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AN EXTENSION OF RESULTS OF A. MCD. MERCER
AND I. GAVREA

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Abstract

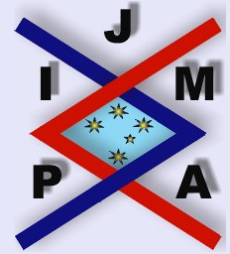
In this note we extend recent results of A. McD. Mercer and I. Gavrea on convex sequences to other classes of sequences.

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Key words: Convex sequence, Polynomial, Convex cone, Dual cone, Farkas lemma, q -class of sequences, Shift operator, Difference operator, Convex sequence of order r .

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1. Introduction

The following result is valid [1, 2]. Let $\mathbf{a} = (a_0, a_1, \dots, a_n)$ be a real sequence. The inequality

$$(1.1) \quad \sum_{k=0}^n a_k u_k \geq 0$$

holds for every convex sequence $\mathbf{u} = (u_0, u_1, \dots, u_n)$ if and only if the polynomial

$$P\mathbf{a}(x) := \sum_{k=0}^n a_k x^k$$

has $x = 1$ as a double root and the coefficients c_k ($k = 0, 1, \dots, n - 2$) of the polynomial

$$\frac{P\mathbf{a}(x)}{(x - 1)^2} = \sum_{k=0}^{n-2} c_k x^k$$

are non-negative. The sufficiency and necessity of this result are due, respectively, to A. McD. Mercer [2] and to I. Gavrea [1].

The purpose of this note is to extend the above result to other classes of sequences \mathbf{u} .



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2. Basic Lemma

A *convex cone* is a non-empty set $C \subset \mathbb{R}^{n+1}$ such that $\alpha C + \beta C \subset C$ for all non-negative scalars α and β . We say that a convex cone C is *generated* by a set $V \subset C$, and write $C = \text{cone } V$, if every vector in C can be expressed as a non-negative linear combination of a finite number of vectors in V .

Let $\langle \cdot, \cdot \rangle$ stand for the standard inner product on \mathbb{R}^{n+1} . The *dual cone* of C is the cone defined by

$$\text{dual } C := \{ \mathbf{u} \in \mathbb{R}^{n+1} : \langle \mathbf{u}, \mathbf{v} \rangle \geq 0 \text{ for all } \mathbf{v} \in C \}.$$

It is well-known that

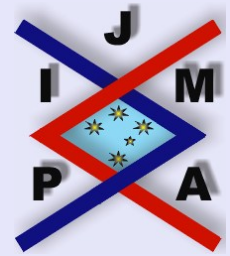
$$(2.1) \quad \text{dual dual } C = C$$

for any closed convex cone $C \subset \mathbb{R}^{n+1}$ (cf. [3, Theorem 14.1, p. 121]). The result below is a key fact in our considerations. It is a consequence of (2.1) for a finitely generated cone $C = \text{cone } \{ \mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_p \}$.

Lemma 2.1 (Farkas lemma). *Let $\mathbf{v}, \mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_p$ be vectors in \mathbb{R}^{n+1} . The following two statements are equivalent:*

- (i) *The inequality $\langle \mathbf{u}, \mathbf{v} \rangle \geq 0$ holds for all $\mathbf{u} \in \mathbb{R}^{n+1}$ such that $\langle \mathbf{u}, \mathbf{v}_i \rangle \geq 0$, $i = 0, 1, \dots, p$.*
- (ii) *There exist non-negative scalars c_i , $i = 0, 1, \dots, p$, such that*

$$\mathbf{v} = c_0 \mathbf{v}_0 + c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p.$$



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3. Main Result

Given a sequence $\mathbf{q} = (q_0, q_1, \dots, q_r) \in \mathbb{R}^{r+1}$ with $0 \leq r \leq n$, we define

$$(3.1) \quad \mathbf{v}_i := (\underbrace{0, \dots, 0}_{i \text{ times}}, q_0, q_1, \dots, q_r, 0, \dots, 0) = S^i \mathbf{v}_0 \in \mathbb{R}^{n+1}$$

for $i = 0, 1, \dots, n - r$.

Here S is the *shift operator* from \mathbb{R}^{n+1} to \mathbb{R}^{n+1} defined by

$$(3.2) \quad S(z_0, z_1, \dots, z_n) := (0, z_0, z_1, \dots, z_{n-1}).$$

A sequence $\mathbf{u} = (u_0, u_1, \dots, u_n) \in \mathbb{R}^{n+1}$ is said to be of \mathbf{q} -class, if

$$(3.3) \quad \langle \mathbf{u}, \mathbf{v}_i \rangle \geq 0 \quad \text{for all } i = 0, 1, \dots, n - r.$$

In other words, the \mathbf{q} -class consists of all vectors of the cone

$$(3.4) \quad D := \text{dual cone } \{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-r}\}.$$

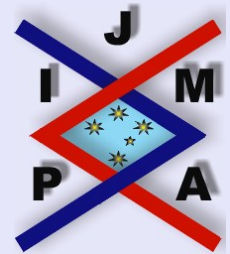
Example 3.1.

(a). Set $r = 0$, $q_0 = 1$ and

$$\mathbf{v}_i = (\underbrace{0, \dots, 0}_{i \text{ times}}, 1, 0, \dots, 0) \in \mathbb{R}^{n+1} \quad \text{for } i = 0, 1, \dots, n.$$

Then (3.3) reduces to

$$u_i \geq 0 \quad \text{for } i = 0, 1, \dots, n.$$



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Thus D is the class of non-negative sequences.

(b). Put $r = 1$, $q_0 = -1$ and $q_1 = 1$, and denote

$$\mathbf{v}_i = (\underbrace{0, \dots, 0}_{i \text{ times}}, -1, 1, 0, \dots, 0) \in \mathbb{R}^{n+1} \text{ for } i = 0, 1, \dots, n-1.$$

Then (3.3) gives

$$u_i \leq u_{i+1} \text{ for } i = 0, 1, \dots, n-1,$$

which means that D is the class of non-decreasing sequences.

(c). Consider $r = 2$, $q_0 = 1$, $q_1 = -2$, $q_2 = 1$ and

$$\mathbf{v}_i = (\underbrace{0, \dots, 0}_{i \text{ times}}, 1, -2, 1, 0, \dots, 0) \in \mathbb{R}^{n+1} \text{ for } i = 0, 1, \dots, n-2.$$

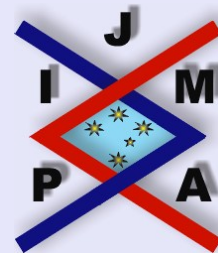
In this case, (3.3) is equivalent to

$$u_{i+1} \leq \frac{u_i + u_{i+2}}{2} \text{ for } i = 0, 1, \dots, n-2.$$

This says that \mathbf{u} is a convex sequence. Therefore D is the class of convex sequences.

Theorem 3.1. Let $\mathbf{a} = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}$ and $\mathbf{q} = (q_0, q_1, \dots, q_r) \in \mathbb{R}^{r+1}$ be given with $0 \leq r \leq n$. Then the inequality

$$(3.5) \quad \sum_{k=0}^n a_k u_k \geq 0$$



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holds for every sequence $\mathbf{u} = (u_0, u_1, \dots, u_n)$ of \mathbf{q} -class if and only if the polynomial

$$P_{\mathbf{a}}(x) := \sum_{k=0}^n a_k x^k$$

is divisible by the polynomial

$$P_{\mathbf{q}}(x) := \sum_{k=0}^r q_k x^k,$$

and the coefficients c_k ($k = 0, 1, \dots, n - r$) of the polynomial

$$\frac{P_{\mathbf{a}}(x)}{P_{\mathbf{q}}(x)} = \sum_{k=0}^{n-r} c_k x^k$$

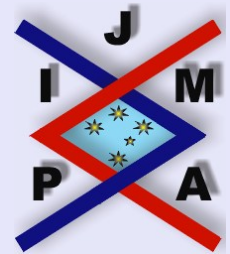
are non-negative.

Proof. The map φ that assigns to each sequence $\mathbf{b} = (b_0, b_1, \dots, b_m)$ the polynomial

$$\varphi(\mathbf{b}) := P_{\mathbf{b}}(x) := \sum_{k=0}^m b_k x^k$$

is a one-to-one linear map from \mathbb{R}^{m+1} to the space of polynomials of degree at most m . Also, $\psi := \varphi^{-1}$ is a one-to-one linear map. It is not difficult to check that

$$\psi(x^k P_{\mathbf{b}}(x)) = S^k \psi(P_{\mathbf{b}}(x)).$$



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Therefore, for any polynomial

$$P_{\mathbf{c}}(x) := c_0 + c_1x + \cdots + c_{n-r}x^{n-r},$$

we have

$$\begin{aligned} \psi(P_{\mathbf{c}}(x)P_{\mathbf{q}}(x)) &= c_0S^0\mathbf{v}_0 + c_1S^1\mathbf{v}_0 + \cdots + c_{n-r}S^{n-r}\mathbf{v}_0 \\ &= c_0\mathbf{v}_0 + c_1\mathbf{v}_1 + \cdots + c_{n-r}\mathbf{v}_{n-r}, \end{aligned}$$

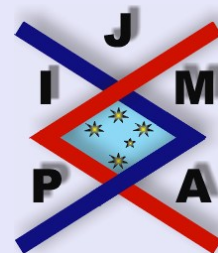
where \mathbf{v}_i are given by (3.1). In other words,

$$(3.6) \quad P_{\mathbf{c}}(x)P_{\mathbf{q}}(x) = \varphi(c_0\mathbf{v}_0 + c_1\mathbf{v}_1 + \cdots + c_{n-r}\mathbf{v}_{n-r})$$

for any $\mathbf{c} = (c_0, c_1, \dots, c_{n-r})$.

We are now in a position to show that the following statements are mutually equivalent:

- (i) Inequality (3.5) holds for every \mathbf{u} of \mathbf{q} -class.
- (ii) $\langle \mathbf{a}, \mathbf{u} \rangle \geq 0$ for every $\mathbf{u} \in \text{dual cone } \{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_{n-r}\}$.
- (iii) There exist non-negative scalars c_0, c_1, \dots, c_{n-r} such that $\mathbf{a} = c_0\mathbf{v}_0 + c_1\mathbf{v}_1 + \cdots + c_{n-r}\mathbf{v}_{n-r}$.
- (iv) There exist non-negative scalars c_0, c_1, \dots, c_{n-r} such that $P_{\mathbf{a}}(x) = (c_0 + c_1x + \cdots + c_{n-r}x^{n-r})P_{\mathbf{q}}(x)$.



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In fact, (ii) is an easy reformulation of (i) (see (3.4)). That (ii) and (iii) are equivalent is a direct consequence of Farkas lemma (see Lemma 2.1). We now show the validity of the implication (iii) \Rightarrow (iv). By (iii) and (3.6), we have

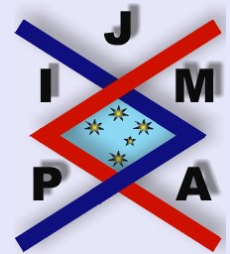
$$P\mathbf{a}(x) = \varphi(\mathbf{a}) = \varphi(c_0\mathbf{v}_0 + c_1\mathbf{v}_1 + \cdots + c_{n-r}\mathbf{v}_{n-r}) = P\mathbf{c}(x)P\mathbf{q}(x)$$

for certain scalars $c_k \geq 0$, $k = 0, 1, \dots, n - r$. Thus (iv) is proved.

To prove the implication (iv) \Rightarrow (iii) assume (iv) holds, that is $P\mathbf{a}(x) = P\mathbf{c}(x)P\mathbf{q}(x)$ with $c_k \geq 0$, $k = 0, 1, \dots, n - r$. Then by (3.6),

$$\begin{aligned} \mathbf{a} &= \psi(P\mathbf{a}(x)) = \psi(P\mathbf{c}(x)P\mathbf{q}(x)) \\ &= \psi\varphi(c_0\mathbf{v}_0 + c_1\mathbf{v}_1 + \cdots + c_{n-r}\mathbf{v}_{n-r}) \\ &= c_0\mathbf{v}_0 + c_1\mathbf{v}_1 + \cdots + c_{n-r}\mathbf{v}_{n-r}. \end{aligned}$$

This completes the proof of Theorem 3.1. □



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4. Applications for Convex Sequences of Order r

In this section we study special types of sequences related to difference calculus and generalized convex sequences.

We introduce the *difference operator* on sequences $\mathbf{z} = (z_0, z_1, \dots, z_m)$ by

$$\Delta \mathbf{z} := (z_1 - z_0, z_2 - z_1, \dots, z_m - z_{m-1}).$$

Notice that $\Delta = \Delta_m$ acts from \mathbb{R}^{m+1} to \mathbb{R}^m . We define

$$\Delta^0 \mathbf{z} := \mathbf{z} \text{ and } \Delta^r \mathbf{z} := \Delta_{m-r+1} \cdots \Delta_{m-1} \Delta_m \mathbf{z} \text{ for } r = 1, 2, \dots, m.$$

A sequence $\mathbf{u} \in \mathbb{R}^{n+1}$ is said to be *convex of order r* (in short, *r -convex*), if

$$\Delta^r \mathbf{u} \geq 0.$$

The last inequality is meant in the componentwise sense in \mathbb{R}^{n+1-r} , that is

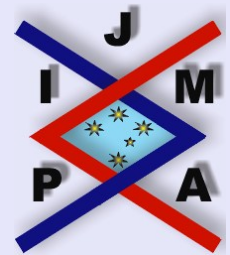
$$(4.1) \quad \langle \Delta^r \mathbf{u}, \mathbf{e}_i \rangle \geq 0 \text{ for } i = 0, 1, \dots, n-r,$$

where

$$\mathbf{e}_i := (\underbrace{0, \dots, 0}_{i \text{ times}}, 1, 0, \dots, 0) \in \mathbb{R}^{n+1-r}.$$

In order to relate the r -convex sequences to the \mathbf{q} -class of Section 3, observe that (4.1) amounts to

$$\langle \mathbf{u}, (\Delta^r)^T \mathbf{e}_i \rangle \geq 0 \text{ for } i = 0, 1, \dots, n-r,$$



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where $(\cdot)^T$ denotes the transpose. By a standard induction argument, we get

$$(\Delta^r)^T \mathbf{e}_i = S^i \mathbf{v}_0 \text{ for } i = 0, 1, \dots, n - r,$$

where S is the shift operator from \mathbb{R}^{n+1} to \mathbb{R}^{n+1} given by (3.2), and

$$(4.2) \quad \mathbf{v}_0 := (\mathbf{q}, 0, \dots, 0) \in \mathbb{R}^{n+1} \quad \text{and} \quad \mathbf{q} := (q_0, q_1, \dots, q_r)$$

$$\text{with } q_j := \binom{r}{j} (-1)^{r-j}.$$

As in (3.1), we set

$$\mathbf{v}_i := S^i \mathbf{v}_0 \text{ for } i = 0, 1, \dots, n - r.$$

In summary, the r -convex sequences form the \mathbf{q} -class for \mathbf{q} given by (4.2). For example, the class of r -convex sequences for $r = 0$ (resp. $r = 1$, $r = 2$) is the class of non-negative (resp. non-decreasing, convex) sequences in \mathbb{R}^{n+1} (cf. Example 3.1).

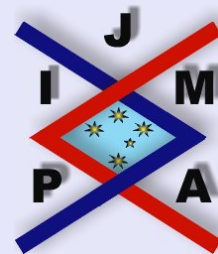
By virtue of (4.2) we get

$$P_{\mathbf{q}}(x) = \sum_{k=0}^r q_k x^k = (x - 1)^r.$$

Therefore we obtain from Theorem 3.1

Corollary 4.1. *Let $\mathbf{a} = (a_0, a_1, \dots, a_n) \in \mathbb{R}^{n+1}$ be given with $0 \leq r \leq n$. Then the inequality*

$$(4.3) \quad \sum_{k=0}^n a_k u_k \geq 0$$



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holds for every r -convex sequence $\mathbf{u} = (u_0, u_1, \dots, u_n)$ if and only if the polynomial

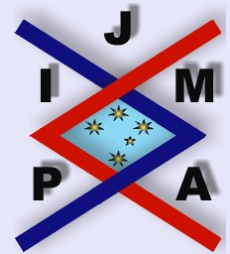
$$P_{\mathbf{a}}(x) = \sum_{k=0}^n a_k x^k$$

has $x = 1$ as a root of multiplicity at least r , and the coefficients c_k ($k = 0, 1, \dots, n - r$) of the polynomial

$$\frac{P_{\mathbf{a}}(x)}{(x - 1)^r} = \sum_{k=0}^{n-r} c_k x^k$$

are non-negative.

Corollary 4.1 extends the mentioned results of A. McD. Mercer and I. Gavrea from $r = 2$ to an arbitrary $0 \leq r \leq n$.



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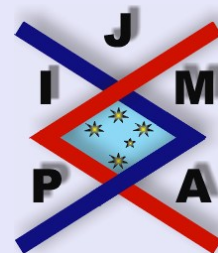
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