THE ALEXANDER TRANSFORMATION OF A SUBCLASS OF SPIRALLIKE FUNCTIONS OF TYPE β

QINGHUA XU
School of Mathematics and Information Science
Jiangxi Normal University
Jiangxi, 330022, China
EMail: xuqhster@gmail.com

SANYA LU
Department of Science
Nanchang Institute of Technology
Jiangxi, 330099, China
EMail: yasanlu@163.com

Received: 13 August, 2008

Accepted: 27 December, 2008

Communicated by: gkohr@math.ubbcluj.ro

2000 AMS Sub. Class.: 30C45.

Key words: Univalent functions, Starlike functions of order α, spirallike functions of type β, Integral transformations.

Abstract: In this paper, a subclass of spirallike function of type β denoted by $\hat{S}_{\alpha}^{\beta}$ is introduced in the unit disc of the complex plane. We show that the Alexander transformation of class of $\hat{S}_{\alpha}^{\beta}$ is univalent when $\cos\beta \leq \frac{1}{2(1-\alpha)}$, which generalizes the related results of some authors.

Acknowledgements: This research has been supported by the Jiangxi Provincial Natural Science Foundation of China (Grant No. 2007GZS0177) and Specialized Research Fund for the Doctoral Program of JiangXi Normal University.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Contents

Alexander Transformation
Qinghua Xu and Sanya Lu
vol. 10, iss. 1, art. 17, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

1. Introduction

Let A denote the class of analytic functions f on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$, S denote the subclass of A consisting of univalent functions, and S^* denote starlike functions on D. Obviously, $S^* \subset S \subset A$ holds.

In [1], Robertson introduced starlike functions of order α on D.

Definition 1.1. *Let $\alpha \in [0, 1)$, $f \in S$ and*

$$\Re e \left[\frac{zf'(z)}{f(z)}\right] > \alpha, \quad z \in D.$$

We say that f is a starlike function of order α. Let $S^*(\alpha)$ denote the whole starlike functions of order α on D.

Spaček [2] extended the class of S^*,and obtained the class of spirallike functions of type β. In the same article, the author gave an analytical characterization of spirallikeness of type β on D.

Theorem 1.2. *Let $f \in S$ and $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $f(z)$ is a spirallike function of type β on D if and only if*

$$\Re e \left[e^{i\beta}\frac{zf'(z)}{f(z)}\right] > 0, \quad z \in D.$$

We denote the whole spirallike functions of type β on D by $\hat{S}_\beta$.

From Definition 1.1 and Theorem 1.2, it is easy to see that starlike functions of order α and spirallike functions of type β have some relationships on geometry. Spirallike functions of type β map D into the right half complex plane by the mapping

Alexander Transformation
Qinghua Xu and Sanya Lu
vol. 10, iss. 1, art. 17, 2009

journal of inequalities in pure and applied mathematics
issn: 1443-5756

$e^{i\beta}\frac{zf'(z)}{f(z)}$, while starlike functions of order α map D into the right half complex plane whose real part is greater than α by the mapping $\frac{zf'(z)}{f(z)}$. Since $\lim_{z\to 0} e^{i\beta}\frac{zf'(z)}{f(z)} = e^{i\beta}$, we can deduce that if we restrict the image of the mapping $e^{i\beta}\frac{zf'(z)}{f(z)}$ in the right complex plane whose real part is greater than a certain constant, then the constant must be smaller than $\cos\beta$. According to this, we introduce the functions class $\hat{S}_\alpha^\beta$ on D.

Definition 1.3. *Let $\alpha \in [0,1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $f \in S$, then $f \in \hat{S}_\alpha^\beta$ if and only if*

$$\Re e\left[e^{i\beta}\frac{zf'(z)}{f(z)}\right] > \alpha\cos\beta, \quad z \in D.$$

Obviously, when $\beta = 0$, $f \in S^*(\alpha)$; while $\alpha = 0$, $f \in \hat{S}_\beta$.

Example 1.1. Let $f(z) = \frac{z}{(1-z)^{\frac{2(1-\alpha)}{1+i\tan\beta}}}$, $z \in D$. The branch of the power function is chosen such that

$$\left[(1-z)\right]^{\frac{2(1-\alpha)}{1+i\tan\beta}}\Big|_{z=0} = 1.$$

It is easily proved that $f \in \hat{S}_\alpha^\beta$. We omit the proof.

For our applications, we set $\hat{S} = \bigcup_\beta \hat{S}_\alpha^\beta$.

In this paper, we first establish the relationships among $\hat{S}_\alpha^\beta$ and some important subclasses of S, then investigate the Alexander transformation of $\hat{S}_\alpha^\beta$ preserving univalence. Furthermore, some other properties of the class of $\hat{S}_\alpha^\beta$ are obtained. These results generalize the related works of some authors.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

2. Integral Transformations and Lemmas

Integral Transformation 1. *The integral transformation*

$$J[f](z) = \int_0^z \frac{f(\zeta)}{\zeta} d\zeta$$

is called the Alexander Transformation and it was introduced by Alexander in [4]. Alexander was the first to observe and prove that the Integral transformation J maps the class $S^*$of starlike functions onto the class K of convex functions in a one-to-one fashion.

In 1960, Biernacki conjectured that $J(S) \subset S$, but Krzyz and Lewandowski disproved it in 1963 by giving the example $f(z) = z(1-iz)^{i-1}$, which is a spirallike function of type $\frac{\pi}{4}$ but is transformed into a non-univalent function by J [4]. In 1969, Robertson studied the Alexander Integral Transformation of spirallike functions of type β. The author showed that $J(\hat{S}_\beta) \subset S$ holds when β satisfies a certain condition, that is $\cos\beta \le x_0$ (a constant). Robertson also noticed that x_0 cannot be replaced by any number greater than $\frac{1}{2}$ and asked about the best value for this [3]. In 2007, Y.C. Kim and T. Sugawa proved that $J(\hat{S}_\beta) \subset S$ holds precisely when $\cos\beta \le \frac{1}{2}$ or $\beta = 0$ [4].

Integral Transformation 2. *Let $\gamma \in \mathbb{C}$, $f(z) \in A$ be locally univalent, and the Integral transformation I_γ [5] be defined by*

$$I_\gamma[f](z) = \int_0^z [f'(\zeta)]^\gamma d\zeta = z\int_0^1 [f'(tz)]^\gamma dt.$$

Based on the definition of I_γ, we may easily show that $I_\gamma \circ I_{\gamma'} = I_{\gamma\gamma'}$.

Let $A(F) = \{\gamma \in \mathbb{C} : I_\gamma(F) \subset S\}$, $F \subset A$ be locally univalent. According to the definition of the $A(F)$, $J(\hat{S}_\alpha^\beta) \subset S$ is equivalent to $1 \in A(J(\hat{S}_\alpha^\beta))$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

For the proof of the theorems in this paper, we need the following lemma, which establishes the relationships among $\hat{S}_\alpha^\beta$ and some important subclasses of S.

Lemma 2.1. *For $\alpha \in [0,1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $c = e^{-i\beta}\cos\beta$, the following assertions hold:*

(i) ([6, 7]) $f \in S^*(\alpha)$ if and only if

$$\frac{f(z)}{z} = \left[\frac{u(z)}{z}\right]^{1-\alpha}, \quad z \in D,$$

where $u(z) \in S^*$. The branch of the power function is chosen such that $\left.\left[\frac{u(z)}{z}\right]^{1-\alpha}\right|_{z=0} = 1$.

(ii) $f \in \hat{S}_\alpha^\beta$ if and only if

$$\frac{f(z)}{z} = \left[\frac{g(z)}{z}\right]^{c}, \quad z \in D,$$

where $g(z) \in S^*(\alpha)$. The branch of the power function is chosen such that $\left.\left[\frac{g(z)}{z}\right]^{c}\right|_{z=0} = 1$.

(iii) $f \in \hat{S}_\alpha^\beta$ if and only if

$$\frac{f(z)}{z} = \left[\frac{s(z)}{z}\right]^{(1-\alpha)c}, \quad z \in D,$$

where $s(z) \in S^*$. The branch of the power function is chosen such that $\left.\left[\frac{s(z)}{z}\right]^{(1-\alpha)c}\right|_{z=0} = 1$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Now we give the proof of (ii) and (iii).

Proof. (ii). First, assume that $f(z) \in \hat{S}_\alpha^\beta$. Setting $g(z) = z\left[\frac{f(z)}{z}\right]^{\frac{e^{i\beta}}{\cos\beta}}$, through simple calculations we may obtain the equality

$$\frac{zg'(z)}{g(z)} = (1 + i\tan\beta)\frac{zf'(z)}{f(z)} - i\tan\beta.$$

Therefore the following inequality holds,

$$\Re e\left[\frac{zg'(z)}{g(z)}\right] = \frac{1}{\cos\beta}\Re e\left[e^{i\beta}\frac{zf'(z)}{f(z)}\right] > \frac{\alpha\cos\beta}{\cos\beta} = \alpha,$$

namely $g(z) \in S^*(\alpha)$.

Conversely, suppose $g(z) \in S^*(\alpha)$, then according to the above calculation, we have the inequality

$$\frac{1}{\cos\beta}\Re e\left[e^{i\beta}\frac{zf'(z)}{f(z)}\right] = \Re e\left[\frac{zg'(z)}{g(z)}\right] > \alpha.$$

This implies

$$\Re e\left[e^{i\beta}\frac{zf'(z)}{f(z)}\right] > \alpha\cos\beta,$$

i.e., $f(z) \in \hat{S}_\alpha^\beta$.

(iii). It is easy to see from (ii) that $f \in \hat{S}_\alpha^\beta$ if and only if $g \in S^*(\alpha)$ such that $\frac{f(z)}{z} = \left[\frac{g(z)}{z}\right]^c$, here $c = e^{-i\beta}\cos\beta$. Noting that $g(z) \in S^*(\alpha)$ if and only if $s(z) \in S^*$ such that $\frac{g(z)}{z} = \left[\frac{s(z)}{z}\right]^{1-\alpha}$ which holds in (i), we may obtain an important relationship

journal of inequalities in pure and applied mathematics
issn: 1443-5756

between the class of $\hat{S}_\alpha^\beta$ and the class of S^* : $f \in \hat{S}_\alpha^\beta$ if and only if there exists $s(z) \in S^*$ such that $\frac{f(z)}{z} = \left[\frac{s(z)}{z}\right]^{(1-\alpha)c}$. Here, $c = e^{-i\beta}\cos\beta$ and the branch of the power function is chosen such that $\left.\left[\frac{s(z)}{z}\right]^{(1-\alpha)c}\right|_{z=0} = 1$. □

Lemma 2.1 expresses the relations of the $\hat{S}_\alpha^\beta$ and S^* classes, which play a key role in this paper.

Lemma 2.2 ([5], [8]). $A(K) = \left\{|\gamma| \le \frac{1}{2}\right\} \cup \left[\frac{1}{2}, \frac{3}{2}\right]$.

Lemma 2.3. *For* $\alpha \in [0,1)$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $J(\hat{S}_\alpha^\beta) = I_{(1-\alpha)e^{-i\beta}\cos\beta}(K)$.

Proof. Let $f \in J(\hat{S}_\alpha^\beta)$, then there exists $g(z) \in \hat{S}_\alpha^\beta$ such that $f(z) = \int_0^z \frac{g(\zeta)}{\zeta} d\zeta$. According to (iii) of Lemma 2.1 there is $s(z) \in S^*$ such that

$$g(z) = z\left[\frac{s(z)}{z}\right]^{(1-\alpha)e^{-i\beta}\cos\beta},$$

therefore

$$f(z) = \int_0^z \left[\frac{s(\zeta)}{\zeta}\right]^{(1-\alpha)e^{-i\beta}\cos\beta} d\zeta.$$

By the relationship of the S^* class and the K class, there exists $u(z) \in K$ such that $s(z) = zu'(z)$, thus

$$f(z) = \int_0^z [u'(\zeta)]^{(1-\alpha)e^{-i\beta}\cos\beta} d\zeta,$$

i.e., $f(z) \in I_{(1-\alpha)e^{-i\beta}\cos\beta}(K)$. As a result, $J(\hat{S}_\alpha^\beta) \subset I_{(1-\alpha)e^{-i\beta}\cos\beta}(K)$ holds.

Conversely, when $f(z) \in I_{(1-\alpha)e^{-i\beta}\cos\beta}(K)$, we can trace back the above procedure to get $f \in J(\hat{S}_\alpha^\beta)$, so $I_{(1-\alpha)e^{-i\beta}\cos\beta}(K) \subset J(\hat{S}_\alpha^\beta)$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

From the above proof, we obtain the assertion. □

Remark 1. If, in the hypothesis of Lemma 2.3, we set $\alpha = 0$, we arrive at Lemma 4 of [4].

journal of inequalities in pure and applied mathematics
issn: 1443-5756

3. The Main Results and Their Proofs

In this section, we let $[z, w]$ denote the closed line segment with endpoints z and w for $z, w \in \mathbb{C}$.

Now we give the main results and their proofs.

Theorem 3.1. *For* $\alpha \in [0, 1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$A(J(\hat{S}_\alpha^\beta)) = \left\{ |\gamma| \le \frac{1}{2(1-\alpha)\cos\beta} \right\} \bigcup \left\{ \frac{e^{i\beta}}{2(1-\alpha)\cos\beta}, \frac{3e^{i\beta}}{2(1-\alpha)\cos\beta} \right\}.$$

Proof. By Lemma 2.3, we have

$$I_\gamma(J(\hat{S}_\alpha^\beta)) = I_\gamma(I_{(1-\alpha)e^{-i\beta}\cos\beta}(K)) = I_{\gamma(1-\alpha)e^{-i\beta}\cos\beta}(K).$$

Therefore, $\gamma \in A(J(\hat{S}_\alpha^\beta))$ if and only if $\gamma(1-\alpha)e^{-i\beta}\cos\beta \in A(K)$, and by Lemma 2.2 we may easily get the result. □

Remark 2. In this theorem, if we set $\alpha = 0$, we obtain Theorem 3 of [4].

Theorem 3.2. *For* $\alpha \in [0, 1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *the inclusion relation* $J(\hat{S}_\alpha^\beta) \subset S$ *holds precisely if either* $\cos\beta \le \frac{1}{2(1-\alpha)}$ *or* $\alpha = \beta = 0$.

Proof. As $\alpha = \beta = 0$, the result holds evidently by Integral transformation 1; while for $\alpha = 0$ and $\beta \ne 0$, the result is Theorem 1 of [4] and was proved by Y.C. Kim and T. Sugawa [4].

If $\alpha \ne 0$ and $\beta = 0$, then $f(z) \in S^*(\alpha)$. By Lemma 2.1(i), there exists $u(z) \in S^*$ such that $u(z) = z\left(\frac{f(z)}{z}\right)^{\frac{1}{1-\alpha}}$. The branch of the power function is chosen such that $\left.\left(\frac{f(z)}{z}\right)^{\frac{1}{1-\alpha}}\right|_{z=0} = 1$. From Integral transformation 1, we can easily see that there

journal of inequalities in pure and applied mathematics
issn: 1443-5756

exists $g(z) \in J(\hat{S}_\alpha^\beta)$ such that

$$g(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^{\frac{1}{1-\alpha}} d\zeta.$$

For

$$\Re e\left[1 + \frac{zg''(z)}{g'(z)}\right] = \Re e\left[\frac{1}{1-\alpha}\,\frac{zf'(z)}{f(z)}\right]$$

and $\Re e\left[\frac{zf'(z)}{f(z)}\right] > \alpha$, we can deduce that $\Re e\left[1 + \frac{zg''(z)}{g'(z)}\right] > 0$. This implies $g(z) \in K$ and $J(S^*(\alpha)) \subset S$.

Now let $\alpha \neq 0$ and $\beta \neq 0$. Since $J(\hat{S}_\alpha^\beta) \subset S$ is equivalent to $1 \in A(J(\hat{S}_\alpha^\beta))$ and $1 \notin \left[\frac{e^{i\beta}}{2(1-\alpha)\cos\beta}, \frac{3e^{i\beta}}{2(1-\alpha)\cos\beta}\right]$, by Theorem 3.1, we deduce that $1 \leq \frac{1}{2(1-\alpha)\cos\beta}$, i.e., $\cos\beta \leq \frac{1}{2(1-\alpha)}$.

Summarizing the above procedure, for $\alpha \in [0,1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $J(\hat{S}_\alpha^\beta) \subset S$ holds when $\cos\beta \leq \frac{1}{2(1-\alpha)}$ or $\alpha = \beta = 0$.This completes the proof. □

Remark 3. This theorem is an extension of Theorem 1 of [4]. Indeed, if we set $\alpha = 0$, we will obtain the result of [4].

Theorem 3.3. *For* $\alpha \in [0,1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$A(J(\hat{S})) = \left\{|\gamma| \leq \frac{1}{2(1-\alpha)\cos\beta}\right\}.$$

Proof. In view of $\hat{S} = \bigcup_\beta \hat{S}_\alpha^\beta$ and $A(F) = \{\gamma \in \mathbb{C} : I_\gamma(F) \subset S\}$, we deduce $A(J(\hat{S})) = \bigcap_\beta (J(\hat{S}_\alpha^\beta))$. With the aid of Theorem 3.1, a simple observation gives $A(J(\hat{S})) = \left\{|\gamma| \leq \frac{1}{2(1-\alpha)\cos\beta}\right\}$. Thus the proof is now complete. □

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Remark 4. For $\alpha = \beta = 0$, Theorem 3.3 implies the Theorem 2 of [4].

At the end of this paper, we mention the norm estimate of pre-Schwarzian derivatives. The hyperbolic norm of the pre-Schwarzian derivative $T_f = f''/f'$ of $f \in A$ is defined to be

$$\|f\| = \sup_{|z|<1}(1-|z|^2)|T_f(z)|.$$

It is known that f is bounded if $\|f\| < 2$ and the bound depends only on the value of $\|f\|$ ([9]). Since

$$\begin{aligned}\|I_\gamma[f]\| &= \sup_{|z|<1}(1-|z|^2)\left|\frac{\left(\int_0^z [f'(\zeta)]^\gamma d\zeta\right)''}{\left(\int_0^z [f'(\zeta)]^\gamma\right)'}\right| \\ &= \sup_{|z|<1}(1-|z|^2)\left|\frac{([f'(z)]^\gamma)'}{f'(z)]^\gamma}\right| \\ &= \sup_{|z|<1}(1-|z|^2)\left|\frac{\gamma f''(z)}{f'(z)}\right| = |\gamma|\|f\|.\end{aligned}$$

We obtain the following assertion.

Proposition 3.4. *For each* $\alpha \in [0,1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *the sharp inequality* $\|f\| \leq 4(1-\alpha)\cos\beta$ *holds for* $f \in J(\hat{S}_\alpha^\beta)$. *Moreover, if* $\cos\beta < \frac{1}{2(1-\alpha)}$, *then a function in* $J(\hat{S}_\alpha^\beta)$ *is bounded by a constant depending on* α *and* β.

Proof. For each $f \in J(\hat{S}_\alpha^\beta)$, by Lemma 2.3, there is a function $k \in K$ such that $f = I_\gamma(k)$, where $\gamma = (1-\alpha)e^{-i\beta}\cos\beta$. Noting that $\|k\| \leq 4$ [10], we obtain the following inequality

$$\|f\| = |\gamma|\|k\| \leq 4|\gamma| = 4(1-\alpha)\cos\beta.$$

journal of inequalities in pure and applied mathematics
issn: 1443-5756

Since the inequality $\|k\| \leq 4$ is sharp, the above inequality is also sharp. If $\cos\beta < \frac{1}{2(1-\alpha)}$, the above inequality implies $\|f\| \leq 4(1-\alpha)\cos\beta < 2$, so f is bounded by a constant depending on α and β. □

Remark 5. If, in the statement of Proposition 3.4, we set $\alpha = 0$, we arrive at the result of [4].

In the above proposition, the bound $\frac{1}{2}$ cannot be replaced by any number greater than $\frac{1}{\sqrt{2(1-\alpha)}}$. Indeed, by the Alexander transformation, if the function

$$g(z) = z(1-z)^{-2(1-\alpha)e^{-i\beta}\cos\beta} \in \hat{S}_\alpha^\beta,$$

then the function

$$f(z) = \frac{(1-z)^{1-2(1-\alpha)e^{-i\beta}\cos\beta} - 1}{2(1-\alpha)e^{-i\beta}\cos\beta - 1} \in J(\hat{S}_\alpha^\beta),$$

and we may verify that the latter is unbounded when $\cos\beta > \frac{1}{\sqrt{2(1-\alpha)}}$.

journal of inequalities in pure and applied mathematics
issn: 1443-5756

References

[1] M.S. ROBERTSON, On the theory of univalent functions, *Ann. Math.*, **37** (1936), 374–408.

[2] L. SPAČEK, Contribution à la théorie des fonctions univalentes, *Casopis Pěst Math.*, **62** (1932), 12–19, (in Russian).

[3] M.S. ROBERTSON, Univalent functions $f(z)$ for which $zf'(z)$ is spirallike, *Michigan Math. J.*, **16** (1969), 97–101.

[4] Y.C. KIM AND T. SUGAWA, The Alexander transform of a spirallike function, *J. Math. Anal. Appl.*, **325**(1) (2007), 608–611.

[5] Y.C. KIM, S. PONNUSAMY AND T. SUGAWA, Mapping properties of nonlinear integral operators and pre-Schwarzian derivatives, *J. Math. Anal. Appl.*, **299** (2004), 433–447.

[6] A.W. GOODMAN, Univalent functions, I-II, *Mariner Publ.Co.,Tampa Florida*,1983.

[7] I. GRAHAM AND G. KOHR, *Geometric function theory in one and higher dimensions*, Marcel Dekker, New York ,2003.

[8] L.A. AKSENT'EV AND I.R. NEZHMETDINOV, Sufficient conditions for univalence of certain integral transforms, *Tr. Semin. Kraev. Zadacham. Kazan*, **18** (1982), 3–11 (in Russian); English translation in: *Amer. Math. Soc. Transl.*, **136**(2) (1987), 1–9.

[9] Y.C. KIM AND T. SUGAWA, Growth and coefficient estimates for uniformly locally univalent functions on the unit disk, *Rocky Mountain J. Math.*, **32** (2002), 179–200.

[10] S. YAMASHITA, Norm estimates for function starlike or convex of order alpha, *Hokkaido Math. J.*, **28** (1999), 217–230.

journal of inequalities in pure and applied mathematics
issn: 1443-5756