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THE LATTICE OF THRESHOLD GRAPHS

RUSSELL MERRIS AND TOM ROBY

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
CALIFORNIA STATE UNIVERSITY
HAYWARD, CA 94542
merris@csuhayward.edu
URL: http://www.sci.csuhayward.edu/∼rmerris

troby@csuhayward.edu
URL: http://seki.csuhayward.edu/∼troby

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Dedicated to the memory of Russ's mother, Joan Diane Seiss, and Tom's stepmother, Mary Lederer Roby.

ABSTRACT. Due in part to their many interesting properties, a family of graphs has been studied under a variety of names, by various authors, since the late 1970's. Only recently has it become apparent that the many different looking definitions for these *threshold* graphs are all equivalent. While the pedigree of *strict partitions* of positive integers is much older, their evolution into the lattice of shifted shapes is relatively recent. In this partly expository article we show, from the perspective of partially ordered sets, that the family of connected threshold graphs is isomorphic to the lattice of shifted shapes, and then discuss some implications of this identification for threshold graphs.

Key words and phrases: Automorphism group, Distributive lattice, Eigenvalue, Graphic partition, Laplacian spectrum, Order ideal, Poset, Projective representation, Saturated chain, Shifted shape, Split graph, Strict partition, Threshold graph, Young subgroup, Young's lattice.

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1. PRELIMINARIES

A *partition* of r is a nonnegative integer sequence $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, where $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_n$, and $r = \pi_1 + \pi_2 + \cdots + \pi_n$. The nonzero π_i are called the *parts* of π and their number,

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denoted $\ell(\pi)$, is the *length* of π. We will write $\pi \vdash r$ to indicate that π is a partition of r, and refer to r as the *rank* of π.

Two partitions of r are *equivalent* if they have the same multiset of parts, i.e., if they differ only in the number of terminal 0's. Thus, e.g.,

$$(6,2,2,1),\ \ (6,2,2,1,0),\ \ (6,2,2,1,0,0),\ldots$$

are equivalent partitions of 11 each of length 4; $(0,0,0)$ is equivalent to the *empty partition* φ of length and rank 0.

Example 1.1. Suppose $G=(V,E)$ is a (simple) graph with vertex set $V=\{1,2,\ldots,n\}$ and edge set E of cardinality $o(E)=m$. Denote by $d_G(i)$ the **degree** of vertex i, that is, the number of edges of G incident with i. Suppose $d_1\geq d_2\geq\cdots\geq d_n\geq 0$ are these vertex degrees (re)arranged in nonincreasing order. By what has come to be known as the "first theorem of graph theory", $d(G)=(d_1,d_2,\ldots,d_n)\vdash 2m$.

Say that partition $\pi=(\pi_1,\pi_2,\ldots,\pi_n)$ is *graphic* if there is a graph H with $\pi=d(H)$. Not every partition is graphic. If π is graphic, its rank must be even and, because (simple) graphs have no loops or multiple edges, $\pi_1\leq\ell(\pi)-1$. That these obvious necessary conditions are not sufficient is illustrated, e.g., by $\rho=(5,4,4,2,2,1)$.

The unifying theme of the present paper is the notion of a "maximal" graphic partition. To make this idea precise, suppose $\alpha=(a_1,a_2,\ldots,a_s)$ and $\beta=(b_1,b_2,\ldots,b_t)$ are nonincreasing sequences of real numbers. Then β *weakly majorizes* α, written $\beta \underset{w}{\succ} \alpha$, if $s\geq t$,

$$\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} a_i, \qquad 1\leq k\leq t, \tag{1.1}$$

and

$$\sum_{i=1}^{t} b_i \geq \sum_{i=1}^{s} a_i. \tag{1.2}$$

If β weakly majorizes α, and equality holds in Inequality (1.2), then β *majorizes* α, written $\beta\succ\alpha$. If $\beta\succ\alpha$ and β is not equivalent to α, then β *strictly majorizes* α. (The standard reference for variations on the theme of majorization is [16].)

For nonnegative integer sequences, majorization has a useful geometric description. Suppose $\pi\vdash r>0$. The *Ferrers* (or *Young*) *diagram* $F(\pi)$ is a left-justified array consisting of $\ell(\pi)$ rows of "boxes"; the i^{th} row of $F(\pi)$ contains a total of π_i boxes. The Ferrers diagram afforded, e.g., by $\tau=(4,3,3,2,2,2)\vdash 16$ is illustrated in Fig. 1.1. Because rows that contain zero boxes do not explicitly appear in $F(\pi)$, equivalent partitions afford the same Ferrers diagram. For the most part, we will treat equivalent partitions as if they were equal.

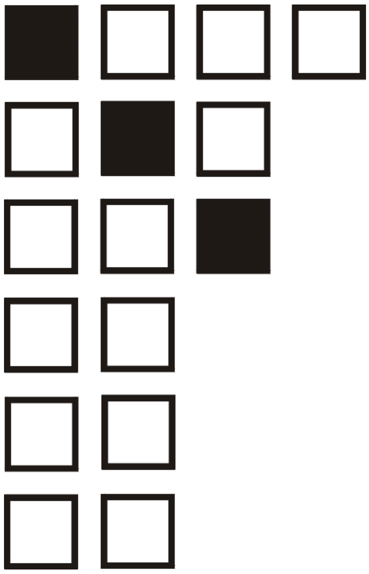

Figure 1.1: $F(\tau)$.

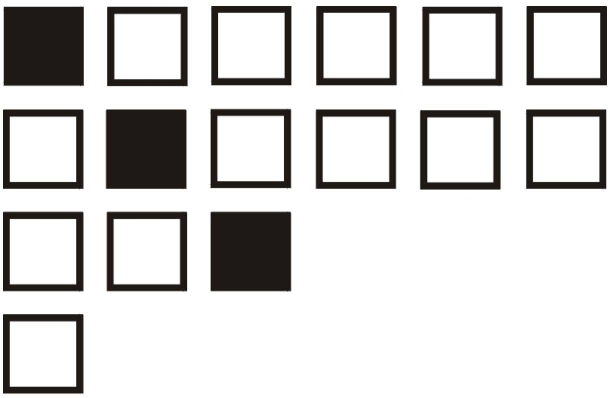

Figure 1.2: $F(\tau^*) = F(\tau)^t$.

Lemma 1.1 (Muirhead's Lemma [16, p. 135])**.** *If* π,$\gamma \vdash r$*, then* $\pi \succ \gamma$ *if and only if* $F(\pi)$ *can be obtained from* $F(\gamma)$ *by moving boxes up (to lower numbered rows).*

A little care must be taken when moving boxes to ensure that the resulting array is a legitimate Ferrers diagram. With this caveat in mind, it follows easily from Lemma 1.1 that majorization induces a partial order on $\{F(\pi) : \pi \vdash r\}$. In other words, the set of (equivalence classes of) partitions of r is partially ordered by majorization.

Lemma 1.2 ([22])**.** *Suppose* π,$\gamma \vdash r$*. If* π *is graphic and if* π *majorizes* γ*, then* γ *is graphic.*

Suppose $d(G) = \pi$. While the details may be a little awkward to write down, the proof of Lemma 1.2 amounts to showing how moving boxes down in $F(\pi)$ can be made to correspond to moving edges around in a graph obtained from G by adding sufficiently many isolated vertices.

Definition 1.1. A graphic partition $\pi \vdash r$ is *maximal* provided no graphic partition strictly majorizes π.

There are several well known criteria for a partition to be graphic (see, e.g., [23], but be wary of misprints). For our purposes, the most useful necessary and sufficient conditions are those commonly attributed to Hässelbarth [12], but first published by Ruch and Gutman [22].

Suppose $\pi \vdash r$. The *conjugate* of π is the partition π^* whose Ferrers diagram $F(\pi^*) = F(\pi)^t$, the *transpose* of $F(\pi)$. In other words, $\pi^* \vdash r$ is the partition whose i^{th} part is $\pi_i^* = o(\{j : \pi_j \geq i\})$, the number of boxes in the i^{th} *column* of $F(\pi)$. If $\tau = (4,3,3,2,2,2)$ then (see Fig. 1.2) $\tau^* = (6,6,3,1)$.

The number of *diagonal boxes* in $F(\pi)$ is $f(\pi) = o(\{i : \pi_i \geq i\})$. The diagonal boxes in Fig.s 1.1 – 1.2 have been filled (darkened), making it easy to see that $f(\tau) = 3 = f(\tau^*)$. Note that $F(\pi)$ is completely determined by its first $f(\pi)$ rows and columns.

Theorem 1.3 (Ruch-Gutman Theorem [22])**.** *Suppose $\pi \vdash 2m$. Then π is graphic if and only if*

$$\sum_{i=1}^{k} \pi_i \leq \sum_{i=1}^{k} (\pi_i^* - 1), \qquad 1 \leq k \leq f(\pi). \tag{1.3}$$

If $\tau = (4,3,3,2,2,2) \vdash 16$ then, as we have seen, $f(\tau) = 3$ and $\tau^* = (6,6,3,1)$. Because $4 < 6-1$, $4+3 < (6-1)+(6-1)$, and $4+3+3 < (6-1)+(6-1)+(3-1)$, the Ruch-Gutman inequalities are satisfied: τ is graphic. Two nonisomorphic graphs with degree sequence τ are exhibited in Fig. 1.3. If $\pi = (5,4,3,2,1) \vdash 15$ then, because 15 is odd, π is not graphic. If $\rho = (5,4,4,2,2,1) \vdash 18$, then $f(\rho) = 3$ and $\rho^* = (6,5,3,3,1)$. While $5 = (6-1)$ and $5+4 = (6-1)+(5-1)$, the third inequality in (1.3) is not satisfied; $5+4+4 > (6-1)+(5-1)+(3-1)$. Because ρ does not satisfy the Ruch-Gutman inequalities, it is not graphic (confirming an earlier observation).

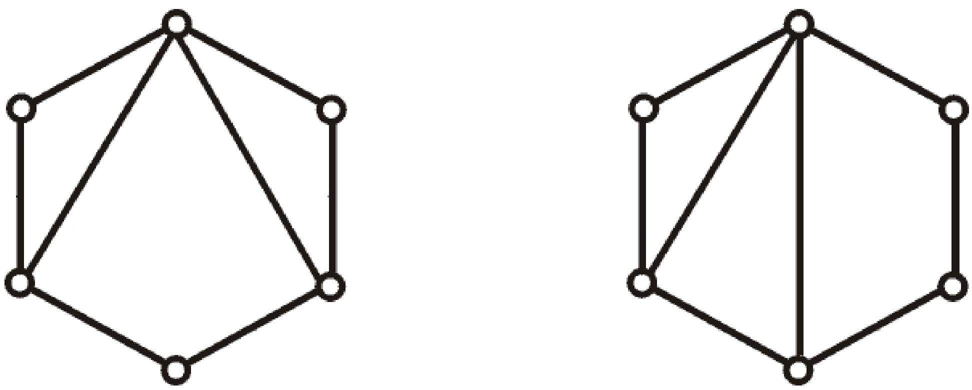

Figure 1.3: Graphs satisfying $d(G) = \tau = (4,3,3,2,2,2)$.

Definition 1.2. A *threshold* partition is a graphic partition for which equality holds throughout (1.3), i.e., $\pi \vdash 2m$ is a threshold partition if and only if

(1.4) $$\pi_i = \pi_i^* - 1, \qquad 1 \leq i \leq f(\pi).$$

Geometrically, π is a threshold partition if and only if $F(\pi)$ can be decomposed, as in Fig. 1.4, into an $f(\pi) \times f(\pi)$ array of boxes in the upper left-hand corner, called the *Durfee square*, a row of $f(\pi)$ boxes directly below the Durfee square, darkened in Fig. 1.4, and a piece below row $f(\pi) + 1$ that is the transpose of the piece to the right of the Durfee square. It follows, for $f(\pi) < k < \ell(\pi)$, that $\pi_k^* = \pi_{k+1} \leq \pi_k$. Thus, for any threshold partition π, of length $n = \ell(\pi)$,

(1.5) $$\pi_k^* \leq \pi_k + 1, \qquad 1 \leq k < n.$$

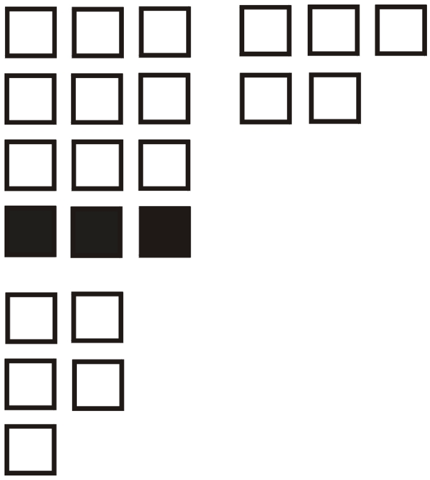

Figure 1.4: Decomposition of $F(6, 5, 3, 3, 2, 2, 1)$.

Theorem 1.4. *Suppose* $\pi \vdash 2m$*. Then* π *is a maximal graphic partition if and only if* π *is a threshold partition.*

The idea of the proof is that Inequalities (1.3) precisely limit the extent to which boxes can be moved up in a Ferrers diagram and maintain the property that the corresponding partition is graphic. Details can be found, e.g., in [22].

A *threshold graph* is one whose degree sequence is a threshold (maximal) partition. First introduced in connection with set packing and knapsack problems [3] and, independently, in the analysis of parallel processes in computer programming [13], threshold graphs have been rediscovered in a variety of contexts, leading to numerous equivalent definitions. (See, e.g., [1], [5], [11], [15], [17], [18], [20], and [21].)

Suppose $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \vdash 2m$ is a threshold partition of length t (so that $\pi_t > 0 = \pi_{t+1} = \cdots = \pi_n$). Let G be a threshold graph with $d(G) = \pi$. Then G has $n - t$ isolated vertices (that go unrepresented in $F(\pi)$). Moreover, because $\pi_1 + 1 = \pi_1^* = t$, it must be that some vertex of G is adjacent to every other vertex of positive degree. So, if G is a threshold graph then it can have at most one nontrivial component (consisting of more than one vertex), and that component must have at least one *dominating* vertex.

Say that two graphs are *equivalent* if they are isomorphic, to within isolated vertices; that is, H_1 and H_2 are equivalent if they are both edgeless graphs or if $H_1' \cong H_2'$, where H_i' is the graph obtained from H_i by deleting all of its isolated vertices, $i = 1, 2$. In particular, every threshold graph is equivalent to a connected threshold graph.

Theorem 1.5. *If π is a threshold partition then, up to isomorphism, there is exactly one connected threshold graph G that satisfies $d(G) = \pi$.*[†]

For the sake of completeness, we sketch a proof of this well-known result. Suppose u is a dominating vertex of a graph G. Let $H = G - u$ be the graph obtained from G by deleting vertex u (and all the edges incident with it). Because the Ferrers diagram $F(d(H))$ is obtained from $F(d(G))$ by deleting its first row and column, G is a threshold graph if and only if H is a threshold graph. (See, e.g., Fig. 1.5.) The result now follows by induction and the fact that every graph on fewer than five vertices is uniquely determined by its degree sequence.

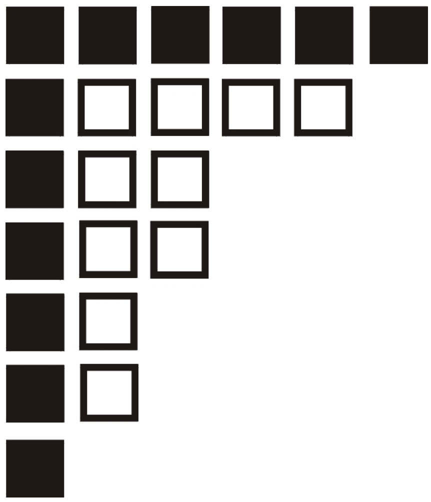

Figure 1.5: $F(6, 5, 3, 3, 2, 2, 1)$

The idea for the proof of Theorem 1.5 can be used to construct a threshold graph having a prescribed (threshold) degree sequence.

Algorithm 1.1 (Threshold Algorithm)**.** Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \vdash 2m$ be a threshold partition.
Set $V = \{1, 2, \ldots, n\}$ and $E = \phi$

[†]Indeed, more is true: Apart from isolated vertices, there is a unique *labeled* graph with degree sequence π. As present purposes do not require this stronger result, we say no more about it.

For $i = 1$ to $f(\pi)$
    For $j = i$ to π_i
        $E = E \cup \{\{i, j+1\}\}$
    Next j
Next i
End

Suppose $\pi = (6, 5, 3, 3, 2, 2, 1)$ is the threshold partition whose Ferrers diagram appears in Fig. 1.5. If π is used as input for the Threshold Algorithm, the output is illustrated in Fig. 1.6.

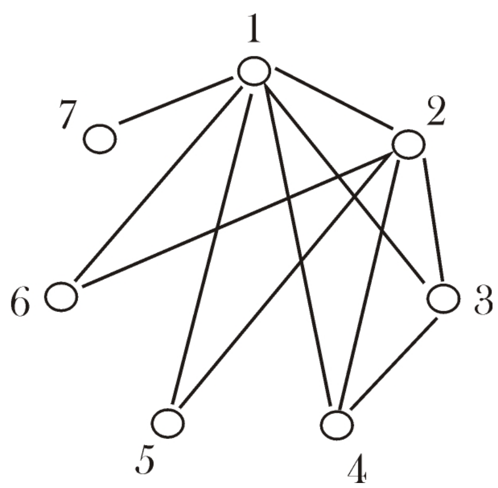

Figure 1.6: A threshold graph.

The reader may verify that if $\tau = (4, 3, 3, 2, 2, 2)$ were used as input, the output of the Threshold Algorithm would be a graph with degree sequence $(4, 3, 3, 3, 1, 0)$. (While τ is graphic, it is not maximal.)

Recall that the *complement* of $G = (V, E)$ is the graph $G^c = (V, E^c)$, where $uv \in E^c$ if and only if $uv \notin E$, i.e., the edges of G^c are the edges of the complete graph, K_n, that do not belong to G. If Fig. 1.6 is viewed as a clockwise application of the Threshold Algorithm, the edges of K_7 that are "missing" from Fig. 1.6 may be construed as a counterclockwise application, constructing G^c. Note that the degree sequence, $d(G^c)$, corresponds to the shape complementary to $F(d(G))$ inside the $n \times (n-1)$ rectangle. For the threshold graph of Figures 1.5 and 1.6, we get $d(G^c) = (5, 4, 4, 3, 3, 1)$. These observations yield the well known fact that G is a threshold graph if and only if G^c is a threshold graph.

2. THRESHOLD GRAPHS

Suppose G is a threshold graph. It is convenient to denote by $F(G)$ the Ferrers diagram corresponding to $d(G)$. Similarly, let $f(G) = f(d(G))$ be the number of boxes on the main diagonal of $F(G)$.

For any graph $G = (V, E)$, the set of *neighbors* of $u \in V$ is $N_G(u) = \{v \in V : uv \in E\}$. The Threshold Algorithm produces a graph G on vertex set $V = \{1, 2, \ldots, n\}$ that satisfies

$$N_G(i) = \{1, 2, \ldots, i-1, i+1, \ldots, \pi_i + 1\}, \qquad i \le f(\pi), \tag{2.1}$$

and

$$N_G(i) = \{1, 2, \ldots, \pi_i\}, \qquad i > f(\pi), \tag{2.2}$$

where $\pi = d(G)$. In particular, the k^{th} largest vertex degree of G is $d_k = d_G(k)$, $1 \le k \le n$.

Lemma 2.1. *Let $G = (V, E)$ be a connected threshold graph on $n \ge 3$ vertices. If $G \ne K_n$, then there is a nonadjacent pair of vertices $i, j \in V$ such that $H = (V, E \cup \{ij\})$ is a threshold graph.*

Proof. Without loss of generality we may assume that $V = \{1, 2, \ldots, n\}$ and that $d_k = d_G(k)$, $1 \le k \le n$. Because $G \ne K_n$, $f(G) < n - 1$. Let i be minimal such that $d_i < n - 1$. Then $2 \le i \le f(G) + 1$. If $i = f(G) + 1$ then, because $d(G)$ is a threshold sequence, $d_i = f(G) = i-1$. (See Fig. 1.4.) Choose $j = i+1$. By (2.2), $ij \notin E$. Since $d_{i-1} = n-1$ forces $d_n \ge i - 1$, it must be that $d_1 = d_2 = \cdots = d_{i-1} = n - 1$ and $d_i = d_{i+1} = \cdots = d_n = i - 1$. In this case $F(H)$ is obtained from $F(G)$ by adding new boxes in positions (i, i) and $(i+1, i)$, the first at the end of row i and the second at the end of column i. In particular, $d(H)$ is a threshold sequence.

If $i \le f(G)$, then $d_i \ge i$. Choose $j = d_i + 2 \le n$. By (2.1), $ij \notin E$. Since $d_i = j - 2$ forces $d_i^* = j - 1$, it must be that $d_{j-1} \ge i$ and $d_j < i$. Because $d_{i-1} = n - 1$, $d_j \ge d_n \ge i - 1$. Therefore, $d_j = i - 1$. In this case, $F(H)$ is obtained from $F(G)$ by adding two new boxes in positions $(i, j-1)$ and (j, i), one at the end of row i, and a second at the end of column i. Thus, $d(H)$ is a threshold sequence. □

Denote by T_n, $n \ge 1$, the set of connected threshold graphs on n vertices. If $n \ge 2$, then $o(T_n) = 2^{n-2}$ (an observation, implicit in [15, p. 468], made explicit in [18]). Let Θ_n be the graph with vertex set T_n, in which $G, H \in T_n$ are adjacent if and only if (up to isomorphism) G can be obtained from H by the addition or deletion of a single edge. (The graph Θ_n is an undirected variation on a theme of Balińska and Quintas [2]. When extended to include disconnected threshold graphs, it becomes the 1-skeleton of the polytope of degree sequences studied in [21].)

Theorem 2.2. *If $n \ge 1$ then Θ_n is connected.*

Proof. Let $G \in T_n$. If $G \neq K_n$ then (Lemma 2.1) there is a path in Θ_n from G to K_n. □

Definition 2.1. If G and H are graphs, write $G \leq H$ to indicate that G is equivalent to a subgraph of H.

Strictly speaking, Definition 2.1 partially orders not the family of graphs, but the set of all equivalence classes of graphs. Like flies swarming around a thoroughbred horse, isolated vertices associated with threshold graphs are a trivial but annoying complication. From this point on, we will treat equivalent threshold graphs as if they were equal. Consistent with our treatment of equivalent partitions, this amounts to little more than choosing the connected threshold graphs as a system of distinct representatives for the equivalence classes of all threshold graphs. Given this identification, the restriction of "$\leq$" to T_n is a partial order, and Θ_n may be viewed as a "Hasse diagram" for the partially ordered set (poset) T_n.

Recall that a poset P is *locally finite* if the interval $[x, z] = \{y \in P : x \leq y \leq z\}$ is finite for all $x, z \in P$. If $x, z \in P$ and $[x, z] = \{x, z\}$, then z *covers* x. A *Hasse diagram* of P is a graph whose vertices are the elements of P, whose edges are the cover relations, and such that z is drawn "above" x whenever $x < z$.

A *lattice* is a poset P in which every pair of elements $x, y \in P$ has a least upper bound (or *join*), $x \vee y \in P$, and a greatest lower bound (or *meet*), $x \wedge y \in P$. Lattice P is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in P$. (An excellent reference for variations on the theme of posets is [27].)

Denote by Y the set of all (equivalence classes of) partitions. If $\mu, \nu \in Y$, define $\mu \leq \nu$ to mean that $\ell(\mu) \leq \ell(\nu)$ and $\mu_i \leq \nu_i$, $1 \leq i \leq \ell(\mu)$. Informally, $\mu \leq \nu$ if $F(\mu) \subset F(\nu)$ in the sense that $F(\mu)$ fits inside $F(\nu)$. With respect to this partial ordering, Y is a locally finite distributive lattice, commonly known as *Young's lattice*. (See, e.g., [7], [25], or [27].) The unique smallest element of Y is $\hat{0} = \varphi$, the empty partition.

Definition 2.2. For each $n \geq 2$, denote by Y_n the induced subposet of Y corresponding to the threshold partitions of length n, i.e., the even rank partitions π that satisfy (1.4) and whose first part is $\pi_1 = n - 1$. Let $Y_1 = \{\varphi\}$.

The poset Y_n is half of the "minuscule poset" $M(n)$ discussed, e.g., in [24, §5].

Lemma 2.3. *Suppose $G, H \in T_n$. Then $G \leq H$ (in T_n) if and only if $d(G) \leq d(H)$ (in Y_n).*

Proof. We begin by extending the partial order of Young's lattice to unordered sequences of nonnegative integers: If $A = (a_1, a_2, \ldots, a_r)$ and $B = (b_1, b_2, \ldots, b_s)$, define $A \geq B$ to mean that $r \geq s$, and $a_i \geq b_i$, $1 \leq i \leq s$. If we denote by $\bar{A} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_r)$ the sequence obtained from A by rearranging its elements in nonincreasing order, it follows by induction that $\bar{A} \geq \bar{B}$ whenever $A \geq B$. In particular, if G is obtained from H by deleting one or more edges, then $d_G(i) \leq d_H(i)$, $1 \leq i \leq n$; that is, $G \leq H$ implies $d(G) \leq d(H)$.

Conversely, let $G = (V, E)$ and $H = (W, F)$ be connected threshold graphs on n vertices with $d(G) \leq d(H)$. By the Threshold Algorithm, we may assume $V = W = \{1, 2, \ldots, n\}$; if $d_i = d_G(i)$ and $\delta_i = d_H(i)$, $1 \leq i \leq n$, that $d(G) = (d_1, d_2, \ldots, d_n)$, and $d(H) = (\delta_1, \delta_2, \ldots, \delta_n)$; and that $d_1 = n - 1 = \delta_1$. If $d(G) = d(H)$, then (Theorem 1.5) $G \cong H$. Otherwise, $F(G) \neq F(H)$ and there is a largest positive integer $k \leq f(H)$, such that $d_k < \delta_k$. Let $r = \delta_k^* = \delta_k + 1$. By (2.1), $e = kr \in E(H)$. By (1.5), $r > d_k + 1 \geq d_k^* = o(\{i : d_i \geq k\})$, which implies that $d_r < k$. Similarly, $r = \delta_k^*$ implies that $\delta_r \geq k$. Thus, $\delta_r > d_r$. Let $H' = H - e$. Since $F(H')$ is obtained from $F(H)$ by taking a box from the end of column k, and a second box from the end of row k, H' is a connected threshold graph that satisfies $d(H') \geq d(G)$. Because this process of deleting edges may be continued until the resulting graph has the same degree sequence as G, it follows that H contains a subgraph isomorphic to G, i.e., $G \leq H$. $\square$

Recall that the *dual* of poset P is the poset P^* on the same set as P, but such that $x \leq y$ in P^* if and only if $y \leq x$ in P. If P is isomorphic to P^*, then P is *self-dual*.

Theorem 2.4. *The bijection $G \to d(G)$ is a poset isomorphism from T_n onto Y_n. In particular, T_n is a self-dual distributive lattice.*

That T_n is a lattice was observed previously in [10, Section 4]. (Also see [5] and [15].) Using Theorem 2.4 it is easy to strengthen Lemma 2.1 by identifying, as in [21], exactly which edges can be added to, or deleted from, a threshold graph so that the result is another threshold graph.

Proof of Theorem 2.4. The first statement is immediate from Theorem 1.5 and Lemma 2.3. To prove the second, We first show that the induced subposet Y_n is an induced sublattice of Young's Lattice Y. Suppose $\pi,\sigma \in Y_n$. If $\mu_i = \max\{\pi_i, \sigma_i\}$, $1 \leq i \leq n$, then $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is the join of π and σ in Y. To show that $\mu \in Y_n$, suppose $j \leq f(\mu) = o(\{i : \mu_i \geq i\}) = \max\{f(\pi), f(\sigma)\}$. Because $\mu_s \geq j$ if and only if $\max\{\pi_s, \sigma_s\} \geq j$, $\mu_j^* = o(\{s : \mu_s \geq j\}) = \max\{\pi_j^*, \sigma_j^*\} = \max\{\pi_j, \sigma_j\} + 1 = 1 + \mu_j$. Thus, $\mu \in Y_n$. Replacing maximums with minimums, the same argument shows that the meet in Y of π and σ is an element of Y_n.

Because Y is distributive, the induced sublattice Y_n is distibutive. Thus, from the first statement of the theorem, T_n is distributive.

Duality is easier to check from the perspective of graphs. Suppose $G \in T_n$. Let u be a dominating vertex of G and set $H = G - u$. Recall that H and its complement are (not necessarily connected) threshold graphs. Let $\psi(G) = u{\cdot}H^c$ be the (cone) graph obtained from H^c by adding vertex u and $n - 1$ edges connecting u to every vertex of H^c. Then, up to isomorphism, ψ is well defined, and $\psi : T_n \to T_n$ is injective.

If $G_1, G_2 \in T_n$, then $G_1 \leq G_2$ if and only if G_1 is isomorphic to a graph G_1' that can be obtained from G_2 by deleting some of its edges, but none of its vertices, i.e., to a *spanning* subgraph G_1' of G_2. If u is a dominating vertex of G_1 then u', the vertex of G_1' to which it corresponds, must be a dominating vertex of G_1' and, hence, of G_2. Thus, $G_1 \leq G_2$ if and only if $G_1 - u$ is isomorphic to a spanning subgraph of $G_2 - u'$, if and only if $(G_2 - u')^c$ is isomorphic to a spanning subgraph of $(G_1 - u)^c$, if and only if $\psi(G_2) \leq \psi(G_1)$. □

Since it is a distributive lattice, T_n is isomorphic to the lattice of "order ideals" of P_n, the induced subposet of its "join irreducible" elements [27, Ch. 3]. For the purposes of this article, the relevant conclusion is that the poset T_n is completely determined by P_n. We shall return to this point in Section 4.

Because the partial orderings of T_n and Y_n extend naturally to

$$\Im = \bigcup_{n \geq 1} T_n \qquad \text{and} \qquad \tilde{Y} = \bigcup_{n \geq 1} Y_n,$$

respectively, the following is an immediate consequence of Theorem 2.4.

Corollary 2.5. *The bijection $G \to d(G)$ is a poset isomorphism from $\Im$ onto $\tilde{Y}$.*

3. THE LATTICE OF SHIFTED SHAPES

Up to this point, the focus of our attention has been on the number of vertices of G and the length of π. In what follows, it will sometimes be more convenient to focus instead on the number of edges of G and the rank of π.

Suppose $\pi \vdash 2m$. If $\mu_i = \pi_i^* - 1$ and $\nu_i = \pi_i$, $1 \leq i \leq f(\pi)$, then, from (1.3), π is graphic if and only if μ weakly majorizes ν, an observation that simplifies the statement of the Ruch-Gutman criteria without adding much clarity. Let us see what can be done about that. Begin by dividing $F(\pi)$ into two disjoint pieces. Denote by $B(\pi)$ those boxes of $F(\pi)$ that lie strictly below its diagonal, and let $A(\pi)$ be the rest, i.e., $A(\pi)$ consists of those boxes that lie on the diagonal or lie to the right of a diagonal box. Informally, $A(\pi)$ is the piece of $F(\pi)$ on or *above* the diagonal, and $B(\pi)$ is the piece (strictly) *below* the diagonal. For $\tau = (4, 3, 3, 2, 2, 2)$, the division of $F(\tau)$ into $A(\tau)$ and $B(\tau)$ is illustrated in Fig. 3.1.

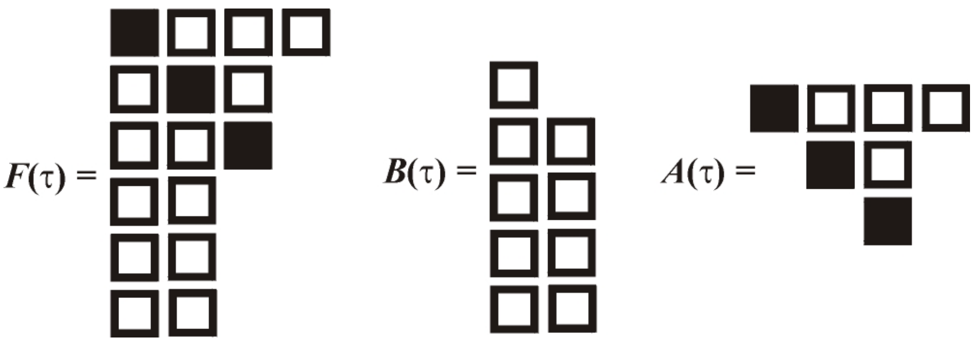

Figure 3.1: Division of $F(\tau)$.

Definition 3.1. Suppose $\pi \vdash r$. Let $\alpha(\pi)$ be the partition whose parts are the lengths of the *rows* of the *shifted shape* $A(\pi)$. Denote by $\beta(\pi)$ the partition whose parts are the lengths of the *columns* of $B(\pi)$.

From Fig. 3.1, $\alpha(\tau) = (4, 2, 1)$ and $\beta(\tau) = (5, 4)$. Together with (1.1) – (1.4), this division of $F(\pi)$ leads to the following variation on the theme of Ruch and Gutman.

Theorem 3.1. *Suppose* $\pi \vdash 2m$*. Then* π *is graphic if and only if* $\beta(\pi)$ *weakly majorizes* $\alpha(\pi)$*. Moreover,* π *is a threshold partition if and only if* $\beta(\pi) = \alpha(\pi)$*.*

We will abbreviate $\alpha(d(G))$ and $\beta(d(G))$ by $\alpha(G)$ and $\beta(G)$, respectively.

Let us look a little more closely at what it means to be a shifted shape. Unlike $F(\pi)$, the rows of $A(\pi)$ are not left-justified. Each successive row is shifted one (more) box to the right. The left-hand boundary of $A(\pi)$ looks like an inverted staircase. On the other hand, because $A(\pi)$ is just the top half of $F(\pi)$, the rules that apply to the right-hand boundary are the same for $A(\pi)$ as for $F(\pi)$, i.e., the last box in row $i + 1$ of $A(\pi)$ can extend no further to the right than the last box in row i. The right-hand boundary rule applied to $F(\pi)$ reflects the fact that the parts of π form a nonincreasing sequence. Because the left-hand boundary rules are different, the same right-hand rule applied to $A(\pi)$ implies that the parts of $\alpha(\pi)$ form a (strictly) decreasing sequence. That is, the parts of $\alpha(\pi)$ are all different. Partitions with distinct parts are called *strict partitions*. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a strict partition of m, denoted $\alpha \vdash m$, then $\alpha_1 > \alpha_2 > \cdots > \alpha_k$, and there is a unique shifted shape whose i^{th} row contains α_i boxes, $1 \le i \le k$.

Corollary 3.2. *The mapping* $\pi \to \alpha(\pi)$ *is a bijection from the threshold partitions of* $2m$ *onto the strict partitions of* m*.*

Representing the connected threshold graph G by the strict partition $\alpha(G)$, the self-dual distributive lattice T_6 (from Theorem 2.4) is illustrated in Fig. 3.2.

It follows from Corollary 3.2 that $\tilde{Y}$ is identical to what has come to be known as the *lattice of shifted shapes*. (See, e.g., [7], or [26, §3].) From this identification (and Corollary 2.5), it follows that $\Im$ is a locally finite distributive lattice with least element $\hat{0} = K_1$, i.e., $\Im$ is a so-called *finitary distributive* lattice.

Recall that a subset C of a poset P is a *chain* if any two elements of C are comparable (in P). A chain is *saturated* if there do not exist $x, z \in C$ and $y \in P \backslash C$ such that $x < y < z$. In a locally finite lattice, a chain $x_0 < x_1 < \cdots < x_k$ (of *length* $k = o(C) - 1$) is saturated if and only if x_i covers x_{i-1}, $1 \le i \le k$.

Because it is a finitary lattice, $\Im$ has a unique rank function $\lambda : \Im \to \mathbb{N}$, where $\lambda(G)$ is the length of any saturated chain from $\hat{0} = K_1$ to G, i.e., $\lambda(G) = m$, the number of edges of G.

Let t_m (not to be confused with T_n) be the number of nonisomorphic connected threshold graphs having m edges. By Corollary 3.2, t_m is equal to the number of strict partitions of rank

m. The generating function for strict partitions has been known at least since the time of Euler:

$$\sum_{m\geq 0} t_m x^m = \prod_{i\geq 1}(1+x^i) \tag{3.1}$$
$$= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + \cdots .$$

Together with Corollaries 2.5 and 3.2, these remarks imply that $\Im$ is a so-called "graded poset" with "rank generating function" given by (3.1).

Definition 3.2. Let $G \in \Im$ be a fixed but arbitrary connected threshold graph. Denote by $e\,(G)$ the *number of saturated chains in $\Im$ from K_1 to G.*

Representing $G \in \Im$ by $\alpha(G)$, the first few levels (ranks) of the graded poset $\Im$ are illustrated in Fig. 3.3. The numbers in the figure are the corresponding values of $e(G)$. (Note that they follow a recurrence reminiscent of Pascal's triangle.)

Starting with an unlimited number of isolated vertices, $e(G)$ is the number of ways to "construct" the threshold graph G by adding edges, one at a time, subject to the condition that every time an edge is added the result is a threshold graph. (The Threshold Algorithm corresponds to constructing $\alpha(G)$ a row at a time.)

Corollary 3.3. *Let G be a threshold graph having m edges and degree sequence $\pi = d(G)$. Suppose $\alpha(\pi) = (\rho_1, \rho_2, \ldots, \rho_r) \vdash m$ where $r = f(G)$, and $\rho_i = \pi_i - i + 1$, $1 \leq i \leq r$. Then*

$$e(G) = \frac{m!}{\rho!} \prod_{i<j} \frac{\rho_i - \rho_j}{\rho_i + \rho_j} \tag{3.2}$$

where $\rho! = \rho_1!\rho_2!\cdots\rho_r!$, i.e., apart from a power of 2 depending on m and r, $e(G)$ is the degree of the projective representation of S_m corresponding to $\alpha(\pi)$.

Proof. The result follows from Corollaries 2.5 and 3.2, and the fact that the number of saturated chains from $\hat{0}$ to $d(G)$ in $\tilde{Y}$ is given by the right-hand side of (3.2). (See, e.g., [14, III.8, Ex. 12].) The natural bijection between projective representations of the symmetric groups and strict partitions is an old result going back to Schur, a modern account of which can be found in [28]. □

4. LATTICE OF ORDER IDEALS

Let I be a (possibly empty) subset of the poset P. If $y \in I$, $x \in P$, and $x < y$, together imply that $x \in I$, then I is an *order ideal* of P. The set of all order ideals of P, ordered by inclusion, is a poset denoted $J(P)$. An element $y \notin \tilde{0}$ of a distributive lattice L is *join irreducible* if y is not the least upper bound of two elements, both of which are strictly less than y (i.e., y is join irreducible if it has exactly one edge below it in any Hasse diagram of L.) The next result follows from the fact that $\Im$ is a finitary distributive lattice [27, Prop. 3.4.3].

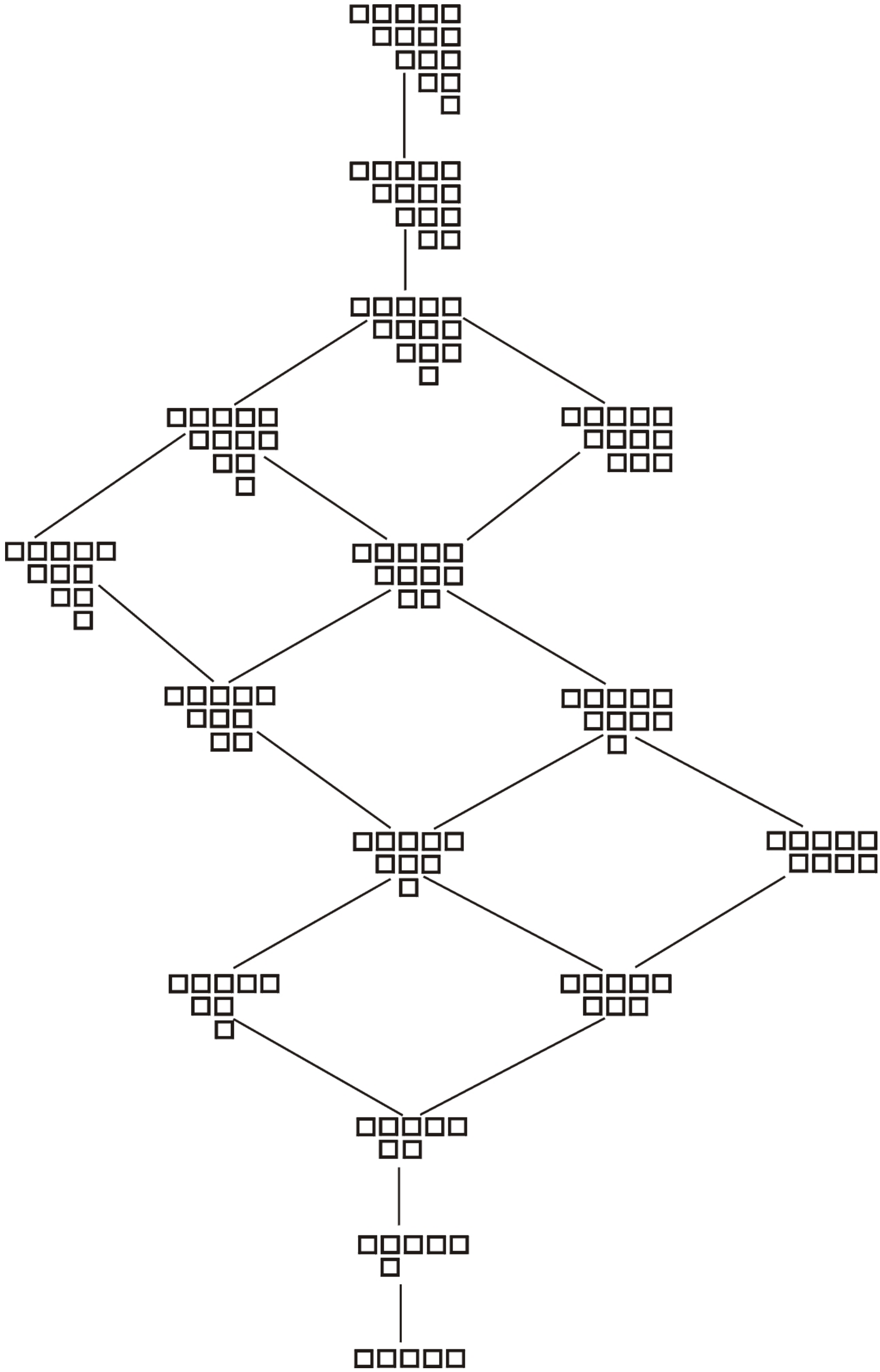

Figure 3.2: Hasse diagram of T_6.

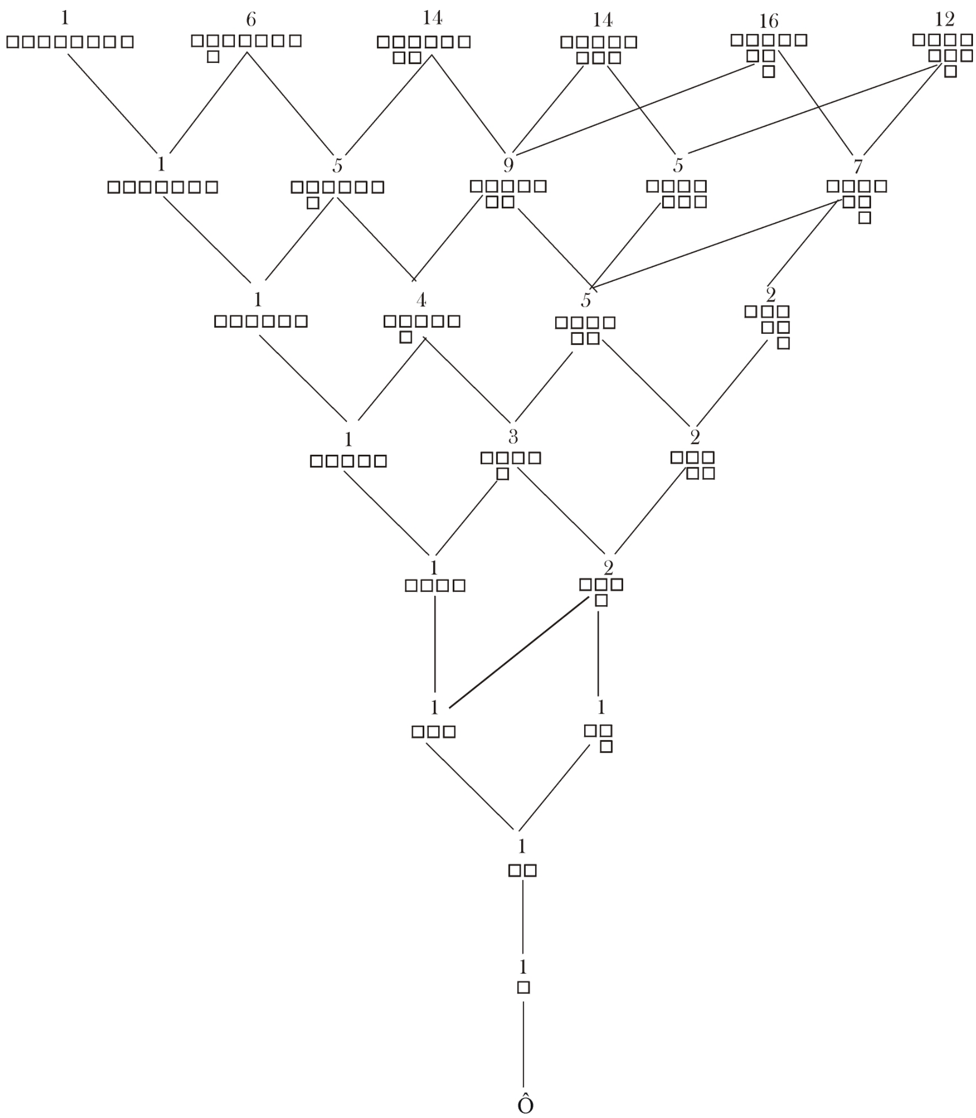

Figure 3.3: $\Im \cong \tilde{Y}$.

Theorem 4.1. *If $\mathcal{P}$ is the induced subposet of join irreducible elements of $\Im$, then $\Im \cong J(\mathcal{P})$, the lattice of order ideals of $\mathcal{P}$.*

Can one give an explicit description of $\mathcal{P}$? Any element that covers $\hat{0}$ is join irreducible. Glancing at Fig. 3.3, one finds only one such shifted shape, namely, □, corresponding to the strict partition (1). Indeed, it is clear from Fig. 3.3, not only that □□ ∼(2), □□□ ∼(3), etc., are join irreducible, but that there are others as well, namely those corresponding to the strict partitions $(2,1)$, $(3,2)$, and $(3,2,1)$. We leave it as an exercise to show that the join irreducible shifted shapes are precisely those that are right-justified.

What about a graph-theoretic interpretation of $\mathcal{P}$? Say that two edges of G are equivalent if there is an automorphism of G that carries one to the other. Then the connected threshold graph G lies in $\mathcal{P}$ if and only if, up to equivalence, there is a unique edge e of G such that

$G - e$ is a threshold graph. This, of course, is not so much an answer as another way of stating the question. A more useful characterization of join irreducible threshold graphs involves the unrelated notion of a "join" of graphs.

Definition 4.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs on disjoint sets of vertices. Their *join* is the graph $G_1 \bullet G_2 = (V, E)$, where $V = V_1 \cup V_2$ and E is the union of $E_1 \cup E_2$ and the set $\{uv : u \in V_1 \text{ and } v \in V_2\}$.

The particular instance of the join of a graph and a single vertex $(u \bullet H^c)$ occurred in the proof of Theorem 2.4.

Theorem 4.2. *Suppose $G \in \Im$. Then $G \in \mathcal{P}$ if and only if, for some pair of positive integers r and s, $G \cong K_r \bullet K_s^c$, the join of a complete graph and the complement of a complete graph.*

Proof. Suppose $G \in \mathcal{P}$. Because $\hat{0} \notin \mathcal{P}$, G has an edge. If $G = K_n$ then $n \geq 2$, $r = n - 1$ and $s = 1$. Otherwise, let $\pi = d(G)$. Because $\alpha(\pi) = \beta(\pi)$ corresponds to a right-justified shifted shape, there exists a positive integer r such that $\pi_1 = \pi_2 = \cdots = \pi_r = n - 1$ and $\pi_{r+1} = \cdots = \pi_n = r$. In other words, r $(< n)$ of the vertices of G are dominating vertices, and the remaining $s = n - r$ of its vertices are adjacent (only) to the dominating vertices; that is, $G \cong K_r \bullet K_s^c$. Conversely, if $G \cong K_r \bullet K_s^c \in \Im$, then $\alpha(G)$ corresponds to a right-justified shifted shape. □

Definition 4.2. Denote by $[n]$ the poset $\{1, 2, \ldots, n\}$ under the natural ordering of the integers. Thus $[n]$ is an n-element chain (of length $n - 1$). Denote by $\mathbb{N}$ the poset of the natural numbers ordered by magnitude.

Recall that the direct (or cartesian) *product* of posets P and Q is the poset $P \times Q = \{(x, y) : x \in P \text{ and } y \in Q\}$, where $(x, y) \leq (p, q)$ if (and only if) $x \leq p$ and $y \leq q$. If $P \cap Q = \phi$, the disjoint *union* of P and Q is the poset $P + Q$, where $x \leq y$ if either $x, y \in P$ and $x \leq y$, or $x, y \in Q$ and $x \leq y$.

The poset of right-justified shifted shapes (the join irreducible elements of $\tilde{Y} \cong \Im$) turns out, itself, to be a finitary distributive lattice. Denote by $\mathcal{P}^1$ the induced subposet of $\mathcal{P}$ consisting of its join irreducible elements, so that $\mathcal{P} \cong J(\mathcal{P}^1)$. Then $\mathcal{P}^1$ is isomorphic to $\mathbb{N} \times [2]$ (making $\mathcal{P}$ "semi-Pascal" [7, p. 381-382]). In the language of strict partitions, $\alpha(\pi) \in \mathcal{P}^1$, if and only if $m \geq 2$ and $\alpha(\pi) = (m)$, corresponding to a shifted shape with a single row of boxes, or $\alpha(\pi) = (r - 1, r - 2, \ldots, 1)$, corresponding to a right-justified "inverted staircase". The connected threshold graph emerging from (m) is the "star", $K_1 \bullet K_m^c$, while the inverted staircase corresponds to K_r.

Because a product of chains is a finitary distributive lattice, $\mathcal{P}^1 \cong J(\mathcal{P}^2)$ where, it turns out, $\mathcal{P}^2 \cong \mathbb{N} + [1]$. In the language of strict partitions, $\alpha(\pi) \in \mathcal{P}^2$ if and only if $\alpha(\pi) = (2, 1)$, or $m \geq 3$ and $\alpha(\pi) = (m)$. Graph theoretically, $G \in \mathcal{P}^2$ if and only if $G = K_3$ or G is a star on $n \geq 4$ vertices. These observations are summarized in the following.

Theorem 4.3. *The lattice $\Im$ of connected threshold graphs is isomorphic to $J(J(J(\mathcal{P}^2)))$, where $\mathcal{P}^2$ is the induced subposet of $\Im$ consisting of K_3 and the stars on $n \geq 4$ vertices.*

For the remainder of this section, we return to the self-dual distributive lattice T_n of connected threshold graphs on n vertices. Our goal is an analog of Theorem 4.3 for T_n. The desired result, stated from the perspective of shifted shapes, can be found in [24]. We merely flesh out some of the details and interpret them from the perspective of threshold graphs.

Denote the poset of join irreducible elements of T_n by P_n, so that $T_n = J(P_n)$. A glance at Fig. 3.2 reveals that $P_6 \not\subset T_6 \cap \mathcal{P}$. Some elements of P_6 correspond to shifted shapes that are not right-justified. This is easily explained. Because a connected threshold graph on 6 vertices has a (dominating) vertex of degree 5, the first row of every shifted shape in Fig. 3.2 must contain 5 boxes. In general, the join-irreducible elements of T_n corresponded to shifted shapes that are right justified *with the possible exception of the first row*.

Definition 4.3. Suppose G is a graph with a dominating vertex u. Denote by $G\#K_t^c$ the graph obtained from G by adding t new vertices v_i, $1 \leq i \leq t$, and t new edges $\{u, v_i\}$, $1 \leq i \leq t$.

If G has two dominating vertices, u_1 and u_2, then the version of $G\#K_t^c$ obtained by adding t neighbors to u_1 is isomorphic to the version obtained by adding t neighbors to u_2. Thus, up to isomorphism, it does not matter which dominating vertex of G is chosen to play the role of u in Definition 4.3. More importantly, if $d(G)$ is a threshold sequence, then $d(G\#K_t^c)$ is a threshold sequence.

Theorem 4.4. *The set of join irreducible elements of T_n is $P_n = \{(K_r \bullet K_s^c)\#K_t^c : r + s + t = n\}$.*

Proof. Because $\alpha(G\#K_t^c)$ is obtained from $\alpha(G)$ by adding t boxes to its first row, the result follows from Theorems 2.4 and 4.2, and the discussion leading up to Definition 4.3. □

Lemma 4.5. *The poset P_n of join irreducible elements of T_n is a distributive lattice.*

Proof. While P_n is an induced subposet of T_n, it is not a sublattice of T_n. Suppose $x, y \in P_n$. From Theorem 2.4 and the proof of Theorem 4.4, we may identify x and y with shifted shapes whose first rows have length $n - 1$ and whose remaining rows (if any) are right-justified. The meet of x and y in T_n is the intersection of their shifted shapes. Because it belongs to P_n, this intersection is the meet of x and y in P_n.

Denote by z and z' the joins of x and y in P_n and T_n, respectively. Then z' is obtained from x and y by superimposing their shifted shapes. If, apart from its first row, z' is right-justified, then $z = z'$. Otherwise, z is obtained from z' by adding a rectangular array of boxes to its lower right hand corner.

The proof that the join and meet of P_n distribute over each other is straightforward. □

It follows from Lemma 4.5 that $P_n = J(P_n^1)$, where P_n^1 is the subposet of join irreducible elements of P_n.

Theorem 4.6. *The subposet of join irreducible elements of P_n is $P_n^1 = \{(K_r \bullet K_s^c)\#K_t^c : r + s + t = n$, and $r = 2$ or $s = 0\}$.*

Proof. From among all possible ways to express $G \in P_n$ in the form $(K_r \bullet K_s^c)\#K_t^c$, choose those for which r is as large as possible and from among those, choose the one for which s is as large as possible. Thus, for example, we choose K_6 over $K_5 \bullet K_1^c$ and $K_2\#K_4^c$ over $(K_1 \bullet K_2^c)\#K_3^c$. Note that, as long as $n \geq 2$, this canonical form results in $r \geq 2$. If the canonical form of $G \in P_n$ is $(K_r \bullet K_s^c)\#K_t^c$ with $r \geq 3$ and $s \geq 1$, then G is a join (in the lattice P_n) of the incomparable graphs $(K_r \bullet K_{s-1}^c)\#K_{t+1}^c$ and $(K_{r-1} \bullet K_{s+1}^c)\#K_t^c$. This proves that the join irreducible elements of (the lattice) P_n are contained in the set identified as P_n^1 in the statement of the theorem.

If the canonical form of $G \in P_n$ is $(K_2 \bullet K_s^c)\#K_t^c$, then the corresponding shifted shape z has two rows, the first of length $n - 1$ and the second of length s. In order for z to be the join of $x, y \in P_n$, neither of x and y can have more than two rows and they must both have first rows of length $n - 1$. Thus, if $x \neq y$, the one with the shorter second row is less than the other one. It follows that z is join irreducible.

If the canonical form of $G \in P_n$ is $K_r\#K_t^c$, with $r > 2$, then, with the possible exception of the first row of length $n - 1$, the corresponding shifted shape z is an inverted staircase. In order for z to be the join of $x, y \in P_n$, at least one of them, say x, must have r rows. But this means $z \leq x \leq z$. This completes the proof that the set identified as P_n^1 in the statement consists only of join irreducible elements of P_n. □

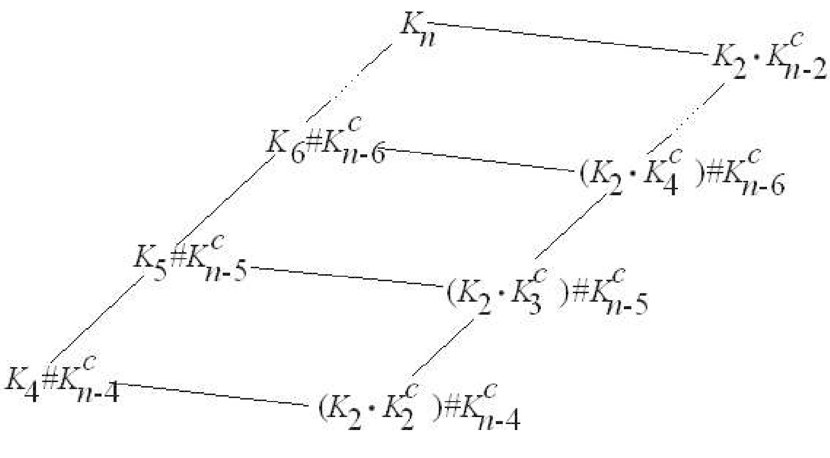

Figure 4.1: P_n^1 for $n \geq 7$.

A few minutes with paper and pencil will show that $T_2 \cong [1]$, $T_3 \cong [2]$ and $T_4 \cong [4]$ are (trivial) chains. It follows from Theorem 4.6 that $P_n^1 \cong [n-3] \times [2]$, $n \geq 5$. (See Fig. 4.1.) From this observation, it is straightforward both to show that P_n is self-dual and that P_n^1 is a distributive lattice. The induced subposet P_n^2, of join irreducible elements of P_n^1, is isomorphic to $[1] + [n-4]$, $n \geq 5$.

Theorem 4.7. *The lattice T_n of connected threshold graphs on $n \geq 5$ vertices is isomorphic to $J(J(J(P_n^2)))$, where P_n^2 is the induced subposet of T_n consisting of $K_4 \# K_{n-4}^c$ and $(K_2 \bullet K_s^c) \# K_{n-2-s}^c$, $3 \leq s \leq n-2$.*

Proof. Immediate from Fig. 4.1 □

5. CONCLUDING REMARKS

If $G = (V, E)$ is a graph, denote by $D(G)$ (not to be confused with $d(G)$) the diagonal matrix of its vertex degrees, i.e., $D(G) = diag(d_G(1), d_G(2), \ldots, d_G(n))$. Let $A(G) = (a_{ij})$ be the $(0,1)-$adjacency matrix (with $a_{ij} = 1$ if and only if $\{i, j\} \in E$). The *Laplacian* matrix of G is $L(G) = D(G) - A(G)$. Then $L(G)$ is a symmetric, positive semidefinite, singular M-matrix. Denote the spectrum of $L(G)$ by $s(G) = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$. Then [17] G is a threshold graph if and only if $s(G) = d(G)^*$, the conjugate of its degree sequence. If $G \in \mathcal{P}$, the induced subposet of join irreducible elements of $\Im$, then (Theorem 4.2), $G \cong K_r \bullet K_s^c$, $r + s = n$. Thus, G has r vertices of degree $n-1$ and s vertices of degree r. Because $d(G)^* = s(G)$, $\lambda_1 = \lambda_2 = \cdots = \lambda_r = n$, $\lambda_{r+1} = \lambda_{r+2} = \cdots = \lambda_{n-1} = r$, and $\lambda_n = 0$. It follows that $\mathcal{P} = \Im \cap \mathscr{L}$ where $\mathscr{L}$ is the set consisting of those graphs G such that $L(G)$ has at most two distinct nonzero eigenvalues. This set of graphs, a natural algebraic generalization of $\mathcal{P}$, has been characterized completely by van Dam [4] and Haemers [9].

Suppose $\pi \vdash 2m$. Then (Theorem 3.1) π is graphic if and only if $\beta(\pi) \underset{w}{\succ} \alpha(\pi)$, and π is threshold if and only if $\beta(\pi) = \alpha(\pi)$. Weaker than equality but stronger than weak majorization is the relation of (ordinary) majorization, the case in which $A(\pi)$ and $B(\pi)$ contain the same number of boxes. What, if anything, can be said about graphs G for which $\beta(G)$ majorizes $\alpha(G)$? It turns out that $\beta(G) \succ \alpha(G)$ if and only if G is a so-called *split graph*. The split graphs have many interesting characterizations, e.g., $G = (V, E)$ is a split graph if and only if V can be partitioned into the disjoint union of a clique and an independent set, if and only if both G and G^c are chordal [6], and so on. A discussion of the many manifestations of split graphs can be found, e.g., in [19].

It is sometimes useful to write partitions "backwards", in nondecreasing notation. In backward notation, $(3, 3, 3, 3, 2, 1, 1, 1)$ becomes $[1, 1, 1, 2, 3, 3, 3, 3]$, which can be abbreviated $[1^3, 2, 3^4]$, where superscripts are used to denote multiplicities.

Theorem 5.1. *Let G be a connected threshold graph with (backward) degree sequence $d(G) = [1^{r_1}, 2^{r_2}, \ldots, (n-1)^{r_{n-1}}]$. As a group of permutations of its n vertices, the automorphism group* *of G is the "Young subgroup" associated with $d(G)$, i.e.,*

$$A(G) \cong S_{r_1} \times S_{r_2} \times \cdots \times S_{r_{n-1}}.$$

This result is a consequence of the structure of threshold graphs displayed in [8] or [15]. As we proceed to demonstrate, it is also an easy consequence of the Threshold Algorithm.

Lemma 5.2. *Let $G = (V, E)$ be a connected threshold graph on $n \geq 2$ vertices. Suppose $i, j \in V$, $i \neq j$. If $d_G(i) = d_G(j)$, then $N_G(i)\backslash j = N_G(j)\backslash i$.*

Proof. We may assume that $V = \{1, 2, \dots, n\}$ and that G emerged from the Threshold Algorithm, so that $d_k = d_G(k)$, $1 \leq k \leq n$. Suppose $i < j$ and let $s = f(G)$. Because $d(G)$ is a threshold partition, $d_{s+1} = s$ and either $d_s > s$, or $d_s = s$ and $d_{s+2} < s$. Thus, because $d_i = d_j$, it cannot happen that both $i \leq s$ and $j \geq s + 2$. This leaves two possibilities: Either $j \leq s + 1$ or $i > s$. In each of these cases, the result follows from Equations (2.1) and (2.2). □

We are grateful to the referee for pointing out that Lemma 5.2 also follows from [3] where it is shown that no subset of vertices of a threshold graph can be arranged in an "alternating cycle" consisting of an edge, a non-edge, an edge, a non-edge, ...

Proof of Theorem 5.1. Fix a positive integer k. Let D_k be the set of vertices of G of degree k. If $o(D_k) \geq 2$ then, by the same approach used to prove Lemma 5.2, either D_k is a clique or it is an independent set. Thus, by Lemma 5.2, any permutation of $V(G)$ that fixes the vertices not contained in D_k is an automorphism of G. Because no automorphism of G can send a vertex of degree k to a vertex of degree $d \neq k$, the proof is complete. □

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