

Journal of Inequalities in Pure and Applied Mathematics

COEFFICIENT INEQUALITIES FOR CLASSES OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS

SHIGEYOSHI OWA, YAŞAR POLATOĞLU AND EMEL YAVUZ

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
EMail: owa@math.kindai.ac.jp

Department of Mathematics and Computer Science
İstanbul Kültür University
Bakırköy 34156, İstanbul, Turkey
EMail: y.polatoglu@iku.edu.tr
EMail: e.yavuz@iku.edu.tr

©2000 Victoria University
ISSN (electronic): 1443-5756
246-06



volume 7, issue 5, article 160,
2006.

*Received 28 September, 2006;
accepted 06 November, 2006.*

Communicated by: N.E. Cho

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)



Abstract

In view of classes of uniformly starlike and convex functions in the open unit disc \mathbb{U} which was considered by S. Shams, S.R. Kulkarni and J.M. Jahangiri, some coefficient inequalities for functions are discussed.

2000 Mathematics Subject Classification: Primary 30C45.

Key words: Uniformly starlike, Uniformly convex.

Contents

1	Introduction	3
2	Coefficient Inequalities	5
	References	



Coefficient Inequalities for Classes of Uniformly Starlike and Convex Functions

Shigeyoshi Owa, Yaşar Polatoğlu
and Emel Yavuz

Title Page

Contents



Go Back

Close

Quit

Page 2 of 11

1. Introduction

Let \mathcal{A} be the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Let $\mathcal{S}^*(\beta)$ denote the subclass of \mathcal{A} consisting of functions $f(z)$ which satisfy

$$(1.2) \quad \operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > \beta \quad (z \in \mathbb{U})$$

for some β ($0 \leq \beta < 1$). A function $f(z) \in \mathcal{S}^*(\beta)$ is said to be starlike of order β in \mathbb{U} . Also let $\mathcal{K}(\beta)$ be the subclass of \mathcal{A} consisting of all functions $f(z)$ which satisfy

$$(1.3) \quad \operatorname{Re} \left(1 + \frac{z f''(z)}{f'(z)} \right) > \beta \quad (z \in \mathbb{U})$$

for some β ($0 \leq \beta < 1$). A function $f(z)$ in $\mathcal{K}(\beta)$ is said to be convex of order β in \mathbb{U} . In view of the class $\mathcal{S}^*(\beta)$, Shams, Kulkarni and Jahangiri [3] have introduced the subclass $\mathcal{SD}(\alpha, \beta)$ of \mathcal{A} consisting of functions $f(z)$ satisfying

$$(1.4) \quad \operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > \alpha \left| \frac{z f'(z)}{f(z)} - 1 \right| + \beta \quad (z \in \mathbb{U})$$



**Coefficient Inequalities for
Classes of Uniformly Starlike
and Convex Functions**

Shigeyoshi Owa, Yaşar Polatoğlu
and Emel Yavuz

Title Page

Contents



Go Back

Close

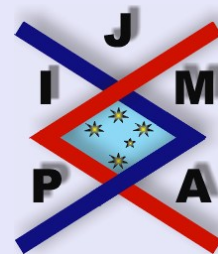
Quit

Page 3 of 11

for some $\alpha \geq 0$ and β ($0 \leq \beta < 1$). We also denote by $\mathcal{KD}(\alpha, \beta)$ the subclass of \mathcal{A} consisting of all functions $f(z)$ which satisfy

$$(1.5) \quad \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \left| \frac{zf''(z)}{f'(z)} \right| + \beta \quad (z \in \mathbb{U})$$

for some $\alpha \geq 0$ and β ($0 \leq \beta < 1$). Then we note that $f(z) \in \mathcal{KD}(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{SD}(\alpha, \beta)$. For such classes $\mathcal{SD}(\alpha, \beta)$ and $\mathcal{KD}(\alpha, \beta)$, Shams, Kulkarni and Jahangiri [3] have shown some sufficient conditions for $f(z)$ to be in the classes $\mathcal{SD}(\alpha, \beta)$ or $\mathcal{KD}(\alpha, \beta)$.



**Coefficient Inequalities for
Classes of Uniformly Starlike
and Convex Functions**

Shigeyoshi Owa, Yaşar Polatoğlu
and Emel Yavuz

Title Page

Contents



Go Back

Close

Quit

Page 4 of 11

2. Coefficient Inequalities

Our first result is contained in

Theorem 2.1. *If $f(z) \in \mathcal{SD}(\alpha, \beta)$ with $0 \leq \alpha \leq \beta$ or $\alpha > \frac{1+\beta}{2}$ then $f(z) \in \mathcal{S}^* \left(\frac{\beta-\alpha}{1-\alpha} \right)$.*

Proof. Since $\operatorname{Re}(w) \leq |w|$ for any complex number w , $f(z) \in \mathcal{SD}(\alpha, \beta)$ implies that

$$(2.1) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \operatorname{Re} \left(\frac{zf'(z)}{f(z)} - 1 \right) + \beta,$$

or that

$$(2.2) \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \frac{\beta - \alpha}{1 - \alpha} \quad (z \in \mathbb{U}).$$

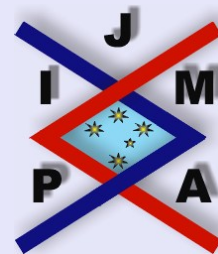
If $0 \leq \alpha \leq \beta$, then we have that

$$0 \leq \frac{\beta - \alpha}{1 - \alpha} < 1,$$

and if $\alpha > \frac{1 + \beta}{2}$, then we have

$$-1 < \frac{\alpha - \beta}{\alpha - 1} \leq 0.$$

□



**Coefficient Inequalities for
Classes of Uniformly Starlike
and Convex Functions**

Shigeyoshi Owa, Yaşar Polatoğlu
and Emel Yavuz

Title Page

Contents



Go Back

Close

Quit

Page 5 of 11

Corollary 2.2. If $f(z) \in \mathcal{KD}(\alpha, \beta)$ with $0 \leq \alpha \leq \beta$ or $\alpha > \frac{1+\beta}{2}$, then $f(z) \in \mathcal{K}\left(\frac{\beta-\alpha}{1-\alpha}\right)$.

Next we derive

Theorem 2.3. If $f(z) \in \mathcal{SD}(\alpha, \beta)$, then

$$(2.3) \quad |a_2| \leq \frac{2(1-\beta)}{|1-\alpha|}$$

and

$$(2.4) \quad |a_n| \leq \frac{2(1-\beta)}{(n-1)|1-\alpha|} \prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|}\right) \quad (n \geq 3).$$

Proof. Note that, for $f(z) \in \mathcal{SD}(\alpha, \beta)$,

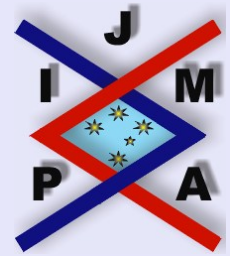
$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \frac{\beta - \alpha}{1 - \alpha} \quad (z \in \mathbb{U}).$$

If we define the function $p(z)$ by

$$(2.5) \quad p(z) = \frac{(1-\alpha)\frac{zf'(z)}{f(z)} - (\beta - \alpha)}{1 - \beta} \quad (z \in \mathbb{U}),$$

then $p(z)$ is analytic in \mathbb{U} with $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$ ($z \in \mathbb{U}$). Letting $p(z) = 1 + p_1z + p_2z^2 + \dots$, we have

$$(2.6) \quad zf'(z) = f(z) \left(1 + \frac{1-\beta}{1-\alpha} \sum_{n=1}^{\infty} p_n z^n \right).$$



**Coefficient Inequalities for
Classes of Uniformly Starlike
and Convex Functions**

Shigeyoshi Owa, Yaşar Polatoğlu
and Emel Yavuz

Title Page

Contents



Go Back

Close

Quit

Page 6 of 11

Therefore, (2.6) implies that

$$(2.7) \quad (n-1)a_n = \frac{1-\beta}{1-\alpha}(p_{n-1} + a_2 p_{n-2} + \cdots + a_{n-1} p_1).$$

Applying the coefficient estimates such that $|p_n| \leq 2$ ($n \geq 1$) (see [1]) for Carathéodory functions, we obtain that

$$(2.8) \quad |a_n| \leq \frac{2(1-\beta)}{(n-1)|1-\alpha|}(1 + |a_2| + |a_3| + \cdots + |a_{n-1}|).$$

Therefore, for $n = 2$,

$$|a_2| \leq \frac{2(1-\beta)}{|1-\alpha|},$$

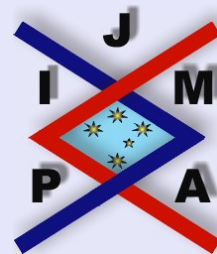
which proves (2.3), and, for $n = 3$,

$$|a_3| \leq \frac{2(1-\beta)}{2|1-\alpha|} \left(1 + \frac{2(1-\beta)}{|1-\alpha|} \right).$$

Thus, (2.4) holds true for $n = 3$.

Supposing that (2.4) is true for $n = 3, 4, 5, \dots, k$, we see that

$$\begin{aligned} |a_{k+1}| &\leq \frac{2(1-\beta)}{k|1-\alpha|} \left\{ 1 + \frac{2(1-\beta)}{|1-\alpha|} + \frac{2(1-\beta)}{2|1-\alpha|} \left(1 + \frac{2(1-\beta)}{|1-\alpha|} \right) \right. \\ &\quad \left. + \cdots + \frac{2(1-\beta)}{(k-1)|1-\alpha|} \prod_{j=1}^{k-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right) \right\} \\ &= \frac{2(1-\beta)}{k|1-\alpha|} \prod_{j=1}^{k-1} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right). \end{aligned}$$



**Coefficient Inequalities for
Classes of Uniformly Starlike
and Convex Functions**

Shigeyoshi Owa, Yaşar Polatoğlu
and Emel Yavuz

Title Page

Contents



Go Back

Close

Quit

Page 7 of 11

Consequently, using mathematical induction, we have proved that (2.4) holds true for any $n \geq 3$. \square

Remark 1. If we take $\alpha = 0$ in Theorem 2.3, then we have

$$|a_n| \leq \frac{\prod_{j=2}^n (j - 2\beta)}{(n - 1)!} \quad (n \geq 2)$$

which was given by Robertson [2].

Since $f(z) \in \mathcal{KD}(\alpha, \beta)$ if and only if $zf'(z) \in \mathcal{SD}(\alpha, \beta)$, we have

Corollary 2.4. If $f(z) \in \mathcal{KD}(\alpha, \beta)$, then

$$(2.9) \quad |a_2| \leq \frac{1 - \beta}{|1 - \alpha|}$$

and

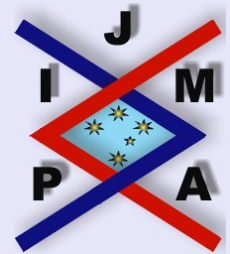
$$(2.10) \quad |a_n| \leq \frac{2(1 - \beta)}{n(n - 1)|1 - \alpha|} \prod_{j=1}^{n-2} \left(1 + \frac{2(1 - \beta)}{j|1 - \alpha|} \right) \quad (n \geq 3).$$

Remark 2. Letting $\alpha = 0$ in Corollary 2.4, we see that

$$|a_n| \leq \frac{\prod_{j=2}^n (j - 2\beta)}{n!} \quad (n \geq 2),$$

given by Robertson [2].

Further applying Theorem 2.3 we derive:



**Coefficient Inequalities for
Classes of Uniformly Starlike
and Convex Functions**

Shigeyoshi Owa, Yaşar Polatoğlu
and Emel Yavuz

Title Page

Contents



Go Back

Close

Quit

Page 8 of 11

Theorem 2.5. If $f(z) \in \mathcal{SD}(\alpha, \beta)$, then

$$\max \left\{ 0, |z| - \frac{2(1-\beta)}{|1-\alpha|} |z|^2 - \sum_{n=3}^{\infty} \frac{2(1-\beta)}{(n-1)|1-\alpha|} \left(\prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right) \right) |z|^n \right\}$$

$$\leq |f(z)| \leq |z| + \frac{2(1-\beta)}{|1-\alpha|} |z|^2 + \sum_{n=3}^{\infty} \frac{2(1-\beta)}{(n-1)|1-\alpha|} \left(\prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right) \right) |z|^n$$

and

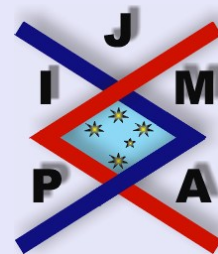
$$\max \left\{ 0, 1 - \frac{4(1-\beta)}{|1-\alpha|} |z| - \sum_{n=3}^{\infty} \frac{2n(1-\beta)}{(n-1)|1-\alpha|} \left(\prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right) \right) |z|^{n-1} \right\}$$

$$\leq |f'(z)| \leq 1 + \frac{4(1-\beta)}{|1-\alpha|} |z| + \sum_{n=3}^{\infty} \frac{2n(1-\beta)}{(n-1)|1-\alpha|} \left(\prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right) \right) |z|^{n-1}.$$

Corollary 2.6. If $f(z) \in \mathcal{KD}(\alpha, \beta)$, then

$$\max \left\{ 0, |z| - \frac{1-\beta}{|1-\alpha|} |z|^2 - \sum_{n=3}^{\infty} \frac{2(1-\beta)}{n(n-1)|1-\alpha|} \left(\prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right) \right) |z|^n \right\}$$

$$\leq |f(z)| \leq |z| + \frac{1-\beta}{|1-\alpha|} |z|^2 + \sum_{n=3}^{\infty} \frac{2(1-\beta)}{n(n-1)|1-\alpha|} \left(\prod_{j=1}^{n-2} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right) \right) |z|^n$$



**Coefficient Inequalities for
Classes of Uniformly Starlike
and Convex Functions**

Shigeyoshi Owa, Yaşar Polatoğlu
and Emel Yavuz

Title Page

Contents



Go Back

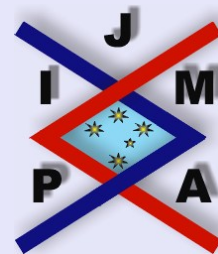
Close

Quit

Page 9 of 11

and

$$\max \left\{ 0, 1 - \frac{2(1-\beta)}{|1-\alpha|} |z| - \sum_{n=3}^{\infty} \frac{2(1-\beta)}{(n-1)|1-\alpha|} \left(\prod_{j=1}^{n-1} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right) \right) |z|^{n-1} \right\}$$
$$\leq |f'(z)| \leq 1 + \frac{2(1-\beta)}{|1-\alpha|} |z| + \sum_{n=3}^{\infty} \frac{2(1-\beta)}{(n-1)|1-\alpha|} \left(\prod_{j=1}^{n-1} \left(1 + \frac{2(1-\beta)}{j|1-\alpha|} \right) \right) |z|^{n-1}.$$



**Coefficient Inequalities for
Classes of Uniformly Starlike
and Convex Functions**

Shigeyoshi Owa, Yaşar Polatoğlu
and Emel Yavuz

Title Page

Contents



Go Back

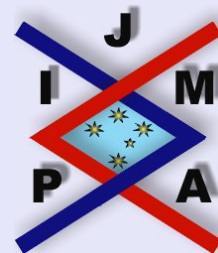
Close

Quit

Page 10 of 11

References

- [1] C. CARATHÉODORY, Über den variabilitätsbereich der Fourier'schen konstanten von positiven harmonischen funktionen, *Rend. Circ. Palermo*, **32** (1911), 193–217.
- [2] M.S. ROBERTSON, On the theory of univalent functions, *Ann. Math.*, **37** (1936), 374–408.
- [3] S. SHAMS, S.R. KULKARNI AND J.M. JAHANGIRI, Classes of uniformly starlike and convex functions, *Internat. J. Math. Math. Sci.*, **55** (2004), 2959–2961.



Coefficient Inequalities for Classes of Uniformly Starlike and Convex Functions

Shigeyoshi Owa, Yaşar Polatoğlu
and Emel Yavuz

Title Page

Contents



Go Back

Close

Quit

Page 11 of 11