journal of inequalities in pure and applied mathematics

http://jipam.vu.edu.au

issn: 1443-5756

Volume 10 (2009), Issue 1, Article 17, 7 pp.

THE ALEXANDER TRANSFORMATION OF A SUBCLASS OF SPIRALLIKE FUNCTIONS OF TYPE β

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Received 13 August, 2008; accepted 27 December, 2008
Communicated by G. Kohr

ABSTRACT. In this paper, a subclass of spirallike function of type β denoted by $\hat{S}_\alpha^\beta$ is introduced in the unit disc of the complex plane. We show that the Alexander transformation of class of $\hat{S}_\alpha^\beta$ is univalent when $\cos\beta \le \frac{1}{2(1-\alpha)}$, which generalizes the related results of some authors.

Key words and phrases: Univalent functions, Starlike functions of order α, spirallike functions of type β, Integral transformations.

2000 *Mathematics Subject Classification.* 30C45.

1. INTRODUCTION

Let A denote the class of analytic functions f on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0$ and $f'(0) = 1$, S denote the subclass of A consisting of univalent functions, and S^* denote starlike functions on D. Obviously, $S^* \subset S \subset A$ holds.

In [1], Robertson introduced starlike functions of order α on D.

Definition 1.1. Let $\alpha \in [0, 1)$, $f \in S$ and

$$\Re e\left[\frac{zf'(z)}{f(z)}\right] > \alpha, \quad z \in D.$$

We say that f is a starlike function of order α. Let $S^*(\alpha)$ denote the whole starlike functions of order α on D.

This research has been supported by the Jiangxi Provincial Natural Science Foundation of China (Grant No. 2007GZS0177) and Specialized Research Fund for the Doctoral Program of JiangXi Normal University.

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Spaček [2] extended the class of S^*,and obtained the class of spirallike functions of type β. In the same article, the author gave an analytical characterization of spirallikeness of type β on D.

Theorem 1.1. *Let $f \in S$ and $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $f(z)$ is a spirallike function of type β on D if and only if*

$$\Re e \left[e^{i\beta} \frac{zf'(z)}{f(z)} \right] > 0, \quad z \in D.$$

We denote the whole spirallike functions of type β on D by $\hat{S}_\beta$.

From Definition 1.1 and Theorem 1.1, it is easy to see that starlike functions of order α and spirallike functions of type β have some relationships on geometry. Spirallike functions of type β map D into the right half complex plane by the mapping $e^{i\beta}\frac{zf'(z)}{f(z)}$, while starlike functions of order α map D into the right half complex plane whose real part is greater than α by the mapping $\frac{zf'(z)}{f(z)}$. Since $\lim_{z\to 0} e^{i\beta}\frac{zf'(z)}{f(z)} = e^{i\beta}$, we can deduce that if we restrict the image of the mapping $e^{i\beta}\frac{zf'(z)}{f(z)}$ in the right complex plane whose real part is greater than a certain constant, then the constant must be smaller than $\cos\beta$. According to this, we introduce the functions class $\hat{S}_\alpha^\beta$ on D.

Definition 1.2. Let $\alpha \in [0, 1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $f \in S$, then $f \in \hat{S}_\alpha^\beta$ if and only if

$$\Re e \left[e^{i\beta} \frac{zf'(z)}{f(z)} \right] > \alpha \cos\beta, \quad z \in D.$$

Obviously, when $\beta = 0$, $f \in S^*(\alpha)$; while $\alpha = 0,\ f \in \hat{S}_\beta$.

Example 1.1. Let $f(z) = \frac{z}{(1-z)^{\frac{2(1-\alpha)}{1+i\tan\beta}}}$, $z \in D$. The branch of the power function is chosen such that

$$\left.[(1-z)]^{\frac{2(1-\alpha)}{1+i\tan\beta}}\right|_{z=0} = 1.$$

It is easily proved that $f \in \hat{S}_\alpha^\beta$. We omit the proof.

For our applications, we set $\hat{S} = \bigcup_\beta \hat{S}_\alpha^\beta$.

In this paper, we first establish the relationships among $\hat{S}_\alpha^\beta$ and some important subclasses of S, then investigate the Alexander transformation of $\hat{S}_\alpha^\beta$ preserving univalence. Furthermore, some other properties of the class of $\hat{S}_\alpha^\beta$ are obtained. These results generalize the related works of some authors.

2. INTEGRAL TRANSFORMATIONS AND LEMMAS

Integral Transformation 1. *The integral transformation*

$$J[f](z) = \int_0^z \frac{f(\zeta)}{\zeta} d\zeta$$

is called the Alexander Transformation and it was introduced by Alexander in [4]. *Alexander was the first to observe and prove that the Integral transformation J maps the class $S^*$of starlike functions onto the class K of convex functions in a one-to-one fashion.*

In 1960, Biernacki conjectured that $J(S) \subset S$, but Krzyz and Lewandowski disproved it in 1963 by giving the example $f(z) = z(1 - iz)^{i-1}$, which is a spirallike function of type $\frac{\pi}{4}$ but is transformed into a non-univalent function by J [4]. In 1969, Robertson studied the Alexander Integral Transformation of spirallike functions of type β. The author showed that $J(\hat{S}_\beta) \subset S$

holds when β satisfies a certain condition, that is $\cos\beta \le x_0$ (a constant). Robertson also noticed that x_0 cannot be replaced by any number greater than $\frac{1}{2}$ and asked about the best value for this [3]. In 2007, Y.C. Kim and T. Sugawa proved that $J(\hat{S}_\beta) \subset S$ holds precisely when $\cos\beta \le \frac{1}{2}$ or $\beta = 0$ [4].

Integral Transformation 2. *Let $\gamma \in \mathbb{C}$, $f(z) \in A$ be locally univalent, and the Integral transformation I_γ [5] be defined by*

$$I_\gamma[f](z) = \int_0^z [f'(\zeta)]^\gamma d\zeta = z\int_0^1 [f'(tz)]^\gamma dt.$$

Based on the definition of I_γ, we may easily show that $I_\gamma \circ I_{\gamma'} = I_{\gamma\gamma'}$.

Let $A(F) = \{\gamma \in \mathbb{C} : I_\gamma(F) \subset S\},\ F \subset A$ be locally univalent. According to the definition of the $A(F)$, $J(\hat{S}_\alpha^\beta) \subset S$ is equivalent to $1 \in A(J(\hat{S}_\alpha^\beta))$.

For the proof of the theorems in this paper, we need the following lemma, which establishes the relationships among $\hat{S}_\alpha^\beta$ and some important subclasses of S.

Lemma 2.1. *For $\alpha \in [0,1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $c = e^{-i\beta}\cos\beta$, the following assertions hold:*

(i) *([6, 7]) $f \in S^*(\alpha)$ if and only if*

$$\frac{f(z)}{z} = \left[\frac{u(z)}{z}\right]^{1-\alpha}, \quad z \in D,$$

where $u(z) \in S^*$. The branch of the power function is chosen such that $\left.\left[\frac{u(z)}{z}\right]^{1-\alpha}\right|_{z=0} = 1$.

(ii) *$f \in \hat{S}_\alpha^\beta$ if and only if*

$$\frac{f(z)}{z} = \left[\frac{g(z)}{z}\right]^{c}, \quad z \in D,$$

where $g(z) \in S^*(\alpha)$. The branch of the power function is chosen such that $\left.\left[\frac{g(z)}{z}\right]^{c}\right|_{z=0} = 1$.

(iii) *$f \in \hat{S}_\alpha^\beta$ if and only if*

$$\frac{f(z)}{z} = \left[\frac{s(z)}{z}\right]^{(1-\alpha)c}, \quad z \in D,$$

where $s(z) \in S^*$. The branch of the power function is chosen such that $\left.\left[\frac{s(z)}{z}\right]^{(1-\alpha)c}\right|_{z=0} = 1$.

Now we give the proof of (ii) and (iii).

Proof. (ii). First, assume that $f(z) \in \hat{S}_\alpha^\beta$. Setting $g(z) = z\left[\frac{f(z)}{z}\right]^{\frac{e^{i\beta}}{\cos\beta}}$, through simple calculations we may obtain the equality

$$\frac{zg'(z)}{g(z)} = (1 + i\tan\beta)\frac{zf'(z)}{f(z)} - i\tan\beta.$$

Therefore the following inequality holds,

$$\Re e\left[\frac{zg'(z)}{g(z)}\right] = \frac{1}{\cos\beta}\Re e\left[e^{i\beta}\frac{zf'(z)}{f(z)}\right] > \frac{\alpha\cos\beta}{\cos\beta} = \alpha,$$

namely $g(z) \in S^*(\alpha)$.

Conversely, suppose $g(z) \in S^*(\alpha)$, then according to the above calculation, we have the inequality

$$\frac{1}{\cos\beta}\Re e\left[e^{i\beta}\frac{zf'(z)}{f(z)}\right] = \Re e\left[\frac{zg'(z)}{g(z)}\right] > \alpha.$$

This implies

$$\Re e\left[e^{i\beta}\frac{zf'(z)}{f(z)}\right] > \alpha\cos\beta,$$

i.e., $f(z) \in \hat{S}_\alpha^\beta$.

(iii). It is easy to see from (ii) that $f \in \hat{S}_\alpha^\beta$ if and only if $g \in S^*(\alpha)$ such that $\frac{f(z)}{z} = \left[\frac{g(z)}{z}\right]^c$, here $c = e^{-i\beta}\cos\beta$. Noting that $g(z) \in S^*(\alpha)$ if and only if $s(z) \in S^*$ such that $\frac{g(z)}{z} = \left[\frac{s(z)}{z}\right]^{1-\alpha}$ which holds in (i), we may obtain an important relationship between the class of $\hat{S}_\alpha^\beta$ and the class of S^* : $f \in \hat{S}_\alpha^\beta$ if and only if there exists $s(z) \in S^*$ such that $\frac{f(z)}{z} = \left[\frac{s(z)}{z}\right]^{(1-\alpha)c}$. Here, $c = e^{-i\beta}\cos\beta$ and the branch of the power function is chosen such that $\left.\left[\frac{s(z)}{z}\right]^{(1-\alpha)c}\right|_{z=0} = 1$. □

Lemma 2.1 expresses the relations of the $\hat{S}_\alpha^\beta$ and S^* classes, which play a key role in this paper.

Lemma 2.2 ([5], [8]). $A(K) = \left\{|\gamma| \le \frac{1}{2}\right\} \cup \left[\frac{1}{2}, \frac{3}{2}\right]$.

Lemma 2.3. *For* $\alpha \in [0,1)$, $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $J(\hat{S}_\alpha^\beta) = I_{(1-\alpha)e^{-i\beta}\cos\beta}(K)$.

Proof. Let $f \in J(\hat{S}_\alpha^\beta)$, then there exists $g(z) \in \hat{S}_\alpha^\beta$ such that $f(z) = \int_0^z \frac{g(\zeta)}{\zeta}d\zeta$. According to (iii) of Lemma 2.1 there is $s(z) \in S^*$ such that

$$g(z) = z\left[\frac{s(z)}{z}\right]^{(1-\alpha)e^{-i\beta}\cos\beta},$$

therefore

$$f(z) = \int_0^z \left[\frac{s(\zeta)}{\zeta}\right]^{(1-\alpha)e^{-i\beta}\cos\beta} d\zeta.$$

By the relationship of the S^* class and the K class, there exists $u(z) \in K$ such that $s(z) = zu'(z)$, thus

$$f(z) = \int_0^z [u'(\zeta)]^{(1-\alpha)e^{-i\beta}\cos\beta} d\zeta,$$

i.e., $f(z) \in I_{(1-\alpha)e^{-i\beta}\cos\beta}(K)$. As a result, $J(\hat{S}_\alpha^\beta) \subset I_{(1-\alpha)e^{-i\beta}\cos\beta}(K)$ holds.

Conversely, when $f(z) \in I_{(1-\alpha)e^{-i\beta}\cos\beta}(K)$, we can trace back the above procedure to get $f \in J(\hat{S}_\alpha^\beta)$, so $I_{(1-\alpha)e^{-i\beta}\cos\beta}(K) \subset J(\hat{S}_\alpha^\beta)$.

From the above proof, we obtain the assertion. □

Remark 1. If, in the hypothesis of Lemma 2.3, we set $\alpha = 0$, we arrive at Lemma 4 of [4].

3. THE MAIN RESULTS AND THEIR PROOFS

In this section, we let $[z, w]$ denote the closed line segment with endpoints z and w for $z, w \in \mathbb{C}$.

Now we give the main results and their proofs.

Theorem 3.1. *For* $\alpha \in [0, 1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$A(J(\hat{S}_\alpha^\beta)) = \left\{ |\gamma| \le \frac{1}{2(1-\alpha)\cos\beta} \right\} \bigcup \left\{ \frac{e^{i\beta}}{2(1-\alpha)\cos\beta}, \frac{3e^{i\beta}}{2(1-\alpha)\cos\beta} \right\}.$$

Proof. By Lemma 2.3, we have

$$I_\gamma(J(\hat{S}_\alpha^\beta)) = I_\gamma(I_{(1-\alpha)e^{-i\beta}\cos\beta}(K)) = I_{\gamma(1-\alpha)e^{-i\beta}\cos\beta}(K).$$

Therefore, $\gamma \in A(J(\hat{S}_\alpha^\beta))$ if and only if $\gamma(1-\alpha)e^{-i\beta}\cos\beta \in A(K)$, and by Lemma 2.2 we may easily get the result. $\square$

Remark 2. In this theorem, if we set $\alpha = 0$, we obtain Theorem 3 of [4].

Theorem 3.2. *For* $\alpha \in [0, 1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *the inclusion relation* $J(\hat{S}_\alpha^\beta) \subset S$ *holds precisely if either* $\cos\beta \le \frac{1}{2(1-\alpha)}$ *or* $\alpha = \beta = 0$.

Proof. As $\alpha = \beta = 0$, the result holds evidently by Integral transformation 1; while for $\alpha = 0$ and $\beta \ne 0$, the result is Theorem 1 of [4] and was proved by Y.C. Kim and T. Sugawa [4].

If $\alpha \ne 0$ and $\beta = 0$, then $f(z) \in S^*(\alpha)$. By Lemma 2.1(i), there exists $u(z) \in S^*$ such that $u(z) = z\left(\frac{f(z)}{z}\right)^{\frac{1}{1-\alpha}}$. The branch of the power function is chosen such that $\left(\frac{f(z)}{z}\right)^{\frac{1}{1-\alpha}}\Big|_{z=0} = 1$. From Integral transformation 1, we can easily see that there exists $g(z) \in J(\hat{S}_\alpha^\beta)$ such that

$$g(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^{\frac{1}{1-\alpha}} d\zeta.$$

For

$$\Re e\left[1 + \frac{zg''(z)}{g'(z)}\right] = \Re e\left[\frac{1}{1-\alpha}\frac{zf'(z)}{f(z)}\right]$$

and $\Re e\left[\frac{zf'(z)}{f(z)}\right] > \alpha$, we can deduce that $\Re e\left[1 + \frac{zg''(z)}{g'(z)}\right] > 0$. This implies $g(z) \in K$ and $J(S^*(\alpha)) \subset S$.

Now let $\alpha \ne 0$ and $\beta \ne 0$. Since $J(\hat{S}_\alpha^\beta) \subset S$ is equivalent to $1 \in A(J(\hat{S}_\alpha^\beta))$ and $1 \notin \left[\frac{e^{i\beta}}{2(1-\alpha)\cos\beta}, \frac{3e^{i\beta}}{2(1-\alpha)\cos\beta}\right]$, by Theorem 3.1, we deduce that $1 \le \frac{1}{2(1-\alpha)\cos\beta}$, i.e., $\cos\beta \le \frac{1}{2(1-\alpha)}$.

Summarizing the above procedure, for $\alpha \in [0, 1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $J(\hat{S}_\alpha^\beta) \subset S$ holds when $\cos\beta \le \frac{1}{2(1-\alpha)}$ or $\alpha = \beta = 0$.This completes the proof. $\square$

Remark 3. This theorem is an extension of Theorem 1 of [4]. Indeed, if we set $\alpha = 0$, we will obtain the result of [4].

Theorem 3.3. *For* $\alpha \in [0, 1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$A(J(\hat{S})) = \left\{ |\gamma| \le \frac{1}{2(1-\alpha)\cos\beta} \right\}.$$

Proof. In view of $\hat{S} = \bigcup_\beta \hat{S}_\alpha^\beta$ and $A(F) = \{\gamma \in \mathbb{C} : I_\gamma(F) \subset S\}$, we deduce $A(J(\hat{S})) = \bigcap_\beta (J(\hat{S}_\alpha^\beta))$. With the aid of Theorem 3.1, a simple observation gives $A(J(\hat{S})) = \left\{ |\gamma| \le \frac{1}{2(1-\alpha)\cos\beta} \right\}$. Thus the proof is now complete. $\square$

Remark 4. For $\alpha = \beta = 0$, Theorem 3.3 implies the Theorem 2 of [4].

At the end of this paper, we mention the norm estimate of pre-Schwarzian derivatives. The hyperbolic norm of the pre-Schwarzian derivative $T_f = f''/f'$ of $f \in A$ is defined to be

$$\|f\| = \sup_{|z|<1}(1-|z|^2)|T_f(z)|.$$

It is known that f is bounded if $\|f\| < 2$ and the bound depends only on the value of $\|f\|$ ([9]). Since

$$\begin{aligned}\|I_\gamma[f]\| &= \sup_{|z|<1}(1-|z|^2)\left|\frac{\left(\int_0^z[f'(\zeta)]^\gamma d\zeta\right)''}{\left(\int_0^z[f'(\zeta)]^\gamma\right)'}\right| \\ &= \sup_{|z|<1}(1-|z|^2)\left|\frac{([f'(z)]^\gamma)'}{f'(z)]^\gamma}\right| \\ &= \sup_{|z|<1}(1-|z|^2)\left|\frac{\gamma f''(z)}{f'(z)}\right| = |\gamma|\|f\|.\end{aligned}$$

We obtain the following assertion.

Proposition 3.4. *For each $\alpha \in [0,1)$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the sharp inequality $\|f\| \le 4(1-\alpha)\cos\beta$ holds for $f \in J(\hat{S}_\alpha^\beta)$. Moreover, if $\cos\beta < \frac{1}{2(1-\alpha)}$, then a function in $J(\hat{S}_\alpha^\beta)$ is bounded by a constant depending on α and β.*

Proof. For each $f \in J(\hat{S}_\alpha^\beta)$, by Lemma 2.3, there is a function $k \in K$ such that $f = I_\gamma(k)$, where $\gamma = (1-\alpha)e^{-i\beta}\cos\beta$. Noting that $\|k\| \le 4$ [10], we obtain the following inequality

$$\|f\| = |\gamma|\|k\| \le 4|\gamma| = 4(1-\alpha)\cos\beta.$$

Since the inequality $\|k\| \le 4$ is sharp, the above inequality is also sharp. If $\cos\beta < \frac{1}{2(1-\alpha)}$, the above inequality implies $\|f\| \le 4(1-\alpha)\cos\beta < 2$, so f is bounded by a constant depending on α and β. □

Remark 5. If, in the statement of Proposition 3.4, we set $\alpha = 0$, we arrive at the result of [4].

In the above proposition, the bound $\frac{1}{2}$ cannot be replaced by any number greater than $\frac{1}{\sqrt{2(1-\alpha)}}$. Indeed, by the Alexander transformation, if the function

$$g(z) = z(1-z)^{-2(1-\alpha)e^{-i\beta}\cos\beta} \in \hat{S}_\alpha^\beta,$$

then the function

$$f(z) = \frac{(1-z)^{1-2(1-\alpha)e^{-i\beta}\cos\beta} - 1}{2(1-\alpha)e^{-i\beta}\cos\beta - 1} \in J(\hat{S}_\alpha^\beta),$$

and we may verify that the latter is unbounded when $\cos\beta > \frac{1}{\sqrt{2(1-\alpha)}}$.

REFERENCES

[1] M.S. ROBERTSON, On the theory of univalent functions, *Ann. Math.*, **37** (1936), 374–408.

[2] L. SPAČEK, Contribution à la théorie des fonctions univalentes, *Casopis Pěst Math.*, **62** (1932), 12–19, (in Russian).

[3] M.S. ROBERTSON, Univalent functions $f(z)$ for which $zf'(z)$ is spirallike, *Michigan Math. J.*, **16** (1969), 97–101.

[4] Y.C. KIM AND T. SUGAWA, The Alexander transform of a spirallike function, *J. Math. Anal. Appl.*, **325**(1) (2007), 608–611.

[5] Y.C. KIM, S. PONNUSAMY AND T. SUGAWA, Mapping properties of nonlinear integral operators and pre-Schwarzian derivatives, *J. Math. Anal. Appl.*, **299** (2004), 433–447.

[6] A.W. GOODMAN, Univalent functions, I-II, *Mariner Publ.Co.,Tampa Florida*,1983.

[7] I. GRAHAM AND G. KOHR, *Geometric function theory in one and higher dimensions*, Marcel Dekker, New York ,2003.

[8] L.A. AKSENT'EV AND I.R. NEZHMETDINOV, Sufficient conditions for univalence of certain integral transforms, *Tr. Semin. Kraev. Zadacham. Kazan*, **18** (1982), 3–11 (in Russian); English translation in: *Amer. Math. Soc. Transl.*, **136**(2) (1987), 1–9.

[9] Y.C. KIM AND T. SUGAWA, Growth and coefficient estimates for uniformly locally univalent functions on the unit disk, *Rocky Mountain J. Math.*, **32** (2002), 179–200.

[10] S. YAMASHITA, Norm estimates for function starlike or convex of order alpha, *Hokkaido Math. J.*, **28** (1999), 217–230.