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ON CERTAIN SUBCLASS OF BAZILEVIČ FUNCTIONS

DONG GUO AND MING-SHENG LIU

SCHOOL OF MATHEMATICAL SCIENCES,
SOUTH CHINA NORMAL UNIVERSITY,
GUANGZHOU 510631, GUANGDONG,
P. R. CHINA

liumsh@scnu.edu.cn

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ABSTRACT. Let $\mathcal{H}$ be the class of functions $f(z)$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, which are analytic in the unit disk $U = \{z : |z| < 1\}$. In this paper, the authors introduce a subclass $\mathscr{M}(\alpha, \lambda, \rho)$ of $\mathcal{H}$ and study its some properties. The subordination relationships, inclusion relationships, coefficient estimates, the integral operator and covering theorem are proven here for each of the function classes. Furthermore, some interesting Fekete-Szegö inequalities are obtained. Some of the results, presented in this paper, generalize the corresponding results of earlier authors.

Key words and phrases: Starlike function, Bazilevič Function, Subordination relationships, Inclusion relationship, Coefficient estimates, Integral operator, Covering theorem, Fekete-Szegö inequalities.

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1. INTRODUCTION

Let $\mathcal{H}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$, and let $\mathcal{S}$ denote the class of all functions in $\mathcal{H}$ which are univalent in the disk U. Suppose also that $\mathcal{S}^*, \mathcal{K}$ and $\alpha - \mathcal{K}$ denote the familiar subclasses of $\mathcal{H}$ consisting of functions which are, respectively, starlike in $\mathcal{U}$, convex in $\mathcal{U}$ and $\alpha-$ convex in $\mathcal{U}$. Thus we have

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{H} \text{ and } \mathcal{R}\left\{ \frac{zf'(z)}{f(z)} \right\} > 0, \quad z \in \mathcal{U} \right\},$$

$$\mathcal{K} = \left\{ f : f \in \mathcal{H} \text{ and } \mathcal{R}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in \mathcal{U} \right\}$$

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and

$$\alpha-\mathcal{K}=\left\{f: f\in\mathcal{H} \text{ and } \mathcal{R}\left\{\alpha\left[1+\frac{zf''(z)}{f'(z)}\right]+(1-\alpha)\frac{zf'(z)}{f(z)}\right\}>0, \quad z\in\mathcal{U}\right\}.$$

Let $f(z)$ and $F(z)$ be analytic in U. Then we say that the function $f(z)$ is subordinate to $F(z)$ in U, if there exists an analytic function $\omega(z)$ in U such that $|\omega(z)|\le|z|$ and $f(z)=F(\omega(z))$, denoted $f\prec F$ or $f(z)\prec F(z)$. If $F(z)$ is univalent in U, then the subordination is equivalent to $f(0)=F(0)$ and $f(U)\subset F(U)$ (see [18]).

Assuming that $\alpha>0,\ \lambda\ge 0,\ \rho<1$, a function $p(z)=1+p_1z+p_2z^2+\cdots$ is said to be in the class $\mathcal{P}_\rho$ if and only if $p(z)$ is analytic in the unit disk U and $\mathcal{R}p(z)>\rho, z\in U$. A function $f(z)\in H$ is said to be in the class $B(\lambda,\alpha,\rho)$ if and only if it satisfies

$$\mathcal{R}(1-\lambda)\left[\left(\frac{f(z)}{z}\right)^\alpha+\lambda\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\alpha\right]>\rho, \quad z\in\mathcal{U}, \tag{1.2}$$

where we choose the branch of the power $\left(\frac{f(z)}{z}\right)^\alpha$ such that $\left.\left(\frac{f(z)}{z}\right)^\alpha\right|_{z=0}=1$. It is obvious that the subclass $B(1,\alpha,0)$ is the subclass of Bazilevič functions, which is the subclass of univalent functions $\mathcal{S}$, we set $B(\alpha,\rho)\equiv B(1,\alpha,\rho)$. The function class $B(\lambda,\alpha,\rho)$ was introduced and studied by Liu [10]. Some special cases of the function class $B(\lambda,\alpha,\rho)$ had been studied by Bazilevič [1], Chichra [2], Ding, Ling and Bao [3], Liu [9] and Singh [19], respectively.

Liu [11] introduced the following class $B(\lambda,\alpha,A,B,g(z))$ of analytic functions, and studied its some properties.

$$\begin{aligned}&B(\lambda,\alpha,A,B,g(z))\\&\qquad=\left\{f\in\mathcal{H}:\left(1-\lambda\frac{zg'(z)}{g(z)}\right)\left(\frac{f(z)}{g(z)}\right)^\alpha+\lambda\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^\alpha\prec\frac{1+Az}{1+Bz}\right\},\end{aligned}$$

where $\alpha>0,\lambda\ge0,-1\le B<A\le1,\ g(z)\in S^*$.

Fekete and Szegö [4] showed that for $f\in S$ given by (1.1),

$$|a_3-\mu a_2^2|\le\begin{cases}3-4u, & \text{if } \mu\le 0,\\ 1+2e^{-2/(1-\mu)}, & \text{if } 0\le\mu<1,\\ 4-3\mu, & \text{if } \mu\ge1.\end{cases}$$

As a result, many authors studied similar problems for some subclasses of $\mathcal{H}$ or $\mathcal{S}$ (see [6, 7, 8, 13, 14, 15, 20]), which is popularly referred to as the Fekete-Szegö inequality or the Fekete-Szegö problem. Li and Liu [12] obtained the Fekete-Szegö inequality for the function class $B(\lambda,\alpha,\rho)$.

Recently, Patel [17] introduced the following subclass $\mathcal{M}_p(\lambda,\mu,A,B)$ of $p-$valent Bazilevič functions, and studied some of its properties.

An analytic function $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ is said to be in the class $\mathcal{M}_p(\lambda,\mu,A,B)$ if and only if there exists a $p-$valent starlike function $g(z)=z^p+\sum_{n=p+1}^{\infty}b_nz^n$ such that

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^\mu+\lambda\left[1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}+\mu\left(\frac{zf'(z)}{f(z)}-\frac{zg'(z)}{g(z)}\right)\right]\prec p\frac{1+Az}{1+Bz},$$

where $\mu\ge0,\lambda>0,-1\le B<A\le1$.

In the present paper, we introduce the following subclass of analytic functions, and obtain some interesting results.

Definition 1.1. Assume that $\alpha \geq 0,\ \lambda \geq 0,\ 0 \leq \rho < 1,\ f \in \mathcal{H}$. We say that $f(z) \in \mathscr{M}(\alpha, \lambda, \rho)$ if and only if $f(z)$ satisfies the following inequality:

$$\mathcal{R}\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right]\right\} > \rho, \quad z \in \mathcal{U}.$$

It is evident that $\mathscr{M}(\alpha, 0, \rho) = B(\alpha, \rho)(\alpha \geq 0)$ and $\mathscr{M}(0, \alpha, 0) = \alpha - \mathcal{K}(\alpha \geq 0)$.

2. PRELIMINARIES

To derive our main results, we shall require the following lemmas.

Lemma 2.1 ([16]). *If $-1 \leq B < A \leq 1, \beta > 0$ and the complex number γ satisfies $\mathcal{R}(\lambda) \geq \frac{-\beta(1-A)}{1-B}$, then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1 + Az}{1 + Bz}, \quad z \in \mathcal{U},$$

has a univalent solution in $\mathcal{U}$ given by

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}dt} - \frac{\gamma}{\beta}, & B \neq 0, \\[2ex] \dfrac{z^{\beta+\gamma}\exp^{(\beta Az)}}{\beta\int_0^z t^{\beta+\gamma-1}\exp(\beta At)dt} - \frac{\gamma}{\beta}, & B = 0. \end{cases} \tag{2.1}$$

If $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathcal{U}$ and satisfies

$$\phi(z) + \frac{z\phi'(z)}{\beta\phi(z) + \gamma} \prec \frac{1 + Az}{1 + Bz}, \quad (z \in \mathcal{U}), \tag{2.2}$$

then

$$\phi(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz}, \quad (z \in \mathcal{U}),$$

and $q(z)$ is the best dominant of (2.2).

Lemma 2.2 ([11]). *Suppose that $F(z)$ is analytic and convex in $\mathcal{U}$, and $0 \leq \lambda \leq 1$, $f(z) \in \mathcal{H}$, $g(z) \in \mathcal{H}$. If $f(z) \prec F(z)$ and $g(z) \prec F(z)$. Then*

$$\lambda f(z) + (1 - \lambda)g(z) \prec F(z).$$

Lemma 2.3 ([18]). *Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}_0$. Then*

$$\left|p_2 - \frac{1}{2}p_1^2\right| \leq 2 - \frac{1}{2}|p_1^2|$$

and $|p_n| \leq 2$ for all $n \in \mathbb{N}_+$.

Lemma 2.4 ([1]). *Let $\alpha \geq 0$, $f \in \mathcal{H}$ and for $|z| < R \leq 1$,*

$$\mathcal{R}\left[\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}\right] > 0,$$

then $f(z)$ is univalent in $|z| < R$.

3. MAIN RESULTS AND THEIR PROOFS

Theorem 3.1. *Let $\alpha \geq 0$ and $\lambda > 0$. If $f(z) \in \mathscr{M}(\alpha, \lambda, \rho)$. Then*

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} \prec q(z) \prec \frac{1+(1-2\rho)z}{1-z}, \quad (z \in \mathcal{U}), \tag{3.1}$$

where

$$q(z) = \frac{\lambda z^{1/\lambda}(1-z)^{-2(1-\rho)/\lambda}}{\int_0^z t^{(1-\lambda)/\lambda}(1-t)^{-2(1-\rho)/\lambda} dt},$$

and $q(z)$ is the best dominant of (3.1).

Proof. By applying the method of the proof of Theorem 3.1 in [17] mutatis mutandis, we can prove this theorem. □

With the aid of Lemma 2.4, from Theorem 3.1, we have the following inclusion relation.

Corollary 3.2. *Let $\alpha \geq 0$, $0 \leq \rho < 1$ and $\lambda \geq 0$, then*

$$\mathscr{M}(\alpha, \lambda, \rho) \subset \mathscr{M}(\alpha, 0, \rho) \subset \mathscr{M}(\alpha, 0, 0) \subset \mathcal{S}.$$

Theorem 3.3. *Let $\alpha \geq 0$ and $\lambda_2 > \lambda_1 \geq 0$, $1 > \rho_2 \geq \rho_1 \geq 0$, then*

$$\mathscr{M}(\alpha, \lambda_2, \rho_2) \subset \mathscr{M}(\alpha, \lambda_1, \rho_1).$$

Proof. Suppose that $f(z) \in \mathscr{M}(\alpha, \lambda_2, \rho_2)$. Then, by the definition of $\mathscr{M}(\alpha, \lambda_2, \rho_2)$, we have

$$\mathcal{R}\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda_2\left[1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)}-1\right)\right]\right\} > \rho_2 \qquad (z \in \mathcal{U}). \tag{3.2}$$

Since $\alpha \geq 0$ and $\lambda_2 > \lambda_1 \geq 0$, by Theorem 3.1, we obtain

$$\mathcal{R}\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}\right\} > \rho_2 \quad (z \in \mathcal{U}). \tag{3.3}$$

Setting $\lambda = \frac{\lambda_1}{\lambda_2}$, so that $0 \leq \lambda < 1$, we find from (3.2) and (3.3) that

$$\begin{aligned}
&\mathcal{R}\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda_1\left[1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)}-1\right)\right]\right\} \\
&\qquad = \lambda\mathcal{R}\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda_2[1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)}-1\right)\right\} \\
&\qquad\qquad + (1-\lambda)\mathcal{R}\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} > \rho_2 \geq \rho_1 \quad (z \in \mathcal{U}),
\end{aligned}$$

that is, $f(z) \in \mathscr{M}(\alpha, \lambda_1, \rho_1)$. Hence, we have $\mathscr{M}(\alpha, \lambda_2, \rho_2) \subset \mathscr{M}(\alpha, \lambda_1, \rho_1)$, and the proof of Theorem 3.3 is complete. □

Remark 3.4. Theorem 3.3 obviously provides a refinement of Corollary 3.2. Setting $\alpha = 0, \rho_2 = \rho_1 = 0$ in Theorem 3.3, we get Theorem 9.4 of [5].

With the aid of Lemma 2.2, by using the method of our proof of Theorem 3.3, we can prove the following inclusion relation.

Theorem 3.5. *Let $\mu \geq 0$, $-1 \leq B < A \leq 1$ and $\lambda_2 > \lambda_1 \geq 0$, then*

$$\mathcal{M}_p(\lambda_2, \mu, A, B) \subset \mathcal{M}_p(\lambda_1, \mu, A, B).$$

By applying the method of the proof of Theorem 3.13, Theorem 3.6 and Theorem 3.11 in [17] mutatis mutandis, we can prove the following three results.

Theorem 3.6. *Let $\alpha \geq 0$, $\lambda > 0$ and $\gamma > 0$. If $f(z) \in \mathcal{H}$ satisfies*

$$\gamma \left[\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\alpha} \right] + \lambda \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf'(z)}{f(z)} - 1 \right) \right] \neq it, \quad (z \in \mathcal{U}),$$

where t is a real number satisfying $|t| > \sqrt{\lambda(\lambda + 2\gamma)}$, then

$$\mathcal{R} \left\{ \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\alpha} \right\} > 0, \quad (z \in \mathcal{U}).$$

Theorem 3.7. *Suppose that $\alpha > 0$ and $0 \leq \rho < 1$. If $f(z) \in H$ satisfies*

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\alpha} \right\} > \rho, \quad (z \in \mathcal{U}),$$

then $f(z) \in \mathscr{M}(\alpha, \lambda, \rho)$ for $|z| < R(\lambda, \rho)$, where $\lambda > 0$, and

$$R(\lambda, \rho) = \begin{cases} \frac{(1+\lambda-\rho) - \sqrt{(1+\lambda-\rho)^2 - (1-2\rho)}}{1-2\rho}, & \rho \neq \frac{1}{2}, \\ \frac{1}{1+2\lambda}, & \rho = \frac{1}{2}. \end{cases}$$

The bound $R(\lambda, \rho)$ is the best possible.

For a function $f \in \mathcal{H}$, we define the integral operator $F_{\alpha,\delta}$ as follows:

$$F_{\alpha,\delta}(f) = F_{\alpha,\delta}(f)(z) = \left(\frac{\alpha + \delta}{z^{\delta}} \int_0^z t^{\delta - 1} f(t)^{\alpha} dt \right)^{\frac{1}{\alpha}} \quad (z \in \mathcal{U}), \tag{3.4}$$

where α and δ are real numbers with $\alpha > 0$, $\delta > -\alpha$.

Theorem 3.8. *Let α and δ be real numbers with $\alpha > 0$, $0 \leq \rho < 1$, $\delta > \max\{-\alpha, -\alpha\rho\}$ and let $f(z) \in \mathcal{H}$. If*

$$\left| \arg \left[\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\alpha} - \rho \right] \right| \leq \frac{\pi}{2} \beta \quad (0 \leq \rho < 1; 0 < \beta \leq 1),$$

then

$$\left| \arg \left[\frac{zF'_{\alpha,\delta}(f)}{F_{\alpha,\delta}(f)} \left(\frac{F_{\alpha,\delta}(f)}{z} \right)^{\alpha} - \rho \right] \right| \leq \frac{\pi}{2} \beta,$$

where $F_{\alpha,\delta}(f)$ is the operator given by (3.4).

Now we derive the Fekete-Szegö inequality for the function class $\mathscr{M}(\alpha, \lambda, \rho)$.

Theorem 3.9. *Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{M}(\alpha, \lambda, \rho)$. Then*

$$|a_2| \leq \frac{2(1-\rho)}{(1+\lambda)(1+\alpha)},$$

and for each $\mu \in C$, the following bound is sharp

$$|a_3 - \mu a_2^2| \leq \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)} \times \max \left\{ 1, \left| 1 + \frac{(1-\rho)[2\lambda(3+\alpha) - (2+\alpha)(\alpha - 1 + 2\mu + 4\mu\lambda)]}{(1+\lambda)^2(1+\alpha)^2} \right| \right\}.$$

Proof. Since $f(z) \in \mathscr{M}(\alpha, \lambda, \rho)$, by Definition 1.1, there exists a function $p(z) = 1+\sum_{k=1}^{+\infty} p_k z^k \in \mathcal{P}_0$, such that

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)} - 1\right)\right] = (1-\rho)p(z) + \rho, \quad z \in \mathcal{U}.$$

Equating coefficients, we obtain

$$a_2 = \frac{1-\rho}{(1+\lambda)(1+\alpha)}p_1,$$

$$a_3 = \frac{1-\rho}{(1+2\lambda)(2+\alpha)}p_2 + \frac{(1-\rho)^2\left[\lambda(3+\alpha) - \frac{(\alpha+2)(\alpha-1)}{2}\right]}{(1+\lambda)^2(1+\alpha)^2(1+2\lambda)(2+\alpha)}p_1^2.$$

Thus, we have

$$a_3 - \mu a_2^2 = \frac{1-\rho}{(1+2\lambda)(2+\alpha)}\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{(1-\rho)^2[2\lambda(3+\alpha) - (2+\alpha)(\alpha-1) - 2\mu(1+2\lambda)(2+\alpha)] + (1-\rho)(1+\lambda)^2(1+\alpha)^2}{2(1+\lambda)^2(1+\alpha)^2(1+2\lambda)(2+\alpha)}p_1^2.$$

By Lemma 2.3, we obtain that $|a_2| = \frac{1-\rho}{(1+\lambda)(1+\alpha)}|p_1| \leq \frac{2(1-\rho)}{(1+\lambda)(1+\alpha)}$, and

$$|a_3 - \mu a_2^2| \leq H(x) = A + \frac{ABx^2}{4},$$

where $x = |p_1| \leq 2$,

$$A = \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)}, \qquad B = \frac{|C| - (1+\lambda)^2(1+\alpha)^2}{(1+\lambda)^2(1+\alpha)^2},$$

and

$$C = (1+\lambda)^2(1+\alpha)^2 + (1-\rho)[2\lambda(3+\alpha) - (2+\alpha)(\alpha - 1 + 2\mu + 4\mu\lambda)].$$

So, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} H(0) = A, & |c| \leq (1+\lambda)^2(1+\alpha)^2, \\ H(2) = \frac{A|C|}{(1+\lambda)^2(1+\alpha)^2}, & |c| \geq (1+\lambda)^2(1+\alpha)^2. \end{cases}$$

Here equality is attained for the function given by

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} = \begin{cases} \dfrac{\lambda z^{1/\lambda}(1-z^2)^{(\rho-1)/\lambda}}{\int_0^z t^{(1-\lambda)/\lambda}(1-t^2)^{(\rho-1)/\lambda}dt}, & \lambda > 0,\ |c| \leq (1+\lambda)^2(1+\alpha)^2, \\ \dfrac{1+(1-2\rho)z^2}{1-z^2}, & \lambda = 0,\ |c| \leq (1+\lambda)^2(1+\alpha)^2 \\ \dfrac{\lambda z^{1/\lambda}(1-z)^{2(\rho-1)/\lambda}}{\int_0^z t^{(1-\lambda)/\lambda}(1-t)^{2(\rho-1)/\lambda}dt}, & \lambda > 0,\ |c| \geq (1+\lambda)^2(1+\alpha)^2, \\ \dfrac{1+(1-2\rho)z}{1-z}, & \lambda = 0,\ |c| \geq (1+\lambda)^2(1+\alpha)^2. \end{cases} \tag{3.5}$$

□

Setting $\lambda = 0$ in Theorem 3.9, we have the following corollary.

Corollary 3.10. *If $f(z) \in B(\alpha, \rho)$ given by (1.1), then*

$$|a_2| \leq \frac{2(1-\rho)}{1+\alpha},$$

and for each $\mu \in C$, the following bound is sharp

$$|a_3 - \mu a_2^2| \leq \frac{2(1-\rho)}{2+\alpha} \max\left\{1, \left|1 + \frac{(1-\rho)(2+\alpha)(1-2\mu-\alpha)}{(1+\alpha)^2}\right|\right\}.$$

Notice that $\mathscr{M}(0, \alpha, 0) \equiv \alpha - \mathcal{K}$, and from Theorem 3.9, we have the following corollary.

Corollary 3.11. *Let $\alpha \geq 0$. If $f(z) \in \alpha - \mathcal{K}$ given by (1.1). Then*

$$|a_2| \leq \frac{2}{1+\alpha},$$

and for each $\mu \in C$, the following bound is sharp

$$|a_3 - \mu a_2^2| \leq \frac{1}{1+2\alpha} \max\left\{1, \left|1 + \frac{6\alpha + 2 - 4\mu - 8\mu\alpha}{(1+\alpha)^2}\right|\right\}.$$

Theorem 3.12 (Covering Theorem). *Let $\alpha \geq 0, \lambda \geq 0$ and $f(z) \in \mathscr{M}(\alpha, \lambda, \rho)$, then the unit disk U is mapped by $f(z)$ on a domain that contains the disk $|\omega| < r_1$, where*

$$r_1 = \frac{(1+\alpha)(1+\lambda)}{2(1+\alpha)(1+\lambda) + 2(1-\rho)}.$$

Proof. Let ω_0 be any complex number such that $f(z) \neq \omega_0\, (z \in \mathcal{U})$, then $\omega_0 \neq 0$ and (by Corollary 3.2) the function

$$\frac{\omega_0 f(z)}{\omega_0 - f(z)} = z + \left(a_2 + \frac{1}{\omega_0}\right) z^2 + \cdots,$$

is univalent in $\mathcal{U}$, so that

$$\left|a_2 + \frac{1}{\omega_0}\right| \leq 2,$$

Therefore, according to Theorem 3.9, we obtain

$$|\omega_0| \geq \frac{(1+\alpha)(1+\lambda)}{2(1+\alpha)(1+\lambda) + 2(1-\rho)} = r_1.$$

Thus we have completed the proof of Theorem 3.12. □

Remark 3.13. Setting $\alpha = \lambda = \rho = 0$ in Theorem 3.12, we get the well-known $\frac{1}{4}-$ covering theorem for the familiar class $\mathcal{S}^*$ of starlike functions.

If $0 \leq \mu \leq \mu_1$ and μ is a real number, Theorem 3.9 can be improved as follows.

Theorem 3.14. *Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{M}(\lambda, \alpha, \rho)$ and $\mu \in R$. Then*

$$\begin{aligned} (3.6)\quad & |a_3 - \mu a_2^2| + \mu |a_2|^2 \\ & \leq \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)} \left\{1 + \frac{(1-\rho)[2\lambda(3+\alpha) - (2+\alpha)(\alpha-1)]}{(1+\lambda)^2(1+\alpha)^2}\right\}, \quad 0 \leq \mu \leq \mu_0, \end{aligned}$$

$$(3.7)\qquad |a_3 - \mu a_2^2| + (\mu_1 - \mu)|a_2|^2 \leq \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)}, \qquad \mu_0 \leq \mu \leq \mu_1,$$

and these inequalities are sharp, where

$$\mu_0 = \frac{1}{2} + \frac{2\lambda - \alpha(2+\alpha)}{2(1+2\lambda)(2+\alpha)} + \frac{(1+\lambda)^2(1+\alpha)^2}{2(1+2\lambda)(2+\alpha)(1-\rho)},$$

$$\mu_1 = \frac{1}{2} + \frac{2\lambda - \alpha(2+\alpha)}{2(1+2\lambda)(2+\alpha)} + \frac{(1+\lambda)^2(1+\alpha)^2}{(1+2\lambda)(2+\alpha)(1-\rho)}.$$

Proof. From Theorem 3.9, we get

$$(3.8) \quad |a_3 - \mu a_2^2| \le \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)} + \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)} \cdot \left[\frac{|(1-\rho)[2\lambda(3+\alpha) - (2+\alpha)\left[(\alpha-1) + 2\mu(1+2\lambda)\right]] + (1+\lambda)^2(1+\alpha)^2|}{4(1+\lambda)^2(1+\alpha)^2} - \frac{1}{4}\right] |p_1|^2.$$

Using (3.8) and $a_2 = \frac{1-\rho}{(1+\lambda)(1+\alpha)}p_1$, if $0 \le \mu \le \mu_0$, we obtain

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)} + \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)} \\ &\quad \times \frac{(1-\rho)[2\lambda(3+\alpha) - (2+\alpha)(\alpha-1) - 2\mu(1+2\lambda)(2+\alpha)]}{4(1+\lambda)^2(1+\alpha)^2}|p_1|^2 \\ &= \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)} + \frac{2(1-\rho)^2[2\lambda(3+\alpha) - (2+\alpha)(\alpha-1)]}{4(1+2\lambda)(2+\alpha)(1+\lambda)^2(1+\alpha)^2}|p_1|^2 - \mu|a_2|^2. \end{aligned}$$

Hence

$$\begin{aligned} &|a_3 - \mu a_2^2| + \mu|a_2|^2 \\ &\le \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)} + \frac{2(1-\rho)^2[2\lambda(3+\alpha) - (2+\alpha)(\alpha-1)]}{4(1+2\lambda)(2+\alpha)(1+\lambda)^2(1+\alpha)^2}|p_1|^2 \\ &\le \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)}\left\{1 + \frac{(1-\rho)[2\lambda(3+\alpha) - (2+\alpha)(\alpha-1)]}{(1+\lambda)^2(1+\alpha)^2}\right\}, \quad 0 \le \mu \le \mu_0. \end{aligned}$$

If $\mu_0 \le \mu \le \mu_1$, from (3.8), we obtain

$$\begin{aligned} &|a_3 - \mu a_2^2| \\ &\le \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)} + \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)} \\ &\quad \times \frac{-2(1+\lambda)^2(1+\alpha)^2 - (1-\rho)[2\lambda(3+\alpha) - (2+\alpha)(\alpha - 1 + 2\mu + 4\mu\lambda)]}{4(1+\lambda)^2(1+\alpha)^2}|p_1|^2 \\ &= \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)} - (\mu_1 - \mu)|a_2|^2. \end{aligned}$$

Therefore

$$|a_3 - \mu a_2^2| + (\mu_1 - \mu)|a_2|^2 \le \frac{2(1-\rho)}{(1+2\lambda)(2+\alpha)}, \qquad \mu_0 \le \mu \le \mu_1.$$

Here equality is attained for the function given by (3.5), and the proof of Theorem 3.14 is complete. □

Theorem 3.15. *Let $f(z) \in \mathcal{H}$, $\alpha \geq 0$, $\lambda \geq 0$ and $0 < k \leq 1$. If*

$$(3.9)\quad \left| \left\{ \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\alpha} + \lambda \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf'(z)}{f(z)} - 1 \right) \right] \right\} - 1 \right| < k, \quad z \in \mathcal{U},$$

then

$$|a_2| \leq \frac{k}{(1+\lambda)(1+\alpha)},$$

and for each $\mu \in C$, the following bound is sharp

$$|a_3 - \mu a_2^2| \leq \frac{k}{(1+2\lambda)(2+\alpha)} \max \left\{ 1, \frac{k(1+2\lambda)(2+\alpha) \left| 1 - 2\mu - \frac{\alpha}{1+2\lambda} + \frac{2\lambda}{(1+2\lambda)(2+\alpha)} \right|}{2(1+\lambda)^2(1+\alpha)^2} \right\}.$$

Proof. By (3.9), there exists a function $p(z) \in \mathcal{P}_0$ such that for all $z \in \mathcal{U}$

$$\frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\alpha} + \lambda \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \alpha \left(\frac{zf'(z)}{f(z)} - 1 \right) \right] = \frac{2k}{1+p(z)} + 1 - k.$$

Equating the coefficients, we obtain

$$a_2 = -\frac{k}{2(1+\lambda)(1+\alpha)} p_1,$$

$$(1+2\lambda)(2+\alpha)a_3 = -\frac{k}{2} \left(p_2 - \frac{1}{2}p_1^2 \right) + \frac{k^2 \left[\lambda(3+\alpha) - \frac{(2+\alpha)(\alpha-1)}{2} \right]}{4(1+\lambda)^2(1+\alpha)^2} p_1^2.$$

Thus, we have

$$a_3 - \mu a_2^2 = -\frac{k}{2(1+2\lambda)(2+\alpha)} \left(p_2 - \frac{1}{2}p_1^2 \right) + \frac{k^2 \left[\lambda(3+\alpha) - \frac{(2+\alpha)(\alpha-1)}{2} - \mu(1+2\lambda)(2+\alpha) \right]}{4(1+\lambda)^2(1+\alpha)^2(1+2\lambda)(2+\alpha)} p_1^2,$$

so that, by Lemma 2.3, we get that $|a_2| = \frac{k}{2(1+\lambda)(1+\alpha)}|p_1| \leq \frac{k}{(1+\lambda)(1+\alpha)}$, and

$$|a_3 - \mu a_2^2| \leq H(x) = A + \frac{Bx^2}{4},$$

where $x = |p_1| \leq 2$,

$$A = \frac{k}{(1+2\lambda)(2+\alpha)}, \qquad B = \frac{k^2|C|}{[(1+\lambda)^2(1+\alpha)^2]} - \frac{k}{[(1+2\lambda)(2+\alpha)]}$$

and

$$C = \frac{1-2\mu}{2} - \frac{\alpha}{2(1+2\lambda)} + \frac{\lambda}{(1+2\lambda)(2+\alpha)}.$$

Therefore

$$|a_3-\mu a_2^2| \le \begin{cases} H(0)=A, & |c| \le \frac{(1+\lambda)^2(1+\alpha)^2}{k(1+2\lambda)(2+\alpha)}, \\ H(2)=\frac{Ak(1+2\lambda)(2+\alpha)|C|}{(1+\lambda)^2(1+\alpha)^2}, & |c| \ge \frac{(1+\lambda)^2(1+\alpha)^2}{k(1+2\lambda)(2+\alpha)}. \end{cases}$$

Here equality is attained for the function given by

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\alpha = \begin{cases} \frac{\lambda z^{1/\lambda}\exp^{(-kz^2)/(2\lambda)}}{\int_0^z t^{(1-\lambda)/\lambda}\exp^{(-kt^2)/(2\lambda)}\,dt}, & \lambda>0, |c| \le \frac{(1+\lambda)^2(1+\alpha)^2}{k(1+2\lambda)(2+\alpha)}, \\ 1-kz^2, & \lambda=0, |c| \le \frac{(1+\lambda)^2(1+\alpha)^2}{k(1+2\lambda)(2+\alpha)}, \\ \frac{\lambda z^{1/\lambda}\exp^{-kz/\lambda}}{\int_0^z t^{(1-\lambda)/\lambda}\exp^{-kt/\lambda}\,dt}, & \lambda>0, |c| \ge \frac{(1+\lambda)^2(1+\alpha)^2}{k(1+2\lambda)(2+\alpha)}, \\ 1-kz, & \lambda=0, |c| \ge \frac{(1+\lambda)^2(1+\alpha)^2}{k(1+2\lambda)(2+\alpha)}. \end{cases}$$

This completes the proof of Theorem 3.15. □

Setting $\lambda=0$, we get the following corollary.

Corollary 3.16. *Let $f(z)\in\mathcal{H}, \alpha\ge 0$ and $0<k\le 1$. If*

$$\left|\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\alpha - 1\right| < k, \quad z\in\mathcal{U},$$

then

$$|a_2| \le \frac{k}{(1+\alpha)},$$

and for each $\mu\in C$, the following bound is sharp

$$|a_3-\mu a_2^2| \le \frac{k}{2+\alpha}\max\left\{1, \frac{k(2+\alpha)}{2(1+\alpha)^2}|1-2\mu-\alpha|\right\}.$$

Corollary 3.17. *Let $f(z)\in\mathcal{H}$, $\alpha\ge 0$ and $0<k\le 1$. If*

$$\left|(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1+\frac{zf''(z)}{f'(z)}\right] - 1\right| < k, \quad z\in\mathcal{U},$$

then

$$|a_2| \le \frac{k}{1+\alpha},$$

and for each $\mu\in C$ the following bound is sharp

$$|a_3-\mu a_2^2| \le \frac{k}{2(1+2\alpha)}\max\left\{1, \frac{k(1+2\alpha)\left|1-2\mu+\frac{\alpha}{1+2\alpha}\right|}{(1+\alpha)^2}\right\}.$$

Setting $\alpha=1$ in Corollary 3.17, we have the following corollary.

Corollary 3.18. *Let $f(z)\in\mathcal{H}$ and $0<k\le 1$. If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < k, \quad z\in\mathcal{U},$$

then

$$|a_2| \le \frac{k}{2},$$

and for each $\mu\in C$ the following bound is sharp

$$|a_3-\mu a_2^2| \le \frac{k}{6}\max\left\{1, \frac{k|4-6\mu|}{4}\right\}.$$

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