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# ON THE DETERMINANTAL INEQUALITIES

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## Abstract

In this paper, we discuss the determinantal inequalities over arbitrary complex matrices, and give some sufficient conditions for

$$d[A+B]^t \geq d[A]^t + d[B]^t,$$

where $t \in \mathrm{R}$ and $t \geq \frac{2}{n}$. If $B$ is nonsingular and $\operatorname{Re}\lambda(B^{-1}A) \geq 0$, the sufficient and necessary condition is given for the above equality at $t = \frac{2}{n}$. The famous Minkowski inequality and many recent results about determinantal inequalities are extended.

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## Contents

# 1. Preliminaries

We use conventional notions and notations, as in [2]. Let $A \in M_n(C)$, $d[A]$ stands for the modulus of $\det(A)$ (or $|A|$), where $\det(A)$ is the determinant of $A$. $\sigma(A)$ is the spectrum of $A$, namely the set of eigenvalues of matrix $A$. A matrix $X \in M_n(C)$ is called complex (semi-) positive definite if $\mathrm{Re}(x^* Ax) > 0$ ($\mathrm{Re}(x^* Ax) \geq 0$) for all nonzero $x \in C^n$ or if $\frac{1}{2}(X + X^*)$ is a complex (semi-)positive definite matrix (see [4, 7, 8, 2]). Throughout this paper, we denote $C = B^{-1}A$ for $A, B \in M_n(C)$ and $B$ is invertible.

The famous Minkowski inequality states:

If $A, B \in M_n(R)$ are real positive definite symmetric matrices, then

$$|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}. \tag{1.1}$$

It is a very interesting work to generalize the Minkowski inequality. Obviously, (1.1) holds if $A, B \in M_n(C)$ are positive definite Hermitian matrices. Recently, (1.1) has been generalized for $A, B \in M_n(C)$ positive definite matrices (see [8], [9], [10], [3]).

In this paper, we discuss determinantal inequalities over arbitrary complex matrices, and give some sufficient conditions for

$$d[A + B]^t \geq d[A]^t + d[B]^t, \tag{1.2}$$

where $t \in \mathrm{R}$.

If $B$ is nonsingular and $\mathrm{Re}\,\lambda(B^{-1}A) \geq 0$, a sufficient and necessary condition has been given for equality as $t = \frac{2}{n}$ in (1.2). The famous Minkowski inequality and many results about determinantal inequalities are extended.

For $c \in C$, $\mathrm{Re}(c)$ denotes the real part of $c$ and $|c|$ denotes the modulus of $c$. Let $t > 0$ be fixed, we have

**Lemma 1.1.** *If $A, B \in M_n(C)$ and $B$ is invertible, $\sigma(C) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, then inequality (1.2) is true if and only if*

$$\prod_{i=1}^{n} |\lambda_i + 1|^t \geq \prod_{i=1}^{n} |\lambda_i|^t + 1, \tag{1.3}$$

*with equality holding in (1.2) if and only if it holds in (1.3).*

*Proof.* Since $d[A+B]^t = d[B]^t d[C+I]^t$ and $d[A]^t + d[B]^t = d[B]^t(1+d[C]^t)$, formula (1.2) is equivalent to

$$d[C+I]^t \geq 1 + d[C]^t. \tag{1.4}$$

Notice $\sigma(C+I) = \{\lambda_k + 1 : k = 1, 2, \dots, n\}$,

$$d[C+I]^t = \prod_{i=1}^{n} |\lambda_i + 1|^t \quad \text{and} \quad d[C]^t = \prod_{i=1}^{n} |\lambda_i|^t,$$

we obtain that formula (1.4) is equivalent to (1.3). Similarly, it is easy to see that the case of equality is true. Thus the lemma is proved. □

**Lemma 1.2 (see [6]).** *If $x_t, y_t \geq 0$ $(t = 1, 2, \dots, n)$, then*

$$\prod_{t=1}^{n} (x_t + y_t)^{\frac{1}{n}} \geq \prod_{t=1}^{n} x_t^{\frac{1}{n}} + \prod_{t=1}^{n} y_t^{\frac{1}{n}},$$

*with equality if and only if there is linear dependence between $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ or $x_t + y_t = 0$ for a certain number $t$.*

**Lemma 1.3 (Jensen's inequality).** *If $a_1$, $a_2$,..., $a_m$ are positive numbers, then*

$$\left(\sum_{i=1}^{n} a_i^s\right)^{\frac{1}{s}} \leq \left(\sum_{i=1}^{n} a_i^r\right)^{\frac{1}{r}} \quad \text{for} \quad 0 < r \leq s,\ n \geq 2.$$

**Lemma 1.4.** *If $P_1$, $P_2$,..., $P_m$ are positive numbers and $T \geq \frac{1}{m}$, then*

$$\prod_{k=1}^{m} (P_k + 1)^T \geq \prod_{k=1}^{m} P_k^T + 1, \tag{1.5}$$

*with equality if and only if $P_k$ $(k = 1, 2, \ldots, n)$ is constant as $T = \frac{1}{m}$.*

*Proof.* By Lemma 1.2, we have

$$\prod_{k=1}^{m} (P_k + 1)^T = \left[\prod_{k=1}^{m} (P_k + 1)^{\frac{1}{m}}\right]^{mT} \geq \left[\prod_{k=1}^{m} \left(P_k^T\right)^{\frac{1}{mT}} + 1\right]^{mT}.$$

On noting that $0 < \frac{1}{mT} \leq 1$, by Lemma 1.3, we obtain

$$\left[\prod_{k=1}^{m} \left(P_k^T\right)^{\frac{1}{mT}} + 1\right]^{mT} \geq \prod_{k=1}^{m} P_k^T + 1,$$

and inequality (1.5) is demonstrated. By Lemma 1.2, it is easy to see that equality holds if and only if $P_k$ $(k = 1, 2, \ldots, n)$ is constant as $T = \frac{1}{m}$. □

**Remark 1.** *Apparently, Lemma 1.3 is tenable for $a_i \geq 0$ $(i = 1, 2, \ldots, n)$, and Lemma 1.4 is tenable for $P_i \geq 0$ $(i = 1, 2, \ldots, n)$.*

## 2. Main Results

**Theorem 2.1.** *Let* $A$, $B \in M_n(C)$. *If* $B$ *is nonsingular and* $\operatorname{Re}\lambda_k \geq 0$ $(k = 1, 2, \ldots, n)$, *where* $\sigma(C) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, *then for* $t \geq \frac{2}{n}$

$$d[A+B]^t \geq d[A]^t + d[B]^t, \tag{2.1}$$

*Proof.* By Lemma 1.1, we need to prove inequality (1.3). Note that $\operatorname{Re}\lambda_k \geq 0$ $(k = 1, 2, \ldots, n)$ and $|\lambda_k + 1|^2 \geq 1 + |\lambda_k|^2$,

$$\prod_{k=1}^{n} |\lambda_k + 1|^t = \left(\prod_{k=1}^{n} |\lambda_k + 1|^2\right)^{\frac{t}{2}} \geq \prod_{k=1}^{n} \left(|\lambda_k|^2 + 1\right)^{\frac{t}{2}}.$$

Applying Lemma 1.4, we can show that

$$\prod_{k=1}^{n} \left(|\lambda_k|^2 + 1\right)^{\frac{t}{2}} \geq \prod_{k=1}^{n} |\lambda_k|^t + 1 \quad \text{for } t \geq \frac{2}{n},$$

with equality if and only if $|\lambda_k|^2$ $(k = 1, 2, \ldots, n)$ is constant as $t = \frac{2}{n}$. The above two inequalities imply formula (1.3). □

When $t = 1$, we have

**Corollary 2.2.** *Let* $A$, $B \in M_n(C)$ $(n \geq 2)$. *If* $B$ *is invertible and* $\operatorname{Re}\lambda_k \geq 0$ $(k = 1, 2, \ldots, n)$, *where* $\sigma(C) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, *then*

$$d[A+B] \geq d[A] + d[B]. \tag{2.2}$$

**Corollary 2.3.** *Let $A$ be an $n$-by-$n$ complex positive definite matrix, and $B$ be an $n$-by-$n$ positive definite Hermitian matrix $(n \geq 2)$. Then for $t \geq \frac{2}{n}$*

$$d[A+B]^t \geq d[A]^t + [\det(B)]^t. \tag{2.3}$$

*Proof.* Observing $C = B^{-1}A$ is similar to $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and $\operatorname{Re}\lambda(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) > 0$, where $\lambda(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$ is an arbitrary eigenvalue of $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$. Therefore, $\operatorname{Re}\lambda_k \geq 0$ and $\sigma(C) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Hence, Theorem 2.1 yields Corollary 2.3. □

When $t = \frac{2}{n}$, inequality (2.3) gives Theorem 4 of [3]. When $t = 1$, inequality (2.3) gives Theorem 1 of [3]. To merit attention, Theorem 2 in [8] proves that if $A$ is real positive definite and $B$ is real positive definite symmetric, then (2.3) holds for $t = \frac{1}{n}$. It is untenable for example: $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Corollary 2.7 and Corollary 2.8 in this paper have been given correction.

**Theorem 2.4.** *Let $A, B \in M_n(C)$. If $B$ is nonsingular, and $\operatorname{Re}\lambda_k \geq 0$ $(k = 1, 2, \ldots, n)$, where $\sigma(C) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, then $n$ eigenvalues of $C$ are pure imaginary complex numbers with the same modulus if and only if*

$$d[A+B]^{\frac{2}{n}} = d[A]^{\frac{2}{n}} + d[B]^{\frac{2}{n}}, \tag{2.4}$$

*Proof.* If $n$ eigenvalues of $C$ are $\pm id$ $(i = \sqrt{-1}, d > o, d \in R)$, then

$$\prod_{i=1}^{n} |\lambda_i + 1|^{\frac{2}{n}} = \prod_{i=1}^{n} (1 + d^2)^{\frac{1}{n}} = 1 + d^2 = \prod_{i=1}^{n} |\lambda_i|^{\frac{2}{n}} + 1.$$

Hence equality (2.4) holds by Lemma 1.1.

Conversely, suppose (2.4) holds, then

$$\prod_{i=1}^{n} |\lambda_i + 1|^{\frac{2}{n}} = \prod_{i=1}^{n} |\lambda_i|^{\frac{2}{n}} + 1.$$

So

$$\prod_{i=1}^{n} (1 + 2\operatorname{Re}\lambda_i + |\lambda_i|^2)^{\frac{1}{n}} = \prod_{i=1}^{n} (|\lambda_i|^2)^{\frac{1}{n}} + 1.$$

Obviously, $\operatorname{Re}\lambda_k = 0$ $(k = 1, 2, \ldots, n)$, otherwise

$$\prod_{i=1}^{n} (1 + 2\operatorname{Re}\lambda_i + |\lambda_i|^2)^{\frac{1}{n}} > \prod_{i=1}^{n} \left(1 + |\lambda_i|^2\right)^{\frac{1}{n}} \geq \prod_{i=1}^{n} (|\lambda_i|^2)^{\frac{1}{n}} + 1,$$

with illogicality. Therefore

$$\prod_{i=1}^{n} \left[1 + (\operatorname{Im}\lambda_i)^2\right]^{\frac{1}{n}} = \prod_{i=1}^{n} \left[(\operatorname{Im}\lambda_i)^2\right]^{\frac{1}{n}} + 1.$$

By Lemma 1.2 we obtain $(\operatorname{Im}\lambda_k)^2 = d^2$ and $\lambda_k = \pm id$ $(k = 1, 2, \ldots, n)$. This completes the proof. □

**Corollary 2.5.** *If* $A$,$B \in M_n(C)$ *with* $B$ *is nonsingular and* $C = B^{-1}A$ *is skew–Hermitian, then formula* (2.4) *holds if and only if* $A = idBUEU^*$, *where* $i^2 = -1$, $d > 0$, $U$ *is a unitary matrix,* $E = \operatorname{diag}(e_1, e_2, \ldots, e_n)$ *with* $e_i = \pm 1$, $i = 1, 2, \ldots, n$.

*Proof.* Since $C$ is skew–Hermitian and its real parts of $n$ eigenvalues are zero, then Theorem 2.4 implies that (2.4) holds if and only if

$$C = B^{-1}A = U\,\mathrm{diag}(\pm id, \pm id, \dots, \pm id)U^{*},$$

where $\sigma(C) = \{\pm id, \pm id, \dots, \pm id\}$, $d > 0$ and $U$ is unitary. Hence $A = idBUEU^{*}$, where $i^2 = -1$, $d > 0$, $U$ is a unitary matrix, $E = \mathrm{diag}(e_1, e_2, \dots, e_n)$ and $e_i = \pm 1$, $i = 1, 2, \dots, n$. □

**Theorem 2.6.** *Suppose $A$, $B \in M_n(C)$ with $B$ nonsingular and $\mathrm{Re}\lambda_k \geq 0$ $(k = 1, 2, \dots, n)$, where $\sigma(C) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$. If the number of the real eigenvalues of $C$ is $r$, and the non-real eigenvalues of $C$ are pair wise conjugate, then inequality (1.2) holds for $t \geq \frac{2}{n+r}$.*

*Proof.* By Lemma 1.1, we need to prove (1.3) for $t \geq \frac{2}{n+r}$. Without loss of generality, suppose $\lambda_j \geq 0$ $(j = 1, 2, \dots, r)$ are the real eigenvalues of $C$ and $\lambda_k$, $\overline{\lambda_k}$ $(k = r+1, r+2, \dots, r+s)$ are $s$ pairs of non-real eigenvalues of $C$, where $n = r + 2s$. Then the right-hand side of (1.3) becomes

$$\prod_{i=1}^{r} \lambda_i^t \prod_{j=r+1}^{r+s} \left(|\lambda_j|^2\right)^{t} + 1, \tag{2.5}$$

and the left-hand side of (1.3) is

$$\prod_{i=1}^{r} (\lambda_i + 1)^t \prod_{j=r+1}^{r+s} \left(|1 + \lambda_j|^2\right)^{t}. \tag{2.6}$$

Given $\mathrm{Re}\,\lambda_k \geq 0$ $(k = 1, 2, \dots, r+s)$, so $|1+\lambda_j|^2 \geq 1 + |\lambda_j|^2$, then

$$\prod_{i=1}^{r}(1+\lambda_i)^t \prod_{j=r+1}^{r+s}\left(|1+\lambda_j|^2\right)^t \geq \prod_{i=1}^{r}(1+\lambda_i)^t \prod_{j=r+1}^{r+s}\left(1+|\lambda_j|^2\right)^t. \tag{2.7}$$

By Lemma 1.2 and (2.7), we obtain that

$$\prod_{i=1}^{r}(\lambda_i+1)^t \prod_{j=r+1}^{r+s}\left(|1+\lambda_j|^2\right)^t \geq \prod_{i=1}^{r}\lambda_i^t \prod_{j=r+1}^{r+s}\left(|\lambda_j|^2\right)^t + 1,$$

$$\text{for} \quad \frac{1}{r+s} = \frac{2}{n+r}.$$

This completes the proof. □

In the following, we present some generalizations of the Minkowski inequality. By Theorem 2.6, it is easy to show:

**Corollary 2.7.** *Let $A, B \in M_n(C)$. If $B$ is nonsingular and $n$ eigenvalues of $C$ are positive numbers, then for $t \geq \frac{1}{n}$*

$$d[A+B]^{\frac{1}{n}} \geq d[A]^{\frac{1}{n}} + d[B]^{\frac{1}{n}}. \tag{2.8}$$

If $A$ is an $n$-by-$n$ complex positive definite matrix and $B$ is an $n$-by-$n$ positive definite Hermitian matrix, with $n$ eigenvalues of $C$ being real numbers, then $\sigma(C) = \sigma(B^{\frac{1}{2}}CB^{-\frac{1}{2}})$, and $B^{\frac{1}{2}}CB^{-\frac{1}{2}} = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ is positive definite, so any eigenvalue of $C$ has a positive real part. Thus $n$ eigenvalues of $C$ are positive numbers. By Corollary 2.7 we have

**Corollary 2.8.** *Suppose $A, B \in M_n(C)$, where $A$ is a complex positive definite matrix and $B$ is a positive definite Hermitian matrix. If $n$ eigenvalues of $C$ are real numbers, then inequality (2.8) holds for $t \geq \frac{1}{n}$.*

**Corollary 2.9 (Minkowski inequality).** *Suppose $A, B \in M_n(C)$ are positive definite Hermitian matrices, then inequality (1.1) holds.*

*Proof.* Note that $C = B^{-1}A$ is similar to a real diagonal matrix, and its eigenvalues are real numbers, using Corollary 2.8 and letting $t = 1$, the proof is completed. □

**Corollary 2.10.** *Suppose $A, B \in M_n(C)$, where $A$ is a complex positive definite matrix and $B$ is a positive definite Hermitian matrix. If the non-real eigenvalues of $C$ are $m$ pairs conjugate complex numbers, then inequality (1.2) holds for $t \geq \frac{1}{n-m}$.*

*Proof.* Obviously $\operatorname{Re}\lambda_k \geq 0$ $(k = 1, 2, \ldots, n)$, where $\sigma(C) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Applying Theorem 2.6 completes the proof. □

Let $A = H + K \in M_n(C)$, where $H = \frac{1}{2}(A + A^*)$, and $K = \frac{1}{2}(A - A^*)$, then we have

**Theorem 2.11.** *Let $A = H + K$ be an $n$-by-$n$ complex positive definite matrix, then for $t \geq \frac{2}{n}$*

$$d[A]^t \geq d[H]^t + d[K]^t, \tag{2.9}$$

*with equality if and only if $K = idHQ^*EQ$ as $t = \frac{2}{n}$, where $i^2 = -1$, $d > 0$, $Q$ is a unitary matrix,$E = \operatorname{diag}(e_1, e_2, \ldots, e_n)$ with $e_i = \pm 1$, $i = 1, 2, \ldots, n$.*

*Proof.* Since $H^{-\frac{1}{2}}KH^{-\frac{1}{2}}$ is a skew-Hermitian matrix and is similar to $H^{-1}K$, $\operatorname{Re}\lambda(H^{-1}K) = \operatorname{Re}\lambda(H^{-\frac{1}{2}}KH^{-\frac{1}{2}}) = 0$. By Theorem 2.1 and Corollary 2.5, we get the desired result. □

Let $t = 1$, we have the following interesting result.

**Corollary 2.12.** *If $A = H + K$ is an $n$-by-$n$ complex positive definite matrix $(n \geq 2)$, then*

$$d[A] \geq d[H] + d[K]. \tag{2.10}$$

**Corollary 2.13 (Ostrowski-Taussky Inequality).** *If $A = H + K$ is an $n$-by-$n$ positive definite matrix $(n \geq 2)$, then $\det H \leq d[A]$ with equality if and only if $A$ is Hermitian.*

**Theorem 2.14.** *Let $A, B$ be two $n$-by-$n$ complex positive definite matrices, and $n$ eigenvalues of $B$ be real numbers. Suppose $A, B$ are simultaneously upper triangularizable, namely, there exists a nonsingular matrix $P$, such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular matrices, then inequality (1.2) holds for any $t \geq \frac{2}{n}$.*

*Proof.* If $P^{-1}AP$ and $P^{-1}BP$ are upper triangular matrices, then

$$P^{-1}B^{-1}AP = (P^{-1}BP)^{-1}(P^{-1}AP)$$

is an upper triangular matrix, with the product of the eigenvalues of $B^{-1}$ and $A$ on its diagonal. We denote the eigenvalue of $X$ by $\lambda(X)$. Notice that positive definiteness of $A$ and $B^{-1}$, $\operatorname{Re}\lambda(A)$ and $\lambda(B^{-1})$ are positive numbers by hypothesis, it is easy to see that $\operatorname{Re}\lambda(B^{-1}A) \geq 0$. By Theorem 2.1, we get the desired result. □

**Corollary 2.15.** *Let $A, B$ be two $n$-by-$n$ complex positive definite matrices, and all the eigenvalues of $B$ be real numbers. If $r([A, B]) \leq 1$, then inequality (1.2) holds for $t \geq \frac{2}{n}$, where $[A, B] = AB - BA$, $r([A, B])$ is the rank of $[A, B]$.*

*Proof.* It is easy to see that $B^{-1}$ is a complex positive definite matrix and $n$ eigenvalues of $B^{-1}$ are real numbers. By the hypothesis and $r[B^{-1}, A] = r[A, B]$, we have $r([B^{-1}, A]) \leq 1$. By the Laffey-Choi Theorem (see [5], [1]), there exists a non-singular matrix $P$, such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular matrices. The result holds by Theorem 2.14. □

**Corollary 2.16.** *Let $A, B$ be two $n$-by-$n$ complex positive definite matrices $(n \geq 2)$. Suppose $AB = BA$ and $n$ eigenvalues of $B$ are real numbers, then inequality (1.2) holds for $t \geq \frac{2}{n}$.*

*Proof.* Follows from Corollary 2.15 and the fact that $r([A, B]) = 0$. □

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