

# IMPROVEMENT OF THE NON-UNIFORM VERSION OF BERRY-ESSEEN INEQUALITY VIA PADITZ-SIGANOV THEOREMS

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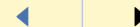
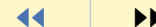
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*Abstract:* We improve the constant in a non-uniform bound of the Berry-Esseen inequality without assuming the existence of the absolute third moment by using the method obtained from the Paditz-Siganov theorems. Our bound is better than the results of Thongtha and Neammanee in 2007 ([14]).

**Berry-Esseen Inequality Via  
Paditz-Siganov Theorems**

**K. Neammanee and P. Thongtha**  
**vol. 8, iss. 4, art. 92, 2007**

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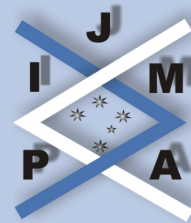
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## 1. Introduction and Main Results

The Berry-Esseen inequality is one of the most important inequalities in the theory of probability. This inequality was independently discovered by two mathematicians, Andrew C. Berry ([2]) and Carl-Gustav Esseen ([5]) in 1941 and 1945 respectively. Let  $X_1, X_2, \dots, X_n$  be independent random variables with zero mean and  $\sum_{i=1}^n EX_i^2 = 1$ . Define  $W_n = X_1 + X_2 + \dots + X_n$ . Then  $\text{Var } W_n = 1$ . Let  $F_n$  be the distribution function of  $W_n$  and  $\Phi$  the standard normal distribution function, i.e.,

$$F_n(x) = P(W_n \leq x) \quad \text{and} \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt.$$

The central limit theorem shows that  $F_n$  converges pointwise to  $\Phi$  as  $n \rightarrow \infty$  and the bounds of this convergence are,

$$(1.1) \quad \sup_{x \in \mathbb{R}} |P(W_n \leq x) - \Phi(x)| \leq C_0 \sum_{i=1}^n E|X_i|^3$$

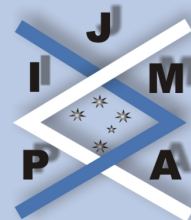
and

$$(1.2) \quad |P(W_n \leq x) - \Phi(x)| \leq \frac{C_1}{1 + |x|^3} \sum_{i=1}^n E|X_i|^3$$

for uniform and non-uniform versions respectively, where both  $C_0$  and  $C_1$  are positive constants and stated under the assumption that  $E|X_i|^3 < \infty$  for  $i = 1, 2, \dots, n$ .

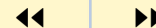
In the case of identical  $X_i$ 's, Siganov ([11]) and Chen ([5]) improved the constant down to 0.7655 and 0.7164, respectively. For non-uniform bounds, Nageav ([7]) was the first to obtain (1.2) and Michel ([6]) calculated the constant to be 30.84.

Without assuming identically distributed  $X_i$ 's, Beek ([15]) sharpened the constant down to 0.7975 in 1972 for the uniform version. The best bound was found by Siganov ([11]) in 1986.



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**Theorem 1.1 (Siganov,1986).** Let  $X_1, X_2, \dots, X_n$  be independent random variables such that  $EX_i = 0$  and  $E|X_i|^3 < \infty$  for  $i = 1, 2, \dots, n$ . Assume that  $\sum_{i=1}^n EX_i^2 = 1$ . Then

$$\sup_{x \in \mathbb{R}} |P(W_n \leq x) - \Phi(x)| \leq 0.7915 \sum_{i=1}^n E|X_i|^3,$$

where  $W_n = X_1 + X_2 + \dots + X_n$ .

For the non-uniform version, Bikelis ([1]) generalized (1.2) to this case and Paditz ([9]) calculated  $C_1$  to be 114.7 in 1977. He also improved his result down to 31.935 in 1989.

**Theorem 1.2 (Paditz ([10]),1989).** Under the assumptions of Theorem 1.1, we have

$$|P(W_n \leq x) - \Phi(x)| \leq \frac{31.935}{1 + |x|^3} \sum_{i=1}^n E|X_i|^3.$$

In 2001, Chen and Shao ([3]) gave new versions of (1.1) and (1.2) without assuming the existence of third moments. Their results are

$$(1.3) \quad \sup_{x \in \mathbb{R}} |P(W_n \leq x) - \Phi(x)| \\ \leq 4.1 \sum_{i=1}^n \left\{ E|X_i|^2 I(|X_i| \geq 1) + E|X_i|^3 I(|X_i| < 1) \right\}$$

and

$$(1.4) \quad |P(W_n \leq x) - \Phi(x)| \\ \leq C_2 \sum_{i=1}^n \left\{ \frac{EX_i^2 I(|X_i| \geq 1 + |x|)}{(1 + |x|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |x|)}{(1 + |x|)^3} \right\},$$

where  $C_2$  is a positive constant and  $I(A)$  is an indicator random variable such that

$$I(A) = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

In 2005, Neammanee ([8]) combined the concentration inequality in ([3]) with a coupling approach to calculate the constant in (1.4), giving,

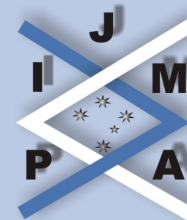
$$(1.5) \quad |P(W_n \leq x) - \Phi(x)| \leq C_3 \sum_{i=1}^n \left\{ \frac{EX_i^2 I(|X_i| \geq 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |\frac{x}{4}|)}{(1 + |\frac{x}{4}|)^3} \right\},$$

where  $C_3$  is 21.44 for large values of  $x$  such that  $|x| \geq 14$ .

Thongtha and Neammanee ([14]) improved the concentration inequality used in ([8]) and gave a better constant, i.e., 9.7 for  $|x| \geq 14$ . The method which was used in ([8]) is Stein's method which was first introduced by Stein ([12]) in 1972. In this work, we provide a better constant by using Paditz-Siganov theorems. The results are as follows.

**Theorem 1.3.** *We have*

$$|P(W_n \leq x) - \Phi(x)| \leq C \sum_{i=1}^n \left\{ \frac{EX_i^2 I(|X_i| \geq 1 + |x|)}{(1 + |x|)^2} + \frac{E|X_i|^3 I(|X_i| < 1 + |x|)}{(1 + |x|)^3} \right\},$$



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where

$$C = \begin{cases} 49.89 & \text{if } 0 \leq |x| < 1.3, \\ 59.45 & \text{if } 1.3 \leq |x| < 2, \\ 73.52 & \text{if } 2 \leq |x| < 3, \\ 76.17 & \text{if } 3 \leq |x| < 7.98, \\ 45.80 & \text{if } 7.98 \leq |x| < 14, \\ 39.39 & \text{if } |x| \geq 14. \end{cases}$$

To compare Theorem 1.3 with the result of Thongtha and Neammanee ([14]) in (1.5), we give Corollary 1.4.

**Corollary 1.4.** *We have*

$$|P(W_n \leq x) - \Phi(x)| \leq C \sum_{i=1}^n \left\{ \frac{EX_i^2 I(|X_i| \geq 1 + \frac{|x|}{4})}{(1 + \frac{|x|}{4})^2} + \frac{E|X_i|^3 I(|X_i| < 1 + \frac{|x|}{4})}{(1 + \frac{|x|}{4})^3} \right\},$$

where

$$C = \begin{cases} 9.54 & \text{if } 0 \leq |x| < 1.3, \\ 19.74 & \text{if } 1.3 \leq |x| < 2, \\ 18.38 & \text{if } 2 \leq |x| < 3, \\ 14.63 & \text{if } 3 \leq |x| < 7.98, \\ 5.13 & \text{if } 7.98 \leq |x| < 14, \\ 3.55 & \text{if } |x| \geq 14. \end{cases}$$

We note from Corollary 1.4 that our result is better than a bound from Thongtha and Neammanee in ([14]).

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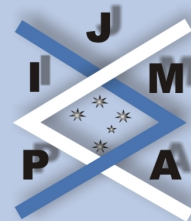
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## 2. Proof of the Main Results

In this section, we will prove Theorem 1.3 by using the Paditz-Siganov theorems. Corollary 1.4 can be obtained easily from Theorem 1.3. To prove these results, let

$$Y_{i,x} = X_i I(|X_i| < 1 + x), \quad S_x = \sum_{i=1}^n Y_{i,x},$$

$$\alpha_x = \sum_{i=1}^n EX_j^2 I(|X_j| \geq 1 + x), \quad \beta_x = \sum_{i=1}^n E|X_j|^3 I(|X_j| < 1 + x),$$

$$\gamma_x = \frac{\beta_x}{2} \quad \text{and} \quad \delta_x = \frac{\alpha_x}{(1+x)^2} + \frac{\beta_x}{(1+x)^3} \quad \text{for } x > 0.$$

**Proposition 2.1.** For each  $n \in \mathbb{N}$ , we have

1.  $\sum_{i=1}^n E|Y_{i,x} - EY_{i,x}|^3 \leq \beta_x + \frac{7\alpha_x}{1+x}$ ,
2.  $1 - 2\alpha_x \leq \text{Var } S_x \leq 1$ , and
3. If  $\alpha_x \leq 0.11$ , then  $0 < \frac{1}{\sqrt{\text{Var } S_x}} \leq 1 + 1.452\alpha_x$ .

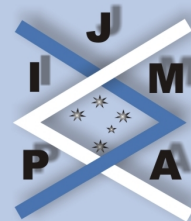
*Proof.* 1. By the fact that

$$(2.1) \quad |EX_i I(|X_i| < 1 + x)| = |EX_i I(|X_i| \geq 1 + x)|,$$

$$E|X_i|^2 \leq \sum_{i=1}^n EX_i^2 = 1 \quad \text{and} \quad E^2 X_i \leq EX_i^2,$$

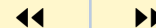
we have

$$\sum_{i=1}^n E|Y_{i,x} - EY_{i,x}|^3 = \sum_{i=1}^n E|X_i I(|X_i| < 1 + x) - EX_i I(|X_i| < 1 + x)|^3$$



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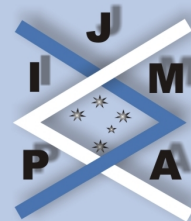
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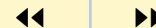
$$\begin{aligned}
 &\leq \sum_{i=1}^n [E|X_i|^3 I(|X_i| < 1+x) + 3EX_i^2 I(|X_i| < 1+x) |EX_i I(|X_i| < 1+x)| \\
 &\quad + 3E|X_i I(|X_i| < 1+x) |E^2 X_i I(|X_i| < 1+x)| + |EX_i I(|X_i| < 1+x)|^3] \\
 &\leq \sum_{i=1}^n E|X_i|^3 I(|X_i| < 1+x) + 3 \sum_{i=1}^n |EX_i I(|X_i| < 1+x)| \\
 &\quad + 3 \sum_{i=1}^n E|X_i| |EX_i I(|X_i| < 1+x)| |EX_i I(|X_i| < 1+x)| \\
 &\quad + \sum_{i=1}^n E|X_i|^2 I(|X_i| < 1+x) |EX_i I(|X_i| < 1+x)| \\
 &\leq \beta_x + 3 \sum_{i=1}^n |EX_i I(|X_i| \geq 1+x)| + 3 \sum_{i=1}^n E|X_i|^2 |EX_i I(|X_i| \geq 1+x)| \\
 &\quad + \sum_{i=1}^n |EX_i I(|X_i| \geq 1+x)| \\
 &\leq \beta_x + 3 \sum_{i=1}^n E|X_i| I(|X_i| \geq 1+x) + 3 \sum_{i=1}^n E|X_i| I(|X_i| \geq 1+x) \\
 &\quad + \sum_{i=1}^n E|X_i| I(|X_i| \geq 1+x) \\
 &= \beta_x + 7 \sum_{i=1}^n E|X_i| I(|X_i| \geq 1+x) \\
 &\leq \beta_x + 7 \sum_{i=1}^n \frac{E|X_i|^2 I(|X_i| \geq 1+x)}{(1+x)} = \beta_x + \frac{7\alpha_x}{(1+x)}.
 \end{aligned}$$





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2. By (2.1), we note that

$$\begin{aligned}\text{Var } S_x &= \sum_{i=1}^n \text{Var } Y_{i,x} = \sum_{i=1}^n (EY_{i,x}^2 - E^2Y_{i,x}) \\ &= \sum_{i=1}^n EX_i^2 I(|X_i| < 1+x) - \sum_{i=1}^n E^2X_i I(|X_i| < 1+x) \\ &= 1 - \sum_{i=1}^n EX_i^2 I(|X_i| \geq 1+x) - \sum_{i=1}^n E^2X_i I(|X_i| \geq 1+x) \\ (2.2) \quad &= 1 - \alpha_x - \sum_{i=1}^n E^2X_i I(|X_i| \geq 1+x).\end{aligned}$$

From this and the fact that  $\alpha_x \geq 0$ , we have  $\text{Var } S_x \leq 1$ .

By (2.2), we have

$$\begin{aligned}\text{Var } S_x &= 1 - \alpha_x - \sum_{i=1}^n E^2X_i I(|X_i| \geq 1+x) \\ &\geq 1 - \alpha_x - \sum_{i=1}^n EX_i^2 I(|X_i| \geq 1+x) = 1 - 2\alpha_x.\end{aligned}$$

Hence,  $1 - 2\alpha_x \leq \text{Var } S_x \leq 1$ .

3. For  $0 < t \leq 0.11$ , by using Taylor's formula, we have

$$\begin{aligned}\frac{1}{\sqrt{1-2t}} &= 1 + \frac{t}{(1-2c)^{\frac{3}{2}}} \text{ for some } c \in (0, 0.11] \\ &\leq 1 + \frac{t}{(1-2(0.11))^{\frac{3}{2}}} \leq 1 + 1.452t.\end{aligned}$$



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From this fact and 2., we have

$$0 < \frac{1}{\sqrt{\text{Var } S_x}} \leq \frac{1}{\sqrt{1 - 2\alpha_x}} \leq 1 + 1.452\alpha_x$$

for  $\alpha_x \leq 0.11$ . □

**Proposition 2.2.** For each  $x > 0$ , let  $\bar{Y}_{i,x} = \frac{Y_{i,x} - EY_{i,x}}{\sqrt{\text{Var } S_x}}$  and  $\bar{S}_x = \sum_{i=1}^n \bar{Y}_{i,x}$ .

1. If  $\alpha_x \leq 0.099$  and  $1.3 \leq x \leq 2$ , then

$$\left| P \left( \bar{S}_x \leq \frac{x - ES_x}{\sqrt{\text{Var } S_x}} \right) - \Phi \left( \frac{x - ES_x}{\sqrt{\text{Var } S_x}} \right) \right| \leq \frac{54.513\alpha_x}{(1+x)^2} + \frac{41.195\beta_x}{(1+x)^3}.$$

2. If  $(1+x)^2\alpha_x < \frac{1}{5}$ , then

$$\left| P \left( \bar{S}_x \leq \frac{x - ES_x}{\sqrt{\text{Var } S_x}} \right) - \Phi \left( \frac{x - ES_x}{\sqrt{\text{Var } S_x}} \right) \right| \leq \frac{C_1\alpha_x}{(1+x)^2} + \frac{C_2\beta_x}{(1+x)^3}$$

where  $C_1 = 57.186$   $C_2 = 73.515$  for  $2 \leq x < 3$ ,

$C_1 = 33.318$   $C_2 = 76.17$  for  $3 \leq x < 7.98$ ,

$C_1 = 3.976$   $C_2 = 45.8$  for  $7.98 \leq x < 14$ , and

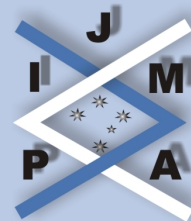
$C_1 = 1.226$   $C_2 = 39.382$  for  $x \geq 14$ .

*Proof.* 1. By Proposition 2.1(1) of ([14]) and Proposition 2.1(2), we have

$$(2.3) \quad |ES_x| \leq \frac{\alpha_x}{1+x} \leq 0.043 \quad \text{and} \quad 1 \geq \text{Var } S_x \geq 0.802$$

which imply

$$(2.4) \quad 0 \leq \frac{x - ES_x}{\sqrt{\text{Var } S_x}} \leq \frac{2 + 0.043}{\sqrt{0.802}} = 2.2813.$$



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By Proposition 2.1(1) and (2.3),

$$\begin{aligned}
 \sum_{i=1}^n E|\bar{Y}_{i,x}|^3 &= \sum_{i=1}^n E \left| \frac{Y_{i,x} - EY_{i,x}}{\sqrt{\text{Var } S_x}} \right|^3 \\
 &= \frac{1}{(\text{Var } S_x)^{\frac{3}{2}}} \sum_{i=1}^n E|Y_{i,x} - EY_{i,x}|^3 \\
 &\leq \frac{1}{(\text{Var } S_x)^{\frac{3}{2}}} \left( \beta_x + \frac{7\alpha_x}{1+x} \right) \\
 (2.5) \qquad &= 1.3923\beta_x + 4.2375\alpha_x.
 \end{aligned}$$

Note that  $\bar{S}_x = \sum_{i=1}^n \bar{Y}_{i,x}$  is the sum of independent random variables whose

$$E\bar{Y}_{i,x} = 0 \quad \text{and} \quad \text{Var } \bar{S}_x = 1.$$

By (2.5) and Theorem 1.1,

$$\begin{aligned}
 |P(\bar{S}_x \leq z) - \Phi(z)| &\leq 0.7915 \sum_{i=1}^n E|\bar{Y}_{i,x}|^3 \\
 &\leq 0.7915(1.3923\beta_x + 4.2375\alpha_x) \\
 &\leq 1.102\beta_x + 3.354\alpha_x
 \end{aligned}$$

for all  $z \in \mathbb{R}$ . From this fact, (2.3) and (2.4), we have

$$\begin{aligned}
 \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) \right| \\
 \leq \frac{\left(1 + \left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right)\right)^3 (1.102\beta_x + 3.354\alpha_x)}{\left(1 + \left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right)\right)^3}
 \end{aligned}$$



$$\begin{aligned} &\leq \frac{(3.2813)^3(1.102\beta_x + 3.354\alpha_x)}{\left(1 + \left(\frac{x-ES_x}{\sqrt{\text{Var } S_x}}\right)\right)^3} \\ &\leq \frac{38.933\beta_x + 118.495\alpha_x}{(0.957 + x)^3} \\ &\leq \frac{41.195\beta_x}{(1+x)^3} + \frac{125.379\alpha_x}{(1+x)^3} \\ &\leq \frac{41.195\beta_x}{(1+x)^3} + \frac{54.513\alpha_x}{(1+x)^2} \end{aligned}$$

where we use the fact that

$$\frac{1+x}{0.957+x} \leq 1.019 \text{ for all } 1.3 < x < 2$$

in the fourth inequality.

**2. Case  $2 \leq x < 3$ .**

We can prove the result of this case by using the same argument as 1.

**Case  $3 \leq x < 7.98$ .**

To bound  $\left|P\left(\bar{S}_x \leq \frac{x-ES_x}{\sqrt{\text{Var } S_x}}\right) - \Phi\left(\frac{x-ES_x}{\sqrt{\text{Var } S_x}}\right)\right|$  in 1., we used Theorem 1.1.

But in this case, we will use Theorem 1.2.

We note that

$$(2.6) \quad 0 \leq \alpha_x \leq 0.0125, \quad 1 \geq \text{Var } S_x \geq 0.975,$$

and, by Proposition 2.1(1) of ([14]),  $|ES_x| \leq 0.00313$ .

Then, for  $3 \leq x \leq 7.98$ ,

$$\frac{1}{1 + \left(\frac{x-ES_x}{\sqrt{\text{Var } S_x}}\right)^3} \leq \frac{2.29}{\left(1 + \frac{x-ES_x}{\sqrt{\text{Var } S_x}}\right)^3} \quad \text{and} \quad \sum_{i=1}^n E|\bar{Y}_{i,x}|^3 \leq 1.039\beta_x + 1.819\alpha_x.$$

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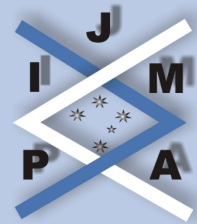
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From these facts, (2.6) and Theorem 1.2, we have

$$\begin{aligned} & \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) \right| \\ & \leq \frac{(31.935) \sum_{i=1}^n E|\bar{Y}_{i,x}|^3}{1 + \left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right)^3} \leq \frac{(31.935)(2.29) \sum_{i=1}^n E|\bar{Y}_{i,x}|^3}{\left(1 + \frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right)^3} \\ & \leq \frac{73.131(1.039\beta_x + 1.818\alpha_x)}{(0.99687 + x)^3} \leq \frac{(1.0008)^3(75.983\beta_x + 132.952\alpha_x)}{(1 + x)^3} \\ & \leq \frac{76.17\beta_x}{(1 + x)^3} + \frac{133.27\alpha_x}{(1 + x)^3} \leq \frac{76.17\beta_x}{(1 + x)^3} + \frac{33.318\alpha_x}{(1 + x)^2}, \end{aligned}$$

where we use the fact that

$$\frac{1 + x}{0.99687 + x} \leq 1.0008 \quad \text{for all } 3 \leq x < 7.98$$

in the fourth inequality.

**Case**  $x \geq 7.98$ .

We can prove the result of this case by using the same argument as the case  $3 \leq x < 7.98$ .  $\square$

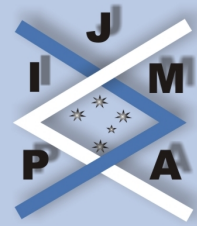
We are now ready to prove Theorem 1.3.

*Proof of Theorem 1.3.* It suffices to consider only  $x \geq 0$  as we can simply apply the results to  $-W_n$  when  $x < 0$ .

**Case 1.**  $0 \leq x < 1.3$ .

Note that for  $x \geq 0$ ,

$$EX_i^2 I(|X_i| \geq 1) + E|X_i|^3 I(|X_i| < 1) \leq EX_i^2 I(|X_i| \geq 1+x) + E|X_i|^3 I(|X_i| < 1+x)$$



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and for  $0 \leq x \leq 1.3$ ,  $(1+x)^3 \leq 12.167$ .

From these facts and (1.3), we have

$$\begin{aligned} & |P(W_n \leq x) - \Phi(x)| \\ & \leq 4.1 \sum_{i=1}^n \left\{ EX_i^2 I(|X_i| \geq 1) + E|X_i|^3 I(|X_i| < 1) \right\} \\ & \leq 4.1 \sum_{i=1}^n \left\{ EX_i^2 I(|X_i| \geq 1+x) + E|X_i|^3 I(|X_i| < 1+x) \right\} \\ & \leq \frac{4.1(12.167)}{(1+x)^3} \sum_{i=1}^n \left\{ EX_i^2 I(|X_i| \geq 1+x) + E|X_i|^3 I(|X_i| < 1+x) \right\} \\ & \leq 49.89 \sum_{i=1}^n \left\{ \frac{EX_i^2 I(|X_i| \geq 1+x)}{(1+x)^2} + \frac{E|X_i|^3 I(|X_i| < 1+x)}{(1+x)^3} \right\}. \end{aligned}$$

Before proving another case, we need the equation

$$(2.7) \quad |P(W_n \leq x) - \Phi(x)| \leq \frac{4.931\alpha_x}{(1+x)^2} + \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) \right|$$

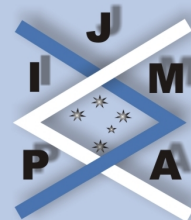
for  $\alpha_x \leq 0.11$  and  $x \geq 1.3$ .

By (2.9) of ([14]), it suffices to show that for  $\alpha_x \leq 0.11$  and  $x \geq 1.3$ ,

$$(2.8) \quad |P(S_x \leq x) - \Phi(x)| \leq \frac{3.319\alpha_x}{(1+x)^2} + \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) \right|.$$

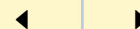
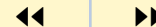
By Proposition 2.1(1) and Proposition 2.1(2), we have

$$\frac{x - ES_x}{\sqrt{\text{Var } S_x}} \geq x - ES_x \geq x - \frac{\alpha_x}{(1+x)},$$



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which implies

$$\min \left\{ x, \frac{x - ES_x}{\sqrt{\text{Var } S_x}} \right\} \geq x - \frac{\alpha_x}{1+x}.$$

From this and the fact that

$$\Phi(b) - \Phi(a) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{t^2}{2}} dt \leq \frac{1}{\sqrt{2\pi}e^{\frac{a^2}{2}}} \int_a^b 1 dt = \frac{(b-a)}{\sqrt{2\pi}e^{\frac{a^2}{2}}}$$

for  $0 < a < b$ , we have

$$\begin{aligned} & |P(S_x \leq x) - \Phi(x)| \\ & \leq \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) \right| + \left| \Phi\left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) - \Phi(x) \right| \\ & \leq \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) \right| \\ & \quad + \frac{1}{\sqrt{2\pi}e^{\frac{1}{2}[\min(x, \frac{x-ES_x}{\sqrt{\text{Var } S_x}})]^2}} \left| \frac{x}{\sqrt{\text{Var } S_x}} - x - \frac{ES_x}{\sqrt{\text{Var } S_x}} \right| \\ & \leq \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) \right| \\ (2.9) \quad & + \frac{1}{\sqrt{2\pi}e^{\frac{1}{2}(x - \frac{\alpha_x}{1+x})^2}} \left| \frac{x}{\sqrt{\text{Var } S_x}} - x - \frac{ES_x}{\sqrt{\text{Var } S_x}} \right|. \end{aligned}$$

Note that for  $x \geq 1.3$

$$e^{\frac{x^2}{2}} \geq 0.933(1+x), \quad e^{\frac{x^2}{2}} \geq 0.193(1+x)^3$$

and

$$e^{\frac{1}{2}(x - \frac{\alpha_x}{1+x})^2} \geq e^{\frac{x^2}{2} - (\frac{x}{1+x})\alpha_x} \geq 0.89e^{\frac{x^2}{2}}.$$



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From these facts, Proposition 2.1(1), Proposition 2.1(3) and  $\alpha_x \leq 0.11$ , we have

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}e^{\frac{1}{2}(x-\frac{\alpha_x}{1+x})^2}} \left| \frac{x}{\sqrt{\text{Var } S_x}} - x - \frac{ES_x}{\sqrt{\text{Var } S_x}} \right| \\ & \leq \frac{1}{\sqrt{2\pi} \left(0.89e^{\frac{x^2}{2}}\right)} \left| \frac{x}{\sqrt{\text{Var } S_x}} - x \right| + \frac{1}{\sqrt{2\pi} \left(0.89e^{\frac{x^2}{2}}\right)} \left| \frac{ES_x}{\sqrt{\text{Var } S_x}} \right| \\ & \leq \frac{1.452\alpha_x x}{\sqrt{2\pi}(0.89)(0.193)(1+x)^3} + \frac{\alpha_x}{(1+x)} \frac{(1+1.452\alpha_x)}{\sqrt{2\pi}(0.89)(0.933)(1+x)} \\ & \leq \frac{3.373\alpha_x x}{(1+x)^3} + \frac{0.558\alpha_x}{(1+x)^2} \leq \frac{3.931\alpha_x}{(1+x)^2}. \end{aligned}$$

From this fact, (2.8) and (2.9), we have (2.7)

**Case 2.**  $1.3 \leq x < 2$ .

By the fact that  $|P(W_n \leq x) - \Phi(x)| \leq 0.55$  ([3, pp. 246]), we can assume  $\frac{\alpha_x}{(1+x)^2} \leq 0.011$ , i.e.  $\alpha_x \leq 0.099$ .

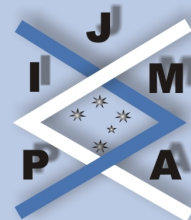
From this fact, (2.7) and Proposition 2.2(1), we have

$$\begin{aligned} |P(W_n \leq x) - \Phi(x)| & \leq \frac{4.931\alpha_x}{(1+x)^2} + \left| P\left(\bar{S}_x \leq \frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) - \Phi\left(\frac{x - ES_x}{\sqrt{\text{Var } S_x}}\right) \right| \\ & \leq \frac{4.931\alpha_x}{(1+x)^2} + \frac{41.195\beta_x}{(1+x)^3} + \frac{54.513\alpha_x}{(1+x)^2} \\ & = \frac{59.444\alpha_x}{(1+x)^2} + \frac{41.195\beta_x}{(1+x)^3} \leq 59.444\delta_x. \end{aligned}$$

**Case 3.**  $2 \leq x \leq 14$ .

**Subcase 3.1.**  $(1+x)^2\alpha_x \geq \frac{1}{5}$ .





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Using the same argument of subcase 1.1 in Theorem 1.2 of ([14]) and the facts that

$$(2.10) \quad \frac{1+x}{x} = 1 + \frac{1}{x} \leq 1.5 \quad \text{and} \quad e^{\frac{x^2}{2}} \geq 0.92x^3 \quad \text{for} \quad 2 \leq x \leq 14,$$

we can show that

$$|P(W_n \leq x) - \Phi(x)| \leq 37.408\delta_x.$$

**Subcase 3.2.**  $(1+x)^2\alpha_x < \frac{1}{5}$ .

Note that for  $x \geq 2$ , we have

$$0 \leq \alpha_x \leq \frac{1}{5(1+x)^2} \leq 0.023 \leq 0.11.$$

By (2.7) and Proposition 2.2(2), we obtain the required bounds.

**Case 4.**  $x > 14$ .

Follows the argument of case 3 on replacing the inequalities

$$e^{\frac{x^2}{2}} \geq 60x^3 \quad \text{and} \quad \frac{1+x}{x} = 1 + \frac{1}{x} \leq 1.071$$

in (2.10). □

*Proof of Corollary 1.4.* If  $0 \leq x < 1.3$ .

We used the same argument as case 1 of Theorem 1.3 and the fact that  $(1 + \frac{x}{4})^3 \leq 2.327$  to get  $C = 9.54$ .

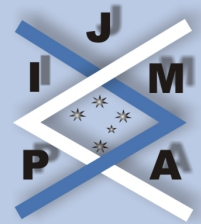
Suppose that  $x \geq 1.3$ . By the fact that

$$\delta_x \leq \left( \frac{1 + \frac{x}{4}}{1+x} \right)^2 \delta_{\frac{x}{4}},$$

we have

$$\delta_x \leq \begin{cases} 0.332\delta_{\frac{x}{4}} & \text{if } 1.3 \leq x < 2, \\ 0.250\delta_{\frac{x}{4}} & \text{if } 2 \leq x < 3, \\ 0.192\delta_{\frac{x}{4}} & \text{if } 3 \leq x < 7.98, \\ 0.112\delta_{\frac{x}{4}} & \text{if } 7.98 \leq x < 14, \\ 0.090\delta_{\frac{x}{4}} & \text{if } x \geq 14. \end{cases}$$

Then Corollary 1.4 follows from this fact and Theorem 1.3.



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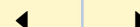
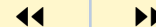
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