L_\infty^+([a,b])$ *such that* $c < f(x) < d$ *for all* $x \in [a,b]$, *where* $-\infty < c < d < \infty$.

(i) *If* g *is a convex increasing or concave decreasing function on* (c,d), *then*

$$A_{[a,b]}(p,f) \le M_{[a,b],g}(p,f). \tag{2.6}$$

(ii) *If* g *is a convex decreasing or concave increasing function on* (c,d), *then*

$$A_{[a,b]}(p,f) \ge M_{[a,b],g}(p,f).$$

Proof. Let g be a convex increasing function. Applying the inverse of g to both sides of Jensen's inequality (2.2) we obtain the desired result (2.6).

Proofs of remaining parts are similar. □

In what follows the following two simple lemmas will be useful.

Lemma 2.4. *Let* $h : [a,b] \to [c,d]$ *and let* h^{-1} *be the inverse function to the function* h.

(i) *If* h *is strictly increasing and convex, or a strictly decreasing and concave function on* $[a,b]$, *then* h^{-1} *is a concave function on* $[c,d]$.

(ii) *If* h *is strictly decreasing and convex, or a strictly increasing and concave function on* $[a,b]$, *then* h^{-1} *is a convex function on* $[c,d]$.

Proof. We will prove only the item (i), the item (ii) may be proved analogously. Suppose that h is a strictly decreasing and convex function on $[a,b]$. Then clearly h^{-1} is strictly decreasing on $[c,d]$. Take $x_1, x_2 \in [a,b]$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$. Since h^{-1} is the inverse to h, there exist $y_1, y_2 \in [c,d]$ such that $y_i = h(x_i)$ and $x_i = h^{-1}(y_i)$, for $i \in \{1,2\}$. Then

$$h^{-1}(\alpha y_1 + \beta y_2) = h^{-1}(\alpha h(x_1) + \beta h(x_2)).$$

Since h is convex, i.e. $\alpha h(x_1) + \beta h(x_2) \ge h(\alpha x_1 + \beta x_2)$, and h^{-1} is strictly decreasing, we have

$$h^{-1}(\alpha y_1 + \beta y_2) \le h^{-1}(h(\alpha x_1 + \beta x_2)) = \alpha h^{-1}(y_1) + \beta h^{-1}(y_2).$$

From this it follows that h^{-1} is a convex function on $[c,d]$. □

Lemma 2.5. *Let* $\varphi : [a,b] \to [c,d]$ *and* $h : [c,d] \to \mathbb{R}$.

(i) *If* φ *is convex on* $[a,b]$ *and* h *is convex increasing on* $[c,d]$, *or* φ *is concave on* $[a,b]$ *and* h *is convex decreasing on* $[c,d]$, *then* $h(\varphi(x))$ *is convex on* $[c,d]$.

(ii) *If* φ *is convex on* $[a,b]$ *and* h *is concave decreasing on* $[c,d]$, *or* φ *is concave on* $[a,b]$ *and* h *is concave increasing on* $[c,d]$, *then* $h(\varphi(x))$ *is concave on* $[c,d]$.

Proof. Let us suppose that φ is concave on $[a,b]$ and h is convex decreasing on $[c,d]$. Taking $x_1, x_2 \in [a,b]$ and $\alpha, \beta \in [0,1] : \alpha + \beta = 1$, we get

$$h(\varphi(\alpha x_1 + \beta x_2)) \le h(\alpha\varphi(x_1) + \beta\varphi(x_2)) \le \alpha h(\varphi(x_1)) + \beta h(\varphi(x_2)),$$

i.e. $h(\varphi(x))$ is a convex function on $[c,d]$. Proofs of the remaining parts are similar. □

3. A GENERALIZATION OF FEJÉR'S RESULT

By the use of Jensen's inequality we obtain the following result involving $M_{[a,b],g}(p,f)$. In what follows g is always a real continuous monotone function on the range of f (in accordance with Definition 2.1).

Theorem 3.1. *Let* $(p, f) \in L_1^+([a,b]) \times L_\infty^+([a,b])$ *and* $\operatorname{Im}(f) = [c,d]$, $-\infty < c < d < \infty$.

(i) *If* $g : [c, d] \to \mathbb{R}$ *is convex increasing or concave decreasing, and* f *is concave, then*

$$f(a)(1-\alpha^*) + f(b)\alpha^* \le M_{[a,b],g}(p, f),$$

where

$$\alpha^* = A_{[a,b]}(p, \alpha), \quad \textit{for} \quad \alpha(x) = \frac{x-a}{b-a}. \tag{3.1}$$

(ii) *If* $g : [c, d] \to \mathbb{R}$ *is concave increasing or convex decreasing, and* f *is convex, then*

$$M_{[a,b],g}(p, f) \le f(a)(1-\alpha^*) + f(b)\alpha^*.$$

Proof. The presented inequalities are intuitively obvious from the geometric meaning of convexity. Let g be a concave increasing and f be a convex function. From Corollary 2.3(ii) we have

$$M_{[a,b],g}(p, f) \le A_{[a,b]}(p, f).$$

Putting

$$\alpha(x) = \frac{x-a}{b-a},$$

we get $x = (1-\alpha(x))a + \alpha(x)b$, for all $x \in [a, b]$. From the convexity of function f we have

$$f\Big((1-\alpha(x))a + \alpha(x)b\Big) \le (1-\alpha(x))f(a) + \alpha(x)f(b),$$

and therefore

$$\begin{aligned} M_{[a,b],g}(p, f) &\le \frac{1}{\|p\|_{[a,b]}} \int_a^b p(x)\Big((1-\alpha(x))f(a) + \alpha(x)f(b)\Big)\, dx \\ &= f(a)\frac{\int_a^b p(x)(1-\alpha(x))\, dx}{\|p\|_{[a,b]}} + f(b)\frac{\int_a^b p(x)\alpha(x)\, dx}{\|p\|_{[a,b]}}. \end{aligned}$$

Using (3.1) the above inequality may be rewritten into

$$M_{[a,b],g}(p, f) \le f(a)(1-\alpha^*) + f(b)\alpha^*.$$

Remaining parts may be proved analogously. □

Remark 3.2. If p is symmetric with respect to the center of the interval $[a, b]$, i.e.

$$p(a+t) = p(b-t), \qquad 0 \le t \le \frac{b-a}{2},$$

then $\alpha^* = 1/2$. It then follows that items (i) and (ii) reduce to

$$\frac{f(a)+f(b)}{2} \le M_{[a,b],g}(p, f), \quad \text{and} \quad M_{[a,b],g}(p, f) \le \frac{f(a)+f(b)}{2},$$

respectively.

The Fejér inequality (1.2) immediately yields the following version of the generalized weighted Hadamard inequality for means $M_{[a,b],g}(p, f)$.

Theorem 3.3. *Let* $(p, f) \in L_1^+([a, b]) \times L_\infty^+([a, b])$ *such that* p *is symmetric with respect to the center of the interval* $[a, b]$. *Let* $\mathrm{Im}(f) = [c, d]$, $-\infty < c < d < \infty$, *and* $g : [c, d] \to \mathbb{R}$.

(i) *If* g *is convex increasing or concave decreasing and* f *is convex, then*

$$f\left(\frac{a+b}{2}\right) \le M_{[a,b],g}(p, f) \le g^{-1}\left(\frac{g(f(a)) + g(f(b))}{2}\right). \tag{3.2}$$

(ii) *If g is concave increasing or convex decreasing and f is concave, then*

$$g^{-1}\left(\frac{g(f(a))+g(f(b))}{2}\right) \le M_{[a,b],g}(p,f) \le f\left(\frac{a+b}{2}\right).$$

Proof. We will prove the item (i). The item (ii) may be proved analogously.

Since g is increasing (decreasing), then assumption (i) of Theorem 3.3 and Lemma 2.5 yield that $h = g \circ f$ is convex (concave). Applying (1.2) for h, then applying the inverse of g to (1.2), the inequalities in (3.2) immediately follow. □

Corollary 3.4. *Let us suppose the functions $p(x) = 1$ and $g(x) = x$. If f is a convex function on $[a, b]$, then we get the celebrated Hadamard inequality (1.1).*

Using our approach from the proof of Theorem 3.1 we are able to prove the following theorem which corresponds to some conversions of the Jensen inequality for convex functions in the case of $M_{[a,b],g}(p, f)$.

Theorem 3.5. *Let $(p, f) \in L_1^+([a, b]) \times L_\infty^+([a, b])$, such that $f : [a, b] \to [k, K]$, and $g : [k, K] \to \mathbb{R}$, where $-\infty < k < K < \infty$.*

(i) *If g is convex on $[k, K]$, then*

$$A_{[a,b]}(p, g\circ f) \le \frac{g(k)\Big(K - A_{[a,b]}(p,f)\Big)}{K-k} + \frac{g(K)\Big(A_{[a,b]}(p,f) - k\Big)}{K-k}.$$

(ii) *If g is concave on $[k, K]$, then*

$$A_{[a,b]}(p, g\circ f) \ge \frac{g(k)\Big(K - A_{[a,b]}(p,f)\Big)}{K-k} + \frac{g(K)\Big(A_{[a,b]}(p,f) - k\Big)}{K-k}.$$

Proof. Let us prove the item (i). Suppose that g is a convex function on the interval $[k, K]$. Let us consider the following integral

$$\int_a^b p(x)g(f(x))\,dx.$$

Since $k \le f(x) \le K$ for all $x \in [a, b]$ and $f(x) = (1 - \alpha_f(x))k + \alpha_f(x)K$, where

$$\alpha_f(x) = \frac{f(x)-k}{K-k}, \tag{3.3}$$

then

$$\begin{aligned}\int_a^b p(x)g(f(x))\,dx &\le \int_a^b p(x)\Big((1-\alpha_f(x))g(k) + \alpha_f(x)g(K)\Big)\,dx \\ &= g(k)\int_a^b p(x)\Big(1-\alpha_f(x)\Big)\,dx + g(K)\int_a^b p(x)\alpha_f(x)\,dx.\end{aligned}$$

By (3.3) we get

$$\int_a^b p(x)\alpha_f(x)\,dx = \frac{1}{K-k}\left(\int_a^b p(x)f(x)\,dx - k\,\|p\|_{[a,b]}\right)$$

and therefore

$$\begin{aligned}\int_a^b p(x)g(f(x))\,dx \le \frac{g(k)}{K-k}&\left(K\,\|p\|_{[a,b]} - \int_a^b p(x)f(x)\,dx\right) \\ &+ \frac{g(K)}{K-k}\left(\int_a^b p(x)f(x)\,dx - k\,\|p\|_{[a,b]}\right).\end{aligned}$$

Since $\|p\|_{[a,b]}$ is positive and finite, we may write

$$\begin{aligned} A_{[a,b]}(p, g\circ f) &\le \frac{g(k)\left(K - \frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)f(x)\,dx\right)}{K-k} \\ &\qquad + \frac{g(K)\left(\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)f(x)\,dx - k\right)}{K-k} \\ &= \frac{g(k)\left(K - A_{[a,b]}(p,f)\right)}{K-k} + \frac{g(K)\left(A_{[a,b]}(p,f)-k\right)}{K-k}. \end{aligned}$$

Hence the result. Item (ii) may be proved analogously. □

4. HADAMARD TYPE INEQUALITY FOR THE PRODUCT OF TWO FUNCTIONS

The main result of this section consists in generalization of a result for two convex functions given in [9]. Observe that symmetry of a weight function p on the interval $[a,b]$ is now not necessarily required. Our approach is based on using of a fairly elementary analysis.

Theorem 4.1. *Let $p \in L_1^+([a,b])$ and h, k be two real-valued nonnegative and integrable functions on $[a,b]$. Let g be a real continuous monotone function defined on the range of hk.*

(i) *If h, k are convex and g is either convex increasing, or concave decreasing, then*

$$\begin{aligned} M_{[a,b],g}(p,hk) \le g^{-1}\Bigg[&(1-2\alpha^*+\beta^*)g\Big(h(a)k(a)\Big) + (\alpha^*-\beta^*) \\ &\times\left(g\Big(h(a)k(b)\Big) + g\Big(h(b)k(a)\Big)\right) + \beta^* g\Big(h(b)k(b)\Big)\Bigg]. \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} M_{[a,b],g}(p,hk) \ge g^{-1}\Bigg[&2g\left(h\left(\frac{a+b}{2}\right)k\left(\frac{a+b}{2}\right)\right) + (\beta^*-\alpha^*) \\ &\times\Big(g\big(h(a)k(a)\big) + g\big(h(b)k(b)\big)\Big) + \left(\alpha^*-\beta^*-\frac{1}{2}\right) \\ &\times\left(g\Big(h(a)k(b)\Big) + g\Big(h(b)k(a)\Big)\right)\Bigg], \end{aligned} \tag{4.2}$$

where

$$\alpha^* = A_{[a,b]}(p,\alpha), \quad \beta^* = A_{[a,b]}(p,\alpha^2) \quad \text{and} \quad \alpha(x) = \frac{x-a}{b-a}. \tag{4.3}$$

(ii) *If h, k are convex and g is either concave increasing, or convex decreasing, then the above inequalities (4.1) and (4.2) are in the reversed order.*

Proof. We will prove only the item (i). The proof of the item (ii) is very similar.
Suppose that g is a convex increasing function and h, k are convex functions on $[a,b]$. Therefore for $t\in[0,1]$, we have

$$h\Big((1-t)a+tb\Big) \le (1-t)h(a) + th(b) \tag{4.4}$$

and

$$k\Big((1-t)a+tb\Big) \le (1-t)k(a) + tk(b). \tag{4.5}$$

From (4.4) and (4.5) we obtain

$$h\Big((1-t)a+tb\Big)k\Big((1-t)a+tb\Big) \\ \le (1-t)^2h(a)k(a)+t(1-t)h(a)k(b)+t(1-t)h(b)k(a)+t^2h(b)k(b).$$

By the Lemma 2.1 the functions $h((1-t)a+tb)$ and $k((1-t)a+tb)$ are convex on the interval $[0,1]$ and therefore they are integrable on $[0,1]$. Consequently the function $h((1-t)a+tb)k((1-t)a+tb)$ is also integrable on $[0,1]$. Similarly since h and k are convex on the interval $[a,b]$, they are integrable on $[a,b]$ and hence hk is also integrable function on $[a,b]$.

Since g is increasing and convex on the range of hk, by applying Jensen's inequality we get

$$\text{(4.6)}\quad g\Big(h\Big((1-t)a+tb\Big)k\Big((1-t)a+tb\Big)\Big) \\ \le (1-t)^2g\Big(h(a)k(a)\Big)+t(1-t)\left(g\Big(h(a)k(b)\Big)+g\Big(h(b)k(a)\Big)\right)+t^2g\Big(h(b)k(b)\Big).$$

Multiplying both sides of the equation (4.6) by $p\Big((1-t)a+tb\Big)/\|p\|_{[a,b]}$ and integrating over the interval $[0,1]$, we have

$$\frac{1}{\|p\|_{[a,b]}}\int_0^1 p\Big((1-t)a+tb\Big)g\Big[h\Big((1-t)a+tb\Big)k\Big((1-t)a+tb\Big)\Big]\,dt \\ \le \frac{1}{\|p\|_{[a,b]}}\,g\Big(h(a)k(a)\Big)\int_0^1 p\Big((1-t)a+tb\Big)(1-t)^2\,dt \\ +\frac{1}{\|p\|_{[a,b]}}\left(g\Big(h(a)k(b)\Big)+g\Big(h(b)k(a)\Big)\right)\int_0^1 p\Big((1-t)a+tb\Big)t(1-t)\,dt \\ +\frac{1}{\|p\|_{[a,b]}}\,g\Big(h(b)k(b)\Big)\int_0^1 p\Big((1-t)a+tb\Big)t^2\,dt.$$

Substituting $(1-t)a+tb=x$ and putting $\alpha(x)=\frac{x-a}{b-a}$ we obtain

$$\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)g\Big(h(x)k(x)\Big)\,dx \le \frac{1}{\|p\|_{[a,b]}}\,g\Big(h(a)k(a)\Big)\int_a^b p(x)(1-\alpha(x))^2\,dx \\ +\frac{1}{\|p\|_{[a,b]}}\left(g\Big(h(a)k(b)\Big)+g\Big(h(b)k(a)\Big)\right)\int_a^b p(x)\alpha(x)(1-\alpha(x))\,dx \\ +\frac{1}{\|p\|_{[a,b]}}\,g\Big(h(b)k(b)\Big)\int_a^b p(x)\alpha^2(x)\,dx.$$

Using notation (4.3) we obtain

$$\frac{1}{\|p\|_{[a,b]}}\int_a^b p(x)g\Big(h(x)k(x)\Big)\,dx \le (1-2\alpha^*+\beta^*)g\Big(h(a)k(a)\Big) \\ +\beta^*g\Big(h(b)k(b)\Big)+(\alpha^*-\beta^*)\left(g\Big(h(a)k(b)\Big)+g\Big(h(b)k(a)\Big)\right).$$

Since g^{-1} is increasing, we get the desired inequality in (4.1).

Now let us show the inequality in (4.2). Since h and k are convex on $[a,b]$, then for $t \in [a,b]$ we observe that

$$\begin{aligned} & h\left(\frac{a+b}{2}\right) k\left(\frac{a+b}{2}\right) \\ & = h\left(\frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}\right) k\left(\frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}\right) \\ & \le \frac{1}{4}\left[h\Big((1-t)a+tb\Big) + h\Big(ta+(1-t)b\Big)\right]\left[k\Big((1-t)a+tb\Big) + k\Big(ta+(1-t)b\Big)\right] \\ & \quad + \frac{1}{4}\Big[2t(1-t)\Big(h(a)k(a)+h(b)k(b)\Big) + \Big(t^2+(1-t)^2\Big)\Big(h(a)k(b)+h(b)k(a)\Big)\Big]. \end{aligned}$$

Since g is increasing and convex, by the use of Jensen's inequality we obtain

$$\begin{aligned} & g\left(h\left(\frac{a+b}{2}\right) k\left(\frac{a+b}{2}\right)\right) \\ & \qquad \le \frac{1}{4}\, g\left(h\Big((1-t)a+tb\Big) k\Big((1-t)a+tb\Big)\right) \\ & \qquad\quad + \frac{1}{4}\, g\left(h\Big(ta+(1-t)b\Big) k\Big(ta+(1-t)b\Big)\right) \\ & \qquad\quad + \frac{1}{2}\, t(1-t)\Big[g\Big(h(a)k(a)\Big) + g\Big(h(b)k(b)\Big)\Big] \\ & \qquad\quad + \frac{1}{2}\left(t^2 - t + \frac{1}{2}\right)\Big[g\Big(h(a)k(b)\Big) + g\Big(h(b)k(a)\Big)\Big]. \end{aligned}$$

Multiplying both sides of the last inequality by $p\Big((1-t)a+tb\Big)/\|p\|_{[a,b]}$ and integrating over the interval $[0,1]$, we have

$$\begin{aligned} & \frac{2}{\|p\|_{[a,b]}} \int_0^1 p\Big((1-t)a+tb\Big) g\left(h\left(\frac{a+b}{2}\right) k\left(\frac{a+b}{2}\right)\right) dt \\ & \quad \le \frac{1}{\|p\|_{[a,b]}} \int_0^1 p\Big((1-t)a+tb\Big) g\left(h\Big((1-t)a+tb\Big) k\Big((1-t)a+tb\Big)\right) dt \\ & \quad\quad + \frac{1}{\|p\|_{[a,b]}} \int_0^1 p\Big((1-t)a+tb\Big) g\left(h\Big(ta+(1-t)b\Big) k\Big(ta+(1-t)b\Big)\right) dt \\ & \quad\quad + \frac{g\Big(h(a)k(a)\Big) + g\Big(h(b)k(b)\Big)}{\|p\|_{[a,b]}} \int_0^1 p\Big((1-t)a+tb\Big) t(1-t)\, dt \\ & \quad\quad + \frac{g\Big(h(a)k(b)\Big) + g\Big(h(b)k(a)\Big)}{\|p\|_{[a,b]}} \int_0^1 p\Big((1-t)a+tb\Big)\left(t^2 - t + \frac{1}{2}\right) dt. \end{aligned}$$

Substituting $(1-t)a+tb = x$ and using notation (4.3), we obtain

$$\begin{aligned} 2g\left(h\left(\frac{a+b}{2}\right) k\left(\frac{a+b}{2}\right)\right) & \le \frac{1}{\|p\|_{[a,b]}} \int_a^b p(x) g\Big(h(x)k(x)\Big)\, dx \\ & \quad + (\alpha^* - \beta^*)\Big(g\Big(h(a)k(a)\Big) + g\Big(h(b)k(b)\Big)\Big) \\ & \quad + \left(\beta^* - \alpha^* + \frac{1}{2}\right)\Big(g\Big(h(a)k(b)\Big) + g\Big(h(b)k(a)\Big)\Big) \end{aligned}$$

Since g^{-1} is increasing, we complete the proof. $\square$

Remark 4.2. If p is a symmetric function with respect to the center of the interval $[a, b]$, then $\alpha^* = 1/2$ and $\beta^* = 1/3$.

As a consequence of Theorem 4.1 we obtain the following main result stated in [9].

Corollary 4.3. *Let us consider $g(x) = x$ and $p(x) \equiv 1$ on $[a, b]$. If h, k are two real-valued nonnegative convex functions on $[a, b]$, then*

$$2h\left(\frac{a+b}{2}\right)k\left(\frac{a+b}{2}\right) - \frac{1}{6}M(a,b) - \frac{1}{3}N(a,b) \leq \frac{1}{b-a}\int_a^b h(x)k(x)\,dx,$$

and

$$\frac{1}{b-a}\int_a^b h(x)k(x)\,dx \leq \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b),$$

where $M(a,b) = h(a)k(a) + h(b)k(b)$ and $N(a,b) = h(a)k(b) + h(b)k(a)$.

Proof. Since p is symmetric on $[a, b]$, then the result follows immediately from Theorem 4.1 (i) and Remark 4.2. $\square$

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