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# SOME ESTIMATES ON THE WEAKLY CONVERGENT SEQUENCE COEFFICIENT IN BANACH SPACES

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ABSTRACT. In this paper, we study the weakly convergent sequence coefficient and obtain its estimates for some parameters in Banach spaces, which give some sufficient conditions for a Banach space to have normal structure.

*Key words and phrases:* Weakly convergent sequence coefficient; James constant; Von Neumann-Jordan constant; Modulus of smoothness.

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## 1. INTRODUCTION

A Banach space $X$ said to have *(weak) normal structure* provided for every (weakly compact) closed bounded convex subset $C$ of $X$ with $\operatorname{diam}(C) > 0$, contains a nondiametral point, i.e., there exists $x_0 \in C$ such that $\sup\{\|x - x_0\| : x \in C\} < \operatorname{diam}(C)$. It is clear that normal structure and weak normal structure coincides when $X$ is reflexive.

The weakly convergent sequence coefficient $WCS(X)$, a measure of weak normal structure, was introduced by Bynum in [3] as the following.

**Definition 1.1.** The weakly convergent sequence coefficient of $X$ is defined by

$$WCS(X) = \inf\left\{\frac{\operatorname{diam}_a(\{x_n\})}{r_a(\{x_n\})} : \{x_n\} \text{ is a weakly convergent sequence}\right\}, \tag{1.1}$$

where $\operatorname{diam}_a(\{x_n\}) = \limsup_{k\to\infty}\{\|x_n - x_m\| : n, m \geq k\}$ is the asymptotic diameter of $\{x_n\}$ and $r_a(\{x_n\}) = \inf\{\limsup_{n\to\infty}\|x_n - y\| : y \in \bar{co}(\{x_n\})\}$ is the asymptotic radius of $\{x_n\}$.

One of the equivalent forms of $WCS(X)$ is

$$WCS(X) = \inf\left\{\lim_{n,m,n\neq m}\|x_n - x_m\| : x_n \xrightarrow{w} 0, \|x_n\| = 1 \quad \text{and} \quad \lim_{n,m,n\neq m}\|x_n - x_m\| \text{ exists}\right\}.$$

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Obviously, $1 \le WCS(X) \le 2$, and it is well known that $WCS(X) > 1$ implies that $X$ has a weak normal structure.

The constant $R(a, X)$, which is a generalized García-Falset coefficient [10], was introduced by Domínguez [7] as: For a given real number $a > 0$,

$$R(a, X) = \sup \left\{ \liminf_{n\to\infty} \|x + x_n\| \right\}, \tag{1.2}$$

where the supremum is taken over all $x \in X$ with $\|x\| \le a$ and all weakly null sequences $\{x_n\} \subseteq B_X$ such that

$$\lim_{n,m,n\neq m} \|x_n - x_m\| \le 1. \tag{1.3}$$

We shall assume throughout this paper that $B_X$ and $S_X$ to denote the unit ball and unit sphere of $X$, respectively. $x_n \xrightarrow{w} x$ stands for weak convergence of sequence $\{x_n\}$ in $X$ to a point $x$ in $X$.

## 2. Main Results

The *James constant*, or the *nonsquare constant*, was introduced by Gao and Lau in [8] as

$$\begin{aligned} J(X) &= \sup\{\|x + y\| \wedge \|x - y\| : x, y \in S_X\} \\ &= \sup\{\|x + y\| \wedge \|x - y\| : x, y \in B_X\}. \end{aligned}$$

A relation between the constant $R(1, X)$ and the James constant $J(X)$ can be found in [6, 12]:

$$R(1, X) \le J(X).$$

We now state an inequality between the James constant $J(X)$ and the weakly convergent sequence coefficient $WCS(X)$.

**Theorem 2.1.** *Let $X$ be a Banach space with the James constant $J(X)$. Then*

$$WCS(X) \ge \frac{J(X) + 1}{(J(X))^2}. \tag{2.1}$$

*Proof.* If $J(X) = 2$, it suffices to note that $WCS(X) \ge 1$. Thus our estimate is a trivial one.

If $J(X) < 2$, then $X$ is reflexive. Let $\{x_n\}$ be a weakly null sequence in $S_X$. Assume that $d = \lim_{n,m,n\neq m} \|x_n - x_m\|$ exists and consider a normalized functional sequence $\{x_n^*\}$ such that $x_n^*(x_n) = 1$. Note that the reflexivity of $X$ guarantees, by passing through the subsequence, that there exists $x^* \in X^*$ such that $x_n^* \xrightarrow{w} x^*$. Let $0 < \epsilon < 1$ and choose $N$ large enough so that $|x^*(x_N)| < \epsilon/2$ and

$$d - \epsilon < \|x_N - x_m\| < d + \epsilon$$

for all $m > N$. Note that

$$\lim_{n,m,n\neq m} \left\| \frac{x_n - x_m}{d + \epsilon} \right\| \le 1 \qquad \text{and} \qquad \left\| \frac{x_N}{d + \epsilon} \right\| \le 1.$$

Then by the definition of $R(1, X)$, we can choose $M > N$ large enough such that

$$\left\| \frac{x_N + x_M}{d + \epsilon} \right\| \le R(1, X) + \epsilon \le J(X) + \epsilon, \qquad |(x_M^* - x^*)(x_N)| < \epsilon/2,$$

and $|x_N^*(x_M)| < \epsilon$. Hence

$$|x_M^*(x_N)| \le |(x_M^* - x^*)(x_N))| + |x^*(x_N)| < \epsilon.$$

Put $\alpha = J(X)$,

$$x = \frac{x_N - x_M}{d + \epsilon}, \qquad \text{and} \qquad y = \frac{x_N + x_M}{(d + \epsilon)(\alpha + \epsilon)}.$$

It follows that $\|x\| \le 1$, $\|y\| \le 1$, and also that

$$\begin{aligned}\|x+y\| &= \frac{1}{(d+\epsilon)(\alpha+\epsilon)}\big\|(\alpha+1+\epsilon)x_N-(\alpha-1+\epsilon)x_M\big\| \\ &\ge \frac{1}{(d+\epsilon)(\alpha+\epsilon)}\big((\alpha+1+\epsilon)x_N^*(x_N)-(\alpha-1+\epsilon)x_N^*(x_M)\big) \\ &\ge \frac{\alpha+1-\epsilon}{(d+\epsilon)(\alpha+\epsilon)}, \\ \|x-y\| &= \frac{1}{(d+\epsilon)(\alpha+\epsilon)}\big\|(\alpha+1+\epsilon)x_M-(\alpha-1+\epsilon)x_N\big\| \\ &\ge \frac{1}{(d+\epsilon)(\alpha+\epsilon)}\big((\alpha+1+\epsilon)x_M^*(x_M)-(\alpha-1+\epsilon)x_M^*(x_N)\big) \\ &\ge \frac{\alpha+1-\epsilon}{(d+\epsilon)(\alpha+\epsilon)}.\end{aligned}$$

Thus, from the definition of the James constant,

$$J(X) \ge \frac{\alpha+1-\epsilon}{(d+\epsilon)(\alpha+\epsilon)} = \frac{J(X)+1-\epsilon}{(d+\epsilon)(J(X)+\epsilon)}.$$

Letting $\epsilon \to 0$, we get

$$d \ge \frac{J(X)+1}{(J(X))^2}.$$

Since the sequence $\{x_n\}$ is arbitrary, we get the inequality (2.1). □

As an application of Theorem 2.1, we can obtain a sufficient condition for $X$ to have normal structure in terms of the James constant.

**Corollary 2.2** ([4, Theorem 2.1]). *Let $X$ be a Banach space with $J(X) < (1+\sqrt{5})/2$. Then $X$ has normal structure.*

The *modulus of smoothness* [14] of $X$ is the function $\rho_X(\tau)$ defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : x, y \in S_X\right\}.$$

It is readily seen that for any $x, y \in S_X$,

$$\|x \pm y\| \le \|x \pm \tau y\| + (1-\tau) \qquad (0 < \tau \le 1),$$

which implies that $J(X) \le \rho_X(\tau) + 2 - \tau$.

In [2], Baronti et al. introduced a constant $A_2(X)$, which is defined by

$$A_2(X) = \rho_X(1) + 1 = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} : x, y \in S_X\right\}.$$

It is worth noting that $A_2(X) = A_2(X^*)$.

We now state an inequality between the modulus of smoothness $\rho_X(\tau)$ and the weakly convergent sequence coefficient $WCS(X)$.

**Theorem 2.3.** *Let $X$ be a Banach space with the modulus of smoothness $\rho_X(\tau)$. Then for any $0 < \tau \le 1$,*

$$WCS(X) \ge \frac{\rho_X(\tau)+2}{(\rho_X(\tau)+1)(\rho_X(\tau)-\tau+2)}. \tag{2.2}$$

*Proof.* Let $0 < \tau \leq 1$. If $\rho_X(\tau) = \tau$, it suffices to note that

$$\frac{\rho_X(\tau) + 2}{(\rho_X(\tau) + 1)(\rho_X(\tau) - \tau + 2)} = \frac{\tau + 2}{2(\tau + 1)} \leq 1.$$

Thus our estimate is a trivial one.

If $\rho_X(\tau) < \tau$, then $X$ is reflexive. Let $\{x_n\}$ be a weakly null sequence in $S_X$. Assume that $d = \lim n, m, n \neq m \|x_n - x_m\|$ exists and consider a normalized functional sequence $\{x_n^*\}$ such that $x_n^*(x_n) = 1$. Note that the reflexivity of $X$ guarantees that there exists $x^* \in X^*$ such that $x_n^* \xrightarrow{w} x^*$. Let $\epsilon > 0$ and $x_M, x_N, x$ and $y$ selected as in Theorem 2.1. Similarly, we get

$$\|x \pm \tau y\| \geq \frac{\alpha(\tau) + \tau - \epsilon}{(d + \epsilon)(\alpha(\tau) + \epsilon)},$$

where $\alpha(\tau) = \rho_X(\tau) + 2 - \tau$. Then by the definition of $\rho_X(\tau)$, we obtain

$$\rho_X(\tau) \geq \frac{\alpha(\tau) + \tau - \epsilon}{(d + \epsilon)(\alpha(\tau) + \epsilon)} - 1.$$

Letting $\epsilon \to 0$,

$$\rho_X(\tau) + 1 \geq \frac{\alpha(\tau) + \tau}{d\alpha(\tau)} = \frac{\rho_X(\tau) + 2}{d(\rho_X(\tau) - \tau + 2)},$$

which gives that

$$d \geq \frac{\rho_X(\tau) + 2}{(\rho_X(\tau) + 1)(\rho_X(\tau) - \tau + 2)}.$$

Since the sequence $\{x_n\}$ is arbitrary, we get the inequality (2.2). □

It is known that if $\rho_X(\tau) < \tau/2$ for some $\tau > 0$, then $X$ has normal structure (see [9]). Using Theorem 2.3, We can improve this result in the following form:

**Corollary 2.4.** *Let $X$ be a Banach space with*

$$\rho_X(\tau) < \frac{\tau - 2 + \sqrt{\tau^2 + 4}}{2}$$

*for some $\tau \in (0, 1]$. Then $X$ has normal structure. In particular, if $A_2(X) < (1 + \sqrt{5})/2$, then $X$ and its dual $X^*$ have normal structure.*

In connection with a famous work of Jordan-von Neumann concerning inner products, the *Jordan-von Neumann constant* $C_{\mathrm{NJ}}(X)$ of $X$ was introduced by Clarkson (cf. [1, 11]) as

$$C_{\mathrm{NJ}}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ and not both zero} \right\}.$$

A relationship between $J(X)$ and $C_{\mathrm{NJ}}(X)$ is found in ([11] Theorem 3): $J(X) \leq \sqrt{2C_{\mathrm{NJ}}(X)}$.

In [5], Dhompongsa et al. proved the following inequality (2.3). We now restate this inequality without the ultra product technique and the fact $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(X^*)$.

**Theorem 2.5** ([5] Theorem 3.8)**.** *Let $X$ be a Banach space with the von Neumann-Jordan constant $C_{\mathrm{NJ}}(X)$. Then*

$$(WCS(X))^2 \geq \frac{2C_{\mathrm{NJ}}(X) + 1}{2(C_{\mathrm{NJ}}(X))^2}. \tag{2.3}$$

*Proof.* If $C_{\mathrm{NJ}}(X) = 2$, it suffices to note that $WCS(X) \geq 1$. Thus our estimates is a trivial one.

If $C_{\mathrm{NJ}}(X) < 2$, then $X$ is reflexive. Let $\{x_n\}$ be a weakly null sequence in $S_X$. Assume that $d = \lim_{n,m,n\neq m} \|x_n - x_m\|$ exists and consider a normalized functional sequence $\{x_n^*\}$ such that $x_n^*(x_n) = 1$. Note that the reflexivity of $X$ gurantees that there esists $x^* \in X^*$ such that $x_n^* \xrightarrow{w} x^*$. Let $\epsilon > 0$ and choose $N$ large enough so that $|x^*(x_N)| < \epsilon/2$ and

$$d - \epsilon < \|x_N - x_m\| < d + \epsilon$$

for all $m > N$. Note that

$$\lim_{n,m,n\neq m} \left\| \frac{x_n - x_m}{d+\epsilon} \right\| \leq 1 \qquad \text{and} \qquad \left\| \frac{x_N}{d+\epsilon} \right\| \leq 1.$$

Then by the definition of $R(1,X)$, we can choose $M > N$ large enough such that

$$\left\| \frac{x_N - x_M}{d+\epsilon} \right\| \leq R(1,X) + \epsilon \leq \sqrt{2C_{\mathrm{NJ}}(X)} + \epsilon, \qquad |(x_M^* - x^*)(x_N)| < \epsilon/2,$$

and $|x_N^*(x_M)| < \epsilon$. Hence

$$|x_M^*(x_N)| < |(x_M^* - x^*)(x_N))| + |x^*(x_N)| < \epsilon.$$

Put $\alpha = \sqrt{2C_{\mathrm{NJ}}(X)}$, $x = \alpha^2(x_N - x_M)$, $y = x_N + x_M$. It follows that $\|x\| \leq \alpha^2(d+\epsilon)$, $\|y\| \leq (\alpha+\epsilon)(d+\epsilon)$, and also that

$$\begin{aligned}
\|x+y\| &= \|(\alpha^2+1)x_N - (\alpha^2-1)x_M\| \\
&\geq (\alpha^2+1)x_N^*(x_N) - (\alpha^2-1)x_N^*(x_M) \\
&\geq \alpha^2 + 1 - 3\epsilon, \\
\|x-y\| &= \|(\alpha^2+1)x_M - (\alpha^2-1)x_N\| \\
&\geq (\alpha^2+1)x_M^*(x_M) - (\alpha^2-1)x_M^*(x_N) \\
&\geq \alpha^2 + 1 - 3\epsilon.
\end{aligned}$$

Thus, from the definition of the von Neumann-Jordan constant,

$$\begin{aligned}
C_{\mathrm{NJ}}(X) &\geq \frac{2(\alpha^2+1-3\epsilon)^2}{2(\alpha^4(d+\epsilon)^2 + (\alpha+\epsilon)^2(d+\epsilon)^2)} \\
&= \frac{1}{(d+\epsilon)^2} \cdot \frac{(\alpha^2+1-3\epsilon)^2}{\alpha^4 + (\alpha+\epsilon)^2}.
\end{aligned}$$

Since $\epsilon$ is arbitrary and $\alpha = \sqrt{2C_{\mathrm{NJ}}(X)}$, we get

$$C_{\mathrm{NJ}}(X) \geq \frac{1}{d^2}\left(1 + \frac{1}{\alpha^2}\right) = \frac{2C_{\mathrm{NJ}}(X)+1}{d^2 \cdot 2C_{\mathrm{NJ}}(X)},$$

which implies that

$$d^2 \geq \frac{2C_{\mathrm{NJ}}(X)+1}{2(C_{\mathrm{NJ}}(X))^2}.$$

Since the sequence $\{x_n\}$ is arbitrary, we obtain the inequality (2.3). □

Using Theorem 2.5, we can get a sufficient condition for $X$ to have normal structure in terms of the von Neumann-Jordan constant.

**Corollary 2.6** ([6, Theorem 3.16], [13, Theorem 2])**.** *Let $X$ be a Banach space with $C_{\mathrm{NJ}}(X) < (1+\sqrt{3})/2$. Then $X$ and its dual $X^*$ have normal structure.*

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