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SOME NEW HILBERT-PACHPATTE'S INEQUALITIES

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ABSTRACT. Some new Hilbert-Pachpatte discrete inequalities and their integral analogues are established in this paper. Other inequalities are also given in remarks.

Key words and phrases: Hilbert-Pachpatte's inequality; Hölder's inequality; Jensen's inequality; Nonnegative sequences.

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1. INTRODUCTION

Let $p \geq 1$, $q \geq 1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \dots, k$ and $n = 1, 2, \dots, r$, where k and r are natural numbers and define $A_m = \sum_{s=1}^{m} a_s$ and $B_n = \sum_{t=1}^{n} b_t$. Then

$$(1.1)\quad \sum_{m=1}^{k}\sum_{n=1}^{r} \frac{A_m^p B_n^q}{m+n} \leq C(p,q,k,r) \left\{ \sum_{m=1}^{k} (k-m+1)(A_m^{p-1} a_m)^2 \right\}^{\frac{1}{2}} \times \left\{ \sum_{n=1}^{r} (r-n+1)(B_n^{q-1} b_n)^2 \right\}^{\frac{1}{2}},$$

unless $\{a_m\}$ or $\{b_n\}$ is null, where $C(p,q,k,r) = \frac{1}{2}pq\sqrt{kr}$.

An integral analogue of (1.1) is given in the following result.

Let $p \geq 1$, $q \geq 1$ and $f(\sigma) \geq 0$, $g(\tau) \geq 0$ for $\sigma \in (0,x)$, $\tau \in (0,y)$, where x, y are positive real numbers and define $F(s) = \int_0^s f(\sigma)d\sigma$ and $G(t) = \int_0^t g(\tau)d\tau$, for $s \in (0,x)$, $t \in (0,y)$. Then

$$(1.2)\quad \int_0^x \int_0^y \frac{F^p(s)G^q(t)dsdt}{s+t} \leq D(p,q,x,y) \left\{ \int_0^x (x-s)(F^{p-1}(s)f(s))^2 ds \right\}^{\frac{1}{2}} \times \left\{ \int_0^y (y-t)(G^{q-1}(t)g(t))^2 dt \right\}^{\frac{1}{2}},$$

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unless $f(\sigma) \equiv 0$ or $g(\tau) \equiv 0$, where $D(p,q,x,y) = \frac{1}{2}pq\sqrt{xy}$.

Inequalities (1.1) and (1.2) are the well known Hilbert-Pachpatte inequalities [1], which gave new estimates on Hilbert type inequalities [2]. It is well known that the Hilbert-Pachpatte inequalities play a dominant role in analysis, so the literature on such inequalities and their applications is vast [3] – [8].

Young-Ho Kim [9] gave new inequalities similar to the Hilbert-Pachpatte inequalities as follows.

Let $p \geq 1$, $q \geq 1$, $\alpha > 0$, and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$, where k and r are natural numbers and define $A_m = \sum_{s=1}^{m} a_s$ and $B_n = \sum_{t=1}^{n} b_t$. Then

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{(m^\alpha + n^\alpha)^{\frac{1}{\alpha}}} \leq C(p,q,k,r;\alpha)\left\{\sum_{m=1}^{k}(k-m+1)(A_m^{p-1}a_m)^2\right\}^{\frac{1}{2}} \times \left\{\sum_{n=1}^{r}(r-n+1)(B_n^{q-1}b_n)^2\right\}^{\frac{1}{2}}, \tag{1.3}$$

unless $\{a_m\}$ or $\{b_n\}$ is null, where $C(p,q,k,r;\alpha) = \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} pq\sqrt{kr}$.

An integral analogue of (1.3) is given in the following result.

Let $p \geq 1$, $q \geq 1$, $\alpha > 0$ and $f(\sigma) \geq 0$, $g(\tau) \geq 0$ for $\sigma \in (0,x)$, $\tau \in (0,y)$, where x, y are positive real numbers and define $F(s) = \int_0^s f(\sigma)d\sigma$ and $G(t) = \int_0^t g(\tau)d\tau$, for $s \in (0,x)$, $t \in (0,y)$. Then

$$\int_0^x\int_0^y \frac{F^p(s)G^q(t)dsdt}{(s^\alpha + t^\alpha)^{\frac{1}{\alpha}}} \leq D(p,q,x,y;\alpha)\left\{\int_0^x (x-s)(F^{p-1}(s)f(s))^2 ds\right\}^{\frac{1}{2}} \times \left\{\int_0^y (y-t)(G^{q-1}(t)g(t))^2 dt\right\}^{\frac{1}{2}}, \tag{1.4}$$

unless $f(\sigma) \equiv 0$ or $g(\tau) \equiv 0$, where $D(p,q,x,y;\alpha) = \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} pq\sqrt{xy}$.

The purpose of the present paper is to derive some new generalized inequalities (1.1) and (1.2) that are similar to (1.3) and (1.4). By applying an elementary inequality, we also obtain some new inequalities similar to some results in [1, 9].

2. MAIN RESULTS

Now we give our results as follows in this paper.

Theorem 2.1. *Let $p \geq 1$, $q \geq 1$, $\alpha > 1$, $\gamma > 1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$, where k and r are natural numbers and define $A_m = \sum_{s=1}^{m} a_s$ and $B_n = \sum_{t=1}^{n} b_t$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \leq C(p,q,k,r;\alpha,\gamma) \times \left\{\sum_{m=1}^{k}(k-m+1)(A_m^{p-1}a_m)^\alpha\right\}^{\frac{1}{\alpha}}\left\{\sum_{n=1}^{r}(r-n+1)(B_n^{q-1}b_n)^\gamma\right\}^{\frac{1}{\gamma}}, \tag{2.1}$$

unless $\{a_m\}$ or $\{b_n\}$ is null, where $C(p,q,k,r;\alpha,\gamma) = \frac{pq}{\alpha+\gamma}k^{\frac{\alpha-1}{\alpha}}r^{\frac{\gamma-1}{\gamma}}$.

Proof. The idea for the proof Theorem 2.1 comes from Theorem 1 of [1] and Theorem 2.1 of [9]. From the hypotheses of Theorem 2.1 and using the following inequality (see [10, 11]),

$$\left\{\sum_{m=1}^{n} z_m\right\}^{\beta} \le \beta \sum_{m=1}^{n} z_m \left\{\sum_{k=1}^{m} z_k\right\}^{\beta-1}, \tag{2.2}$$

where $\beta \ge 1$ is a constant and $z_m \ge 0$, $(m = 1, 2, \dots, n)$, it is easy to observe that

$$\begin{aligned} A_m^p &\le p \sum_{s=1}^{m} A_s^{p-1} a_s, \qquad m = 1, 2, \dots, k, \\ B_n^q &\le q \sum_{t=1}^{n} B_t^{q-1} b_t, \qquad n = 1, 2, \dots, r. \end{aligned} \tag{2.3}$$

From (2.3) and Hölder's inequality, we have

$$\sum_{s=1}^{m} A_s^{p-1} a_s \le m^{\frac{\alpha-1}{\alpha}} \left\{\sum_{s=1}^{m} (A_s^{p-1} a_s)^{\alpha}\right\}^{\frac{1}{\alpha}}, \qquad m = 1, 2, \dots, k, \tag{2.4}$$

and

$$\sum_{t=1}^{n} B_t^{q-1} b_t \le n^{\frac{\gamma-1}{\gamma}} \left\{\sum_{t=1}^{n} (B_t^{q-1} b_t)^{\gamma}\right\}^{\frac{1}{\gamma}}, \qquad n = 1, 2, \dots, r. \tag{2.5}$$

Using the inequality of means [12]

$$\left\{\prod_{i=1}^{n} s_i^{\omega_i}\right\}^{\frac{1}{\Omega_n}} \le \left\{\frac{1}{\Omega_n} \sum_{i=1}^{n} \omega_i s_i^r\right\}^{\frac{1}{r}} \tag{2.6}$$

for $r > 0$, $\omega_i > 0$, $\sum_{i=1}^{n} \omega_i = \Omega_n$, we observe that

$$(s_1^{\omega_1} s_2^{\omega_2})^{r/(\omega_1+\omega_2)} \le \frac{1}{\omega_1 + \omega_2} (\omega_1 s_1^r + \omega_2 s_2^r). \tag{2.7}$$

Let $s_1 = m^{\alpha-1}$, $s_2 = n^{\gamma-1}$, $\omega_1 = \frac{1}{\alpha}$, $\omega_2 = \frac{1}{\gamma}$ and $r = \omega_1 + \omega_2$, from (2.3) – (2.5) and (2.7), we have

$$\begin{aligned} A_m^p B_n^q &\le pq m^{\frac{\alpha-1}{\alpha}} n^{\frac{\gamma-1}{\gamma}} \left\{\sum_{s=1}^{m} (A_s^{p-1} a_s)^{\alpha}\right\}^{\frac{1}{\alpha}} \left\{\sum_{t=1}^{n} (B_t^{q-1} b_t)^{\gamma}\right\}^{\frac{1}{\gamma}} \\ &\le \frac{pq\alpha\gamma}{\alpha+\gamma} \left\{\frac{m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}}{\alpha} + \frac{n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}{\gamma}\right\} \\ &\qquad \times \left\{\sum_{s=1}^{m} (A_s^{p-1} a_s)^{\alpha}\right\}^{\frac{1}{\alpha}} \left\{\sum_{t=1}^{n} (B_t^{q-1} b_t)^{\gamma}\right\}^{\frac{1}{\gamma}}, \end{aligned} \tag{2.8}$$

for $m = 1, 2, \dots, k$, $n = 1, 2, \dots, r$. From (2.8), we observe that

$$\frac{A_m^p B_n^q}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \le \frac{pq}{\alpha+\gamma} \left\{\sum_{s=1}^{m} (A_s^{p-1} a_s)^{\alpha}\right\}^{\frac{1}{\alpha}} \left\{\sum_{t=1}^{n} (B_t^{q-1} b_t)^{\gamma}\right\}^{\frac{1}{\gamma}}, \tag{2.9}$$

for $m = 1, 2, \dots, k$, $n = 1, 2, \dots, r$. Taking the sum on both sides of (2.9) first over n from 1 to r and then over m from 1 to k of the resulting inequality and using Hölder's inequality with indices α, $\alpha/(\alpha-1)$ and γ, $\gamma/(\gamma-1)$ and interchanging the order of summations, we observe that

$$\begin{aligned}
&\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}\\
&\le \frac{pq}{\alpha+\gamma}\left[\sum_{m=1}^{k}\left\{\sum_{s=1}^{m}(A_s^{p-1}a_s)^{\alpha}\right\}^{\frac{1}{\alpha}}\right]\left[\sum_{n=1}^{r}\left\{\sum_{t=1}^{n}(B_t^{q-1}b_t)^{\gamma}\right\}^{\frac{1}{\gamma}}\right]\\
&\le \frac{pq}{\alpha+\gamma}k^{\frac{\alpha-1}{\alpha}}\left\{\sum_{m=1}^{k}\sum_{s=1}^{m}(A_s^{p-1}a_s)^{\alpha}\right\}^{\frac{1}{\alpha}} r^{\frac{\gamma-1}{\gamma}}\left\{\sum_{n=1}^{r}\sum_{t=1}^{n}(B_t^{q-1}b_t)^{\gamma}\right\}^{\frac{1}{\gamma}}\\
&= \frac{pq}{\alpha+\gamma}k^{\frac{\alpha-1}{\alpha}}r^{\frac{\gamma-1}{\gamma}}\left\{\sum_{m=1}^{k}(k-m+1)(A_m^{p-1}a_m)^{\alpha}\right\}^{\frac{1}{\alpha}}\\
&\qquad\times\left\{\sum_{n=1}^{r}(r-n+1)(B_n^{q-1}b_n)^{\gamma}\right\}^{\frac{1}{\gamma}}.
\end{aligned}$$

□

Remark 1. In Theorem 2.1, setting $\alpha=\gamma=2$, we have (1.1). In Theorem 2.1, setting $\frac{1}{\alpha}+\frac{1}{\gamma}=1$, we have

$$\begin{aligned}
\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{\gamma m^{\alpha-1}+\alpha n^{\gamma-1}} &\le C(p,q,k,r;\alpha,\gamma)\\
&\quad\times\left\{\sum_{m=1}^{k}(k-m+1)(A_m^{p-1}a_m)^{\alpha}\right\}^{\frac{1}{\alpha}}\left\{\sum_{n=1}^{r}(r-n+1)(B_n^{q-1}b_n)^{\gamma}\right\}^{\frac{1}{\gamma}},
\end{aligned}$$

unless $\{a_m\}$ or $\{b_n\}$ is null, where $C(p,q,k,r;\alpha,\gamma)=\frac{pq}{\alpha+\gamma}k^{\frac{\alpha-1}{\alpha}}r^{\frac{\gamma-1}{\gamma}}$.

Remark 2. In Theorem 2.1, setting $p=q=1$, we have

$$\begin{aligned}
\text{(2.10)}\quad \sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m B_n}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}&\\
\le C(1,1,k,r;\alpha,\gamma)&\left\{\sum_{m=1}^{k}(k-m+1)a_m^{\alpha}\right\}^{\frac{1}{\alpha}}\left\{\sum_{n=1}^{r}(r-n+1)b_n^{\gamma}\right\}^{\frac{1}{\gamma}},
\end{aligned}$$

unless $\{a_m\}$ or $\{b_n\}$ is null, where $C(1,1,k,r;\alpha,\gamma)=\frac{1}{\alpha+\gamma}k^{\frac{\alpha-1}{\alpha}}r^{\frac{\gamma-1}{\gamma}}$.

In the following theorem we give a further generalization of the inequality (2.10) obtained in Remark 2. Before we give our result, we point out that $\{p_m\}$ and $\{q_n\}$ should be two positive sequences for $m=1,2,\dots,k$ and $n=1,2,\dots,r$ in Theorem 2.3 of [9].

Theorem 2.2. *Let $\alpha>1$, $\gamma>1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers and $\{p_m\}$ and $\{q_n\}$ be positive sequences defined for $m=1,2,\dots,k$ and $n=1,2,\dots,r$, where k and r are natural numbers and define $A_m=\sum_{s=1}^{m}a_s$, $B_n=\sum_{t=1}^{n}b_t$, $P_m=\sum_{s=1}^{m}p_s$*

and $Q_n = \sum_{t=1}^{n} q_t$*. Let* Φ *and* Ψ *be two real-valued, nonnegative, convex, and submultiplicative functions defined on* $\mathbb{R}_+ = [0, \infty)$*. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\Phi(A_m)\Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \le M(k,r;\alpha,\gamma)\left\{\sum_{m=1}^{k}(k-m+1)\left[p_m\Phi\left(\frac{a_m}{p_m}\right)\right]^{\alpha}\right\}^{\frac{1}{\alpha}} \times\left\{\sum_{n=1}^{r}(r-n+1)\left[q_n\Psi\left(\frac{b_n}{q_n}\right)\right]^{\gamma}\right\}^{\frac{1}{\gamma}}, \tag{2.11}$$

where

$$M(k,r;\alpha,\gamma)=\frac{1}{\alpha+\gamma}\left\{\sum_{m=1}^{k}\left[\frac{\Phi(P_m)}{P_m}\right]^{\frac{\alpha}{\alpha-1}}\right\}^{\frac{\alpha-1}{\alpha}}\left\{\sum_{n=1}^{r}\left[\frac{\Psi(Q_n)}{Q_n}\right]^{\frac{\gamma}{\gamma-1}}\right\}^{\frac{\gamma-1}{\gamma}}.$$

Proof. From the hypotheses of Φ and Ψ and by using Jensen's inequality and Hölder's inequality, it is easy to observe that

$$\begin{aligned}\Phi(A_m) &= \Phi\left(\frac{P_m\sum_{s=1}^{m}p_s a_s/p_s}{\sum_{s=1}^{m}p_s}\right)\\ &\le \Phi(P_m)\Phi\left(\frac{\sum_{s=1}^{m}p_s a_s/p_s}{\sum_{s=1}^{m}p_s}\right)\le\frac{\Phi(P_m)}{P_m}\sum_{s=1}^{m}p_s\Phi\left(\frac{a_s}{p_s}\right)\\ &\le\frac{\Phi(P_m)}{P_m}m^{\frac{\alpha-1}{\alpha}}\left\{\sum_{s=1}^{m}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^{\alpha}\right\}^{\frac{1}{\alpha}},\end{aligned} \tag{2.12}$$

and similarly,

$$\Psi(B_n)\le\frac{\Psi(Q_n)}{Q_n}n^{\frac{\gamma-1}{\gamma}}\left\{\sum_{t=1}^{n}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^{\gamma}\right\}^{\frac{1}{\gamma}}. \tag{2.13}$$

Let $s_1 = m^{\alpha-1}$, $s_2 = n^{\gamma-1}$, $\omega_1 = \frac{1}{\alpha}$, $\omega_2 = \frac{1}{\gamma}$ and $r = \omega_1 + \omega_2$, from (2.7), (2.12) and (2.13), we have

$$\begin{aligned}&\Phi(A_m)\Psi(B_n)\\ &\le m^{\frac{\alpha-1}{\alpha}}n^{\frac{\gamma-1}{\gamma}}\left[\frac{\Phi(P_m)}{P_m}\left\{\sum_{s=1}^{m}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^{\alpha}\right\}^{\frac{1}{\alpha}}\right]\\ &\quad\times\left[\frac{\Psi(Q_n)}{Q_n}\left\{\sum_{t=1}^{n}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^{\gamma}\right\}^{\frac{1}{\gamma}}\right]\\ &\le\frac{\alpha\gamma}{\alpha+\gamma}\left\{\frac{m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}}{\alpha}+\frac{n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}{\gamma}\right\}\left[\frac{\Phi(P_m)}{P_m}\left\{\sum_{s=1}^{m}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^{\alpha}\right\}^{\frac{1}{\alpha}}\right]\\ &\quad\times\left[\frac{\Psi(Q_n)}{Q_n}\left\{\sum_{t=1}^{n}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^{\gamma}\right\}^{\frac{1}{\gamma}}\right]\end{aligned} \tag{2.14}$$

for $m = 1, 2, \dots, k$, $n = 1, 2, \dots, r$. From (2.14), we observe that

$$\frac{\Phi(A_m)\Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \leq \frac{1}{\alpha+\gamma}\left[\frac{\Phi(P_m)}{P_m}\left\{\sum_{s=1}^{m}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^{\alpha}\right\}^{\frac{1}{\alpha}}\right]\left[\frac{\Psi(Q_n)}{Q_n}\left\{\sum_{t=1}^{n}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^{\gamma}\right\}^{\frac{1}{\gamma}}\right] \tag{2.15}$$

for $m = 1, 2, \dots, k$, $n = 1, 2, \dots, r$. Taking the sum on both sides of (2.15) first over n from 1 to r and then over m from 1 to k of the resulting inequality and using Hölder's inequality with indices α, $\alpha/(\alpha-1)$ and γ, $\gamma/(\gamma-1)$ and interchanging the order of summations, we observe that

$$\begin{aligned}
&\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\Phi(A_m)\Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}\\
&\leq \frac{1}{\alpha+\gamma}\left[\sum_{m=1}^{k}\frac{\Phi(P_m)}{P_m}\left\{\sum_{s=1}^{m}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^{\alpha}\right\}^{\frac{1}{\alpha}}\right]\left[\sum_{n=1}^{r}\frac{\Psi(Q_n)}{Q_n}\left\{\sum_{t=1}^{n}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^{\gamma}\right\}^{\frac{1}{\gamma}}\right]\\
&\leq \frac{1}{\alpha+\gamma}\left\{\sum_{m=1}^{k}\left[\frac{\Phi(P_m)}{P_m}\right]^{\frac{\alpha}{\alpha-1}}\right\}^{\frac{\alpha-1}{\alpha}}\left\{\sum_{m=1}^{k}\sum_{s=1}^{m}\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^{\alpha}\right\}^{\frac{1}{\alpha}}\\
&\qquad\times\left\{\sum_{n=1}^{r}\left[\frac{\Psi(Q_n)}{Q_n}\right]^{\frac{\gamma}{\gamma-1}}\right\}^{\frac{\gamma-1}{\gamma}}\left\{\sum_{n=1}^{r}\sum_{t=1}^{n}\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^{\gamma}\right\}^{\frac{1}{\gamma}}\\
&= \frac{1}{\alpha+\gamma}\left\{\sum_{m=1}^{k}\left[\frac{\Phi(P_m)}{P_m}\right]^{\frac{\alpha}{\alpha-1}}\right\}^{\frac{\alpha-1}{\alpha}}\left\{\sum_{n=1}^{r}\left[\frac{\Psi(Q_n)}{Q_n}\right]^{\frac{\gamma}{\gamma-1}}\right\}^{\frac{\gamma-1}{\gamma}}\\
&\qquad\times\left\{\sum_{m=1}^{k}(k-m+1)\left[p_s\Phi\left(\frac{a_s}{p_s}\right)\right]^{\alpha}\right\}^{\frac{1}{\alpha}}\left\{\sum_{n=1}^{r}(r-n+1)\left[q_t\Psi\left(\frac{b_t}{q_t}\right)\right]^{\gamma}\right\}^{\frac{1}{\gamma}}.
\end{aligned}$$

□

Remark 3. From the inequality (2.7), we obtain

$$s_1^{\omega_1}s_2^{\omega_2} \leq \frac{1}{\omega_1+\omega_2}\left(\omega_1 s_1^{\omega_1+\omega_2} + \omega_2 s_2^{\omega_1+\omega_2}\right) \tag{2.16}$$

for $\omega_1 > 0$, $\omega_2 > 0$. If we apply the elementary inequality (2.16) on the right-hand sides of (2.1) in Theorem 2.1 and (2.11) in Theorem 2.2, then we get the following inequalities

$$\begin{aligned}
&\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{A_m^p B_n^q}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}\\
&\qquad\leq \frac{\alpha\gamma C(p,q,k,r;\alpha,\gamma)}{\alpha+\gamma}\left[\frac{1}{\alpha}\left\{\sum_{m=1}^{k}(k-m+1)(A_m^{p-1}a_m)^{\alpha}\right\}^{\frac{\alpha+\gamma}{\alpha\gamma}}\right.\\
&\qquad\qquad\left.+\frac{1}{\gamma}\left\{\sum_{n=1}^{r}(r-n+1)(B_n^{q-1}b_n)^{\gamma}\right\}^{\frac{\alpha+\gamma}{\alpha\gamma}}\right],
\end{aligned}$$

where $C(p,q,k,r;\alpha,\gamma)=\frac{pq}{\alpha+\gamma}k^{\frac{\alpha-1}{\alpha}}r^{\frac{\gamma-1}{\gamma}}$. Also,

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{\Phi(A_m)\Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \leq \frac{\alpha\gamma M(k,r;\alpha,\gamma)}{\alpha+\gamma}\left[\frac{1}{\alpha}\left\{\sum_{m=1}^{k}(k-m+1)\left[p_m\Phi\left(\frac{a_m}{p_m}\right)\right]^{\alpha}\right\}^{\frac{\alpha+\gamma}{\alpha\gamma}} + \frac{1}{\gamma}\left\{\sum_{n=1}^{r}(r-n+1)\left[q_n\Psi\left(\frac{b_n}{q_n}\right)\right]^{\gamma}\right\}^{\frac{\alpha+\gamma}{\alpha\gamma}}\right],$$

where

$$M(k,r;\alpha,\gamma)=\frac{1}{\alpha+\gamma}\left\{\sum_{m=1}^{k}\left[\frac{\Phi(P_m)}{P_m}\right]^{\frac{\alpha}{\alpha-1}}\right\}^{\frac{\alpha-1}{\alpha}}\left\{\sum_{n=1}^{r}\left[\frac{\Psi(Q_n)}{Q_n}\right]^{\frac{\gamma}{\gamma-1}}\right\}^{\frac{\gamma-1}{\gamma}}.$$

The following theorems deal with slight variants of the inequality (2.11) given in Theorem 2.2.

Theorem 2.3. *Let $\alpha>1,\gamma>1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers defined for $m=1,2,\dots,k$ and $n=1,2,\dots,r$, where k and r are natural numbers and define $A_m=\frac{1}{m}\sum_{s=1}^{m}a_s$ and $B_n=\frac{1}{n}\sum_{t=1}^{n}b_t$. Let Φ and Ψ be two real-valued, nonnegative, convex functions defined on $\mathbb{R}_+=[0,\infty)$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{mn\Phi(A_m)\Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}\leq C(1,1,k,r;\alpha,\gamma) \times\left\{\sum_{m=1}^{k}(k-m+1)\Phi^{\alpha}(a_m)\right\}^{\frac{1}{\alpha}}\left\{\sum_{n=1}^{r}(r-n+1)\Psi^{\gamma}(b_n)\right\}^{\frac{1}{\gamma}},$$

where $C(1,1,k,r;\alpha,\gamma)=\frac{1}{\alpha+\gamma}k^{\frac{\alpha-1}{\alpha}}r^{\frac{\gamma-1}{\gamma}}$.

Proof. From the hypotheses and by using Jensen's inequality and Hölder's inequality, it is easy to observe that

$$\Phi(A_m)=\Phi\left(\frac{1}{m}\sum_{s=1}^{m}a_s\right) \leq\frac{1}{m}\sum_{s=1}^{m}\Phi(a_s)\leq\frac{1}{m}m^{\frac{\alpha-1}{\alpha}}\left\{\sum_{s=1}^{m}\Phi^{\alpha}(a_s)\right\}^{\frac{\alpha-1}{\alpha}},$$

$$\Psi(B_n)=\Psi\left(\frac{1}{n}\sum_{t=1}^{n}b_t\right) \leq\frac{1}{n}\sum_{t=1}^{n}\Psi(b_t)\leq\frac{1}{n}n^{\frac{\gamma-1}{\gamma}}\left\{\sum_{t=1}^{n}\Psi^{\gamma}(b_t)\right\}^{\frac{\gamma-1}{\gamma}}.$$

The rest of the proof can be completed by following the same steps as in the proofs of Theorems 2.1 and 2.2 with suitable changes and hence we omit the details. □

Theorem 2.4. *Let $\alpha > 1$, $\gamma > 1$ and $\{a_m\}$ and $\{b_n\}$ be two nonnegative sequences of real numbers and $\{p_m\}$ and $\{q_n\}$ be positive sequences defined for $m = 1, 2, \ldots, k$ and $n = 1, 2, \ldots, r$, where k and r are natural numbers and define $P_m = \sum_{s=1}^{m} p_s$, $Q_n = \sum_{t=1}^{n} q_t$, $A_m = \frac{1}{P_m}\sum_{s=1}^{m} p_m a_s$ and $B_n = \frac{1}{Q_n}\sum_{t=1}^{n} q_n b_t$. Let Φ and Ψ be two real-valued, nonnegative, convex functions defined on $\mathbb{R}_+ = [0, \infty)$. Then*

$$\sum_{m=1}^{k}\sum_{n=1}^{r}\frac{P_m Q_n \Phi(A_m)\Psi(B_n)}{\gamma m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \leq C(1,1,k,r;\alpha,\gamma)\left\{\sum_{m=1}^{k}(k-m+1)\left[p_m\Phi\left(a_m\right)\right]^{\alpha}\right\}^{\frac{1}{\alpha}} \times \left\{\sum_{n=1}^{r}(r-n+1)\left[q_n\Psi\left(b_n\right)\right]^{\gamma}\right\}^{\frac{1}{\gamma}},$$

where $C(1,1,k,r;\alpha,\gamma) = \frac{1}{\alpha+\gamma}k^{\frac{\alpha-1}{\alpha}}r^{\frac{\gamma-1}{\gamma}}$.

Proof. From the hypotheses and by using Jensen's inequality and Hölder's inequality, it is easy to observe that

$$\begin{aligned}\Phi(A_m) &= \Phi\left(\frac{1}{P_m}\sum_{s=1}^{m}p_s a_s\right)\\ &\leq \frac{1}{P_m}\sum_{s=1}^{m}p_s\Phi(a_s)\\ &\leq \frac{1}{P_m}m^{\frac{\alpha-1}{\alpha}}\left\{\sum_{s=1}^{m}[p_s\Phi(a_s)]^{\alpha}\right\}^{\frac{\alpha-1}{\alpha}},\end{aligned}$$

$$\begin{aligned}\Psi(B_n) &= \Psi\left(\frac{1}{Q_n}\sum_{t=1}^{n}q_t b_t\right)\\ &\leq \frac{1}{Q_n}\sum_{t=1}^{n}q_t\Psi(b_t)\\ &\leq \frac{1}{Q_n}n^{\frac{\gamma-1}{\gamma}}\left\{\sum_{t=1}^{n}[q_t\Psi(b_t)]^{\gamma}\right\}^{\frac{\gamma-1}{\gamma}}.\end{aligned}$$

The rest of the proof can be completed by following the same steps as in the proofs of Theorems 2.1 and 2.2 with suitable changes and hence we omit the details. □

3. INTEGRAL ANALOGUES

Now we give the integral analogues of the inequalities in Theorems 2.1 – 2.4.

An integral analogue of Theorem 2.1 is given in the following theorem.

Theorem 3.1. *Let $p \geq 0$, $q \geq 0$, $\alpha > 1$, $\gamma > 1$ and $f(\sigma) \geq 0$, $g(\tau) \geq 0$ for $\sigma \in (0, x)$, $\tau \in (0, y)$, where x, y are positive real numbers, define $F(s) = \int_0^s f(\sigma)d\sigma$, $G(t) = \int_0^t g(\tau)d\tau$*

for $s \in (0,x)$, $t \in (0,y)$. *Then*

$$\int_0^x \int_0^y \frac{F^p(s)G^q(t)}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} dsdt \le D(p,q,x,y;\alpha,\gamma) \left\{ \int_0^x (x-s)(F^{p-1}(s)f(s))^\alpha ds \right\}^{\frac{1}{\alpha}} \times \left\{ \int_0^y (y-t)(G^{q-1}(t)g(t))^\gamma dt \right\}^{\frac{1}{\gamma}}, \tag{3.1}$$

unless $f(\sigma) \equiv 0$ *or* $g(\tau) \equiv 0$*, where* $D(p,q,x,y;\alpha,\gamma) = \frac{pq}{\alpha+\gamma} x^{\frac{\alpha-1}{\alpha}} y^{\frac{\gamma-1}{\gamma}}$.

Proof. From the hypotheses of $F(s)$ and $G(t)$, it is easy to observe that

$$F^p(s) = p\int_0^s F^{p-1}(\sigma)f(\sigma)d\sigma, \qquad s \in (0,x), \tag{3.2}$$
$$G^q(t) = q\int_0^t G^{q-1}(\tau)g(\tau)d\tau, \qquad t \in (0,y).$$

From (3.2) and Hölder's inequality, we have

$$\int_0^x F^{p-1}(\sigma)f(\sigma)d\sigma \le s^{\frac{\alpha-1}{\alpha}} \left\{ \int_0^s (F^{p-1}(\sigma)f(\sigma))^\alpha d\sigma \right\}^{\frac{1}{\alpha}}, \qquad s \in (0,s), \tag{3.3}$$

and

$$\int_0^y G^{q-1}(t)g(t)dt \le t^{\frac{\gamma-1}{\gamma}} \left\{ \int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau \right\}^{\frac{1}{\gamma}}, \qquad t \in (0,t). \tag{3.4}$$

Let $s_1 = s^{\alpha-1}$, $s_2 = t^{\gamma-1}$, $\omega_1 = \frac{1}{\alpha}$, $\omega_1 = \frac{1}{\gamma}$, $r = \omega_1 + \omega_2$, from (3.2) – (3.4) and (2.7), we observe that

$$F^p(s)G^q(t) \le pqs^{\frac{\alpha-1}{\alpha}} t^{\frac{\gamma-1}{\gamma}} \left\{ \int_0^s (F^{p-1}(\sigma)f(\sigma))^\alpha d\sigma \right\}^{\frac{1}{\alpha}} \left\{ \int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau \right\}^{\frac{1}{\gamma}} \le \frac{pq\alpha\gamma}{\alpha+\gamma} \left\{ \frac{m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}}{\alpha} + \frac{n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}{\gamma} \right\} \times \left\{ \int_0^s (F^{p-1}(\sigma)f(\sigma))^\alpha d\sigma \right\}^{\frac{1}{\alpha}} \left\{ \int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau \right\}^{\frac{1}{\gamma}} \tag{3.5}$$

for $s \in (0,x)$, $t \in (0,y)$. From (3.5), we observe that

$$\frac{F^p(s)G^q(t)}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} \le \frac{pq}{\alpha+\gamma} \left\{ \int_0^s (F^{p-1}(\sigma)f(\sigma))^\alpha d\sigma \right\}^{\frac{1}{\alpha}} \left\{ \int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau \right\}^{\frac{1}{\gamma}} \tag{3.6}$$

for $s \in (0,x)$, $t \in (0,y)$. Taking the integral on both sides of (3.6) first over t from 0 to y and then over s from 0 to x of the resulting inequality and using Hölder's inequality with indices α,

$\alpha/(\alpha-1)$ and γ, $\gamma/(\gamma-1)$ and interchanging the order of integrals, we observe that

$$\int_0^x\int_0^y \frac{F^p(s)G^q(t)}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}dsdt$$
$$\le \frac{pq}{\alpha+\gamma}\left[\int_0^x\left\{\int_0^s (F^{p-1}(\sigma)f(\sigma))^\alpha d\sigma\right\}^{\frac{1}{\alpha}}ds\right]\left[\int_0^y\left\{\int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau\right\}^{\frac{1}{\gamma}}dt\right]$$
$$\le \frac{pq}{\alpha+\gamma}x^{\frac{\alpha-1}{\alpha}}\left\{\int_0^x\int_0^s (F^{p-1}(\sigma)f(\sigma))^\alpha d\sigma ds\right\}^{\frac{1}{\alpha}} y^{\frac{\gamma-1}{\gamma}}\left\{\int_0^y\int_0^t (G^{q-1}(\tau)g(\tau))^\gamma d\tau dt\right\}^{\frac{1}{\gamma}}$$
$$= \frac{pq}{\alpha+\gamma}x^{\frac{\alpha-1}{\alpha}}y^{\frac{\gamma-1}{\gamma}}\left\{\int_0^x (x-s)(F^{p-1}(s)f(s))^\alpha ds\right\}^{\frac{1}{\alpha}}\left\{\int_0^t (y-t)(G^{q-1}(t)g(t))^\gamma dt\right\}^{\frac{1}{\gamma}}.$$

□

Remark 4. In Theorem 3.1, setting $\alpha=\gamma=2$, we have (1.2). In Theorem 3.1, setting $\frac{1}{\alpha}+\frac{1}{\gamma}=1$, we have

$$\int_0^x\int_0^y \frac{F^p(s)G^q(t)}{\gamma s^{\alpha-1}+\alpha t^{\gamma-1}}dsdt \le D(p,q,x,y;\alpha,\gamma)$$
$$\times\left\{\int_0^x (x-s)(F^{p-1}(s)f(s))^\alpha ds\right\}^{\frac{1}{\alpha}}\left\{\int_0^y (y-t)(G^{q-1}(t)g(t))^\gamma dt\right\}^{\frac{1}{\gamma}},$$

unless $f(\sigma)\equiv 0$ or $g(\tau)\equiv 0$, where $D(p,q,x,y;\alpha,\gamma)=\frac{pq}{\alpha+\gamma}x^{\frac{\alpha-1}{\alpha}}y^{\frac{\gamma-1}{\gamma}}$.

Remark 5. In Theorem 3.1, setting $p=q=1$, we have

(3.7)
$$\int_0^x\int_0^y \frac{F(s)G(t)}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}dsdt$$
$$\le D(1,1,x,y;\alpha,\gamma)\left\{\int_0^x (x-s)f^\alpha(s)ds\right\}^{\frac{1}{\alpha}}\left\{\int_0^y (y-t)g^\gamma(t)dt\right\}^{\frac{1}{\gamma}},$$

unless $f(\sigma)\equiv 0$ or $g(\tau)\equiv 0$, where $D(1,1,x,y;\alpha,\gamma)=\frac{1}{\alpha+\gamma}x^{\frac{\alpha-1}{\alpha}}y^{\frac{\gamma-1}{\gamma}}$.

In the following theorem we give a further generalization of the inequality (3.7) obtained in Remark 5.

Theorem 3.2. *Let $\alpha>1$, $\gamma>1$ and $f(\sigma)\ge 0$, $g(\tau)\ge 0$, $p(\sigma)>0$ and $q(\tau)>0$ for $\sigma\in(0,x)$, $\tau\in(0,y)$, where x, y are positive real numbers. Define $F(s)=\int_0^s f(\sigma)d\sigma$ and $G(t)=\int_0^t g(\tau)d\tau$, $P(s)=\int_0^s p(\sigma)d\sigma$ and $Q(t)=\int_0^t q(\tau)d\tau$ for $s\in(0,x)$, $t\in(0,y)$. Let Φ and Ψ be two real-valued, nonnegative, convex, and submultiplicative functions defined on $\mathbb{R}_+=[0,\infty)$. Then*

(3.8)
$$\int_0^x\int_0^y \frac{\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}dsdt$$
$$\le L(x,y;\alpha,\gamma)\left\{\int_0^x (x-s)\left[p(s)\Phi\left(\frac{f(s)}{p(s)}\right)\right]^\alpha ds\right\}^{\frac{1}{\alpha}}$$
$$\times\left\{\int_0^y (y-t)\left[q(t)\Psi\left(\frac{g(t)}{q(t)}\right)\right]^\gamma dt\right\}^{\frac{1}{\gamma}},$$

where

$$L(x,y;\alpha,\gamma)=\frac{1}{\alpha+\gamma}\left\{\int_0^x\left[\frac{\Phi(P(s))}{P(s)}\right]^{\frac{\alpha}{\alpha-1}}ds\right\}^{\frac{\alpha-1}{\alpha}}\left\{\int_0^y\left[\frac{\Psi(Q(t))}{Q(t)}\right]^{\frac{\gamma}{\gamma-1}}dt\right\}^{\frac{\gamma-1}{\gamma}}.$$

Proof. From the hypotheses of Φ and Ψ and by using Jensen's inequality and Hölder's inequality, it is easy to see that

$$\begin{aligned}\Phi(F(s)))&=\Phi\left(\frac{P(s)\int_0^s p(\sigma)\left(\frac{f(\sigma)}{p(\sigma)}\right)d\sigma}{\int_0^s p(\sigma)d\sigma}\right)\\&\le\Phi(P(s))\Phi\left(\frac{\int_0^s p(\sigma)\left(\frac{f(\sigma)}{p(\sigma)}\right)d\sigma}{\int_0^s p(\sigma)d\sigma}\right)\\&\le\frac{\Phi(P(s))}{P(s)}\int_0^s p(\sigma)\Phi\left(\frac{f(\sigma)}{p(\sigma)}\right)d\sigma\\&\le\frac{\Phi(P(s))}{P(s)}s^{\frac{\alpha-1}{\alpha}}\left\{\int_0^s\left[p(\sigma)\Phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^{\alpha}d\sigma\right\}^{\frac{1}{\alpha}},\end{aligned}\tag{3.9}$$

and similarly,

$$\Psi(G(t))\le\frac{\Psi(Q(t))}{Q(t)}t^{\frac{\gamma-1}{\gamma}}\left\{\int_0^t\left[q(\tau)\Psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^{\gamma}d\tau\right\}^{\frac{1}{\gamma}}.\tag{3.10}$$

Let $s_1=s^{\alpha-1}$, $s_2=t^{\gamma-1}$, $\omega_1=\frac{1}{\alpha}$, $\omega_1=\frac{1}{\gamma}$, $r=\omega_1+\omega_2$, from (3.9), (3.10) and (2.7), we observe that

$$\begin{aligned}\Phi(F(s))\Psi(G(t))&\le s^{\frac{\alpha-1}{\alpha}}t^{\frac{\gamma-1}{\gamma}}\left[\frac{\Phi(P(s))}{P(s)}\left\{\int_0^s\left[p(\sigma)\Phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^{\alpha}d\sigma\right\}^{\frac{1}{\alpha}}\right]\\&\quad\times\left[\frac{\Psi(Q(t))}{Q(t)}\left\{\int_0^t\left[q(\tau)\Psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^{\gamma}d\tau\right\}^{\frac{1}{\gamma}}\right]\\&\le\frac{\alpha\gamma}{\alpha+\gamma}\left\{\frac{m^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}}{\alpha}+\frac{n^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}{\gamma}\right\}\\&\quad\times\left[\frac{\Phi(P(s))}{P(s)}\left\{\int_0^s\left[p(\sigma)\Phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^{\alpha}d\sigma\right\}^{\frac{1}{\alpha}}\right]\\&\quad\times\left[\frac{\Psi(Q(t))}{Q(t)}\left\{\int_0^t\left[q(\tau)\Psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^{\gamma}d\tau\right\}^{\frac{1}{\gamma}}\right]\end{aligned}\tag{3.11}$$

for $s\in(0,x)$, $t\in(0,y)$. From (3.11), we observe that

$$\begin{aligned}\frac{\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}&\le\frac{1}{\alpha+\gamma}\left[\frac{\Phi(P(s))}{P(s)}\left\{\int_0^s\left[p(\sigma)\Phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^{\alpha}d\sigma\right\}^{\frac{1}{\alpha}}\right]\\&\quad\times\left[\frac{\Psi(Q(t))}{Q(t)}\left\{\int_0^t\left[q(\tau)\Psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^{\gamma}d\tau\right\}^{\frac{1}{\gamma}}\right]\end{aligned}\tag{3.12}$$

for $s \in (0,x)$, $t \in (0,y)$. Taking the integral on both sides of (3.12) first over t from 0 to y and then over s from 0 to x of the resulting inequality and using Hölder's inequality with indices α, $\alpha/(\alpha-1)$ and γ, $\gamma/(\gamma-1)$ and interchanging the order of integrals, we observe that

$$\begin{aligned}
&\int_0^x \int_0^y \frac{\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} dsdt \\
&\le \frac{1}{\alpha+\gamma}\left[\int_0^x \frac{\Phi(P(s))}{P(s)}\left\{\int_0^s \left[p(\sigma)\Phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^\alpha d\sigma\right\}^{\frac{1}{\alpha}} ds\right] \\
&\qquad \times \left[\int_0^y \frac{\Psi(Q(t))}{Q(t)}\left\{\int_0^t \left[q(\tau)\Psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^\gamma d\tau\right\}^{\frac{1}{\gamma}} dt\right] \\
&\le \frac{1}{\alpha+\gamma}\left\{\int_0^x \left[\frac{\Phi(P(s))}{P(s)}\right]^{\frac{\alpha}{\alpha-1}} ds\right\}^{\frac{\alpha-1}{\alpha}} \left\{\int_0^x \int_0^s \left[p(\sigma)\Phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^\alpha d\sigma ds\right\}^{\frac{1}{\alpha}} \\
&\qquad \times \left\{\int_0^y \left[\frac{\Psi(Q(t))}{Q(t)}\right]^{\frac{\gamma}{\gamma-1}} dt\right\}^{\frac{\gamma-1}{\gamma}} \left\{\int_0^y \int_0^t \left[q(\tau)\Psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^\gamma d\tau dt\right\}^{\frac{1}{\gamma}} \\
&= \frac{1}{\alpha+\gamma}\left\{\int_0^x \left[\frac{\Phi(P(s))}{P(s)}\right]^{\frac{\alpha}{\alpha-1}} ds\right\}^{\frac{\alpha-1}{\alpha}} \left\{\int_0^y \left[\frac{\Psi(Q(t))}{Q(t)}\right]^{\frac{\gamma}{\gamma-1}} dt\right\}^{\frac{\gamma-1}{\gamma}} \\
&\qquad \times \left\{\int_0^x (x-s)\left[p(s)\Phi\left(\frac{f(s)}{p(s)}\right)\right]^\alpha ds\right\}^{\frac{1}{\alpha}} \left\{\int_0^y (y-t)\left[q(t)\Psi\left(\frac{g(t)}{q(t)}\right)\right]^\gamma dt\right\}^{\frac{1}{\gamma}}.
\end{aligned}$$

□

Remark 6. From the inequality (2.7), we obtain

$$s_1^{\omega_1} s_2^{\omega_2} \le \frac{1}{\omega_1+\omega_2}\left(\omega_1 s_1^{\omega_1+\omega_2} + \omega_2 s_2^{\omega_1+\omega_2}\right) \tag{3.13}$$

for $\omega_1 > 0$, $\omega_2 > 0$. If we apply the elementary inequality (3.13) on the right-hand sides of (3.1) in Theorem 3.1 and (3.8) in Theorem 3.2, then we get the following inequalities

$$\begin{aligned}
&\int_0^x \int_0^y \frac{F^p(s)G^q(t)}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} dsdt \\
&\qquad \le \frac{\alpha\gamma D(p,q,x,y;\alpha,\gamma)}{\alpha+\gamma}\left[\frac{1}{\alpha}\left\{\int_0^x (x-s)(F^{p-1}(s)f(s))^\alpha ds\right\}^{\frac{\alpha+\gamma}{\alpha\gamma}}\right. \\
&\qquad\qquad \left. + \frac{1}{\gamma}\left\{\int_0^y (y-t)(G^{q-1}(t)g(t))^\gamma dt\right\}^{\frac{\alpha+\gamma}{\alpha\gamma}}\right],
\end{aligned} \tag{3.14}$$

where $D(p,q,x,y;\alpha,\gamma) = \frac{pq}{\alpha+\gamma} x^{\frac{\alpha-1}{\alpha}} y^{\frac{\gamma-1}{\gamma}}$. Also,

$$\begin{aligned}
&\int_0^x \int_0^y \frac{\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}} + \alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}} dsdt \\
&\qquad \le \frac{\alpha\gamma L(x,y;\alpha,\gamma)}{\alpha+\gamma}\left[\frac{1}{\alpha}\left\{\int_0^x (x-s)\left[p(s)\Phi\left(\frac{f(s)}{p(s)}\right)\right]^\alpha ds\right\}^{\frac{\alpha+\gamma}{\alpha\gamma}}\right.
\end{aligned}$$

$$+\frac{1}{\gamma}\left\{\int_0^y (y-t)\left[q(t)\Psi\left(\frac{g(t)}{q(t)}\right)\right]^{\gamma}dt\right\}^{\frac{\alpha+\gamma}{\alpha\gamma}}\Bigg],$$

where

$$L(x,y;\alpha,\gamma)=\frac{1}{\alpha+\gamma}\left\{\int_0^x\left[\frac{\Phi(P(s))}{P(s)}\right]^{\frac{\alpha}{\alpha-1}}ds\right\}^{\frac{\alpha-1}{\alpha}}\left\{\int_0^y\left[\frac{\Psi(Q(t))}{Q(t)}\right]^{\frac{\gamma}{\gamma-1}}dt\right\}^{\frac{\gamma-1}{\gamma}}.$$

The following theorems deal with slight variants of (3.8) given in Theorem 3.2. Before we state our next theorem, we point out that "$F(s)=\int_0^s f(\sigma)d\sigma$ and $G(t)=\int_0^t g(\tau)d\tau$" are replaced by "$F(s)=\frac{1}{s}\int_0^s f(\sigma)d\sigma$ and $G(t)=\frac{1}{t}\int_0^t g(\tau)d\tau$" in Theorem 3.4 in [9].

Theorem 3.3. *Let $\alpha>1$, $\gamma>1$ and $f(\sigma)\geq 0$, $g(\tau)\geq 0$ for $\sigma\in(0,x)$, $\tau\in(0,y)$, where x, y are positive real numbers. Define $F(s)=\frac{1}{s}\int_0^s f(\sigma)d\sigma$, $G(t)=\frac{1}{t}\int_0^t g(\tau)d\tau$ for $s\in(0,x)$, $t\in(0,y)$. Let Φ and Ψ be two real-valued, nonnegative, convex functions defined on $\mathbb{R}_+=[0,\infty)$. Then*

$$\int_0^x\int_0^y\frac{st\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}dsdt$$
$$\leq D(1,1,x,y;\alpha,\gamma)\left\{\int_0^x(x-s)\Phi^{\alpha}(f(s))ds\right\}^{\frac{1}{\alpha}}\left\{\int_0^y(y-t)\Psi^{\gamma}(g(t))dt\right\}^{\frac{1}{\gamma}},$$

where $D(1,1,x,y;\alpha,\gamma)=\frac{1}{\alpha+\gamma}x^{\frac{\alpha-1}{\alpha}}y^{\frac{\gamma-1}{\gamma}}$.

Theorem 3.4. *Let $\alpha>1,\gamma>1$ and $f(\sigma)\geq 0, g(\tau)\geq 0, p(\sigma)>0$ and $q(\tau)>0$ for $\sigma\in(0,x)$, $\tau\in(0,y)$, where x, y are positive real numbers. Define $P(s)=\int_0^s p(\sigma)d\sigma$, $Q(t)=\int_0^t q(\tau)d\tau$, $F(s)=\frac{1}{P(s)}\int_0^s p(\sigma)f(\sigma)d\sigma$ and $G(t)=\frac{1}{Q(t)}\int_0^t q(\tau)g(\tau)d\tau$ for $s\in(0,x)$, $t\in(0,y)$. Let Φ and Ψ be two real-valued, nonnegative, convex functions defined on $\mathbb{R}_+=[0,\infty)$. Then*

$$\int_0^x\int_0^y\frac{P(s)Q(t)\Phi(F(s))\Psi(G(t))}{\gamma s^{\frac{(\alpha-1)(\alpha+\gamma)}{\alpha\gamma}}+\alpha t^{\frac{(\gamma-1)(\alpha+\gamma)}{\alpha\gamma}}}dsdt$$
$$\leq D(1,1,x,y;\alpha,\gamma)\left\{\int_0^x(x-s)\left[p(s)\Phi\left(f(s)\right)\right]^{\alpha}ds\right\}^{\frac{1}{\alpha}}\left\{\int_0^y(y-t)\left[q(t)\Psi\left(g(t)\right)\right]^{\gamma}dt\right\}^{\frac{1}{\gamma}},$$

where $D(1,1,k,r;\alpha,\gamma)=\frac{1}{\alpha+\gamma}x^{\frac{\alpha-1}{\alpha}}y^{\frac{\gamma-1}{\gamma}}$.

The proofs of Theorems 3.3 and 3.4 are similar to the proof of Theorem 3.2 and similar to the proofs of Theorems 2.3 and 2.4. Hence, we leave out the details.

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