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HILBERT-PACHPATTE TYPE INEQUALITIES FROM BONSALL'S FORM OF HILBERT'S INEQUALITY

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ABSTRACT. The main objective of this paper is to deduce Hilbert-Pachpatte type inequalities using Bonsall's form of Hilbert's and Hardy-Hilbert's inequalities, both in discrete and continuous case.

Key words and phrases: Inequalities, Hilbert's inequality, sequences and functions, homogeneous kernels, conjugate and non-conjugate exponents, the Beta function.

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1. INTRODUCTION

An interesting feature of one of the forms of Hilbert-Pachpatte type inequalities, is that it controls the size (in the sense of L^p or l^p spaces) of the modified Hilbert transform of a function or of a series with the size of its derivate or its backward differences, respectively. We start with the following results of Zhongxue Lü from [9], for both continuous and discrete cases. For a sequence $a : \mathbb{N}_0 \to \mathbb{R}$, the sequence $\nabla a : \mathbb{N} \to \mathbb{R}$ is defined by $\nabla a(n) = a(n) - a(n-1)$. For a function $u : (0, \infty) \to \mathbb{R}$, u' denotes the usual derivative of u.

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Theorem A. *Let* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $s > 2-\min\{p,q\}$, *and* $f(x)$, $g(y)$ *be real-valued continuous functions defined on* $[0,\infty)$, *respectively, and let* $f(0) = g(0) = 0$, *and*

$$0 < \int_0^\infty \int_0^x x^{1-s}|f'(\tau)|^p d\tau dx < \infty, \quad 0 < \int_0^\infty \int_0^y y^{1-s}|g'(\delta)|^q d\delta dy < \infty,$$

then

$$\begin{aligned}(1.1)\quad & \int_0^\infty \int_0^\infty \frac{|f(x)||g(y)|}{(qx^{p-1}+py^{q-1})(x+y)^s} dxdy \\ & \le \frac{B\left(\frac{q+s-2}{q}, \frac{p+s-2}{p}\right)}{pq} \cdot \left(\int_0^\infty \int_0^x x^{1-s}|f'(\tau)|^p d\tau dx\right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^y y^{1-s}|g'(\delta)|^q d\delta dy\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem B. *Let* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $s > 2-\min\{p,q\}$, *and* $\{a(m)\}$ *and* $\{b(n)\}$ *be two sequences of real numbers where* $m, n \in \mathbb{N}_0$, *and* $a(0) = b(0) = 0$, *and*

$$0 < \sum_{m=1}^\infty \sum_{\tau=1}^m m^{1-s}|\nabla a(\tau)|^p < \infty, \quad 0 < \sum_{n=1}^\infty \sum_{\delta=1}^n n^{1-s}|\nabla b(\delta)|^q < \infty,$$

then

$$\begin{aligned}(1.2)\quad & \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{|a_m||b_n|}{(qm^{p-1}+pn^{q-1})(m+n)^s} \\ & \le \frac{B\left(\frac{q+s-2}{q}, \frac{p+s-2}{p}\right)}{pq} \cdot \left(\sum_{m=1}^\infty \sum_{\tau=1}^m m^{1-s}|\nabla a(\tau)|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty \sum_{\delta=1}^n n^{1-s}|\nabla b(\delta)|^q\right)^{\frac{1}{q}}.\end{aligned}$$

Note that the condition $s > 2-\min\{p,q\}$ from Theorem B is not sufficient. Namely, the author of the proof of Theorem B used the following result

$$(1.3)\quad \sum_{n=1}^\infty \frac{1}{(m+n)^s}\left(\frac{m}{n}\right)^{\frac{2-s}{q}} < B\left(\frac{q+s-2}{q}, \frac{p+s-2}{p}\right) m^{1-s},$$

for $m \in \{1,2,\dots\}$ and $s > 2-\min\{p,q\}$. For $p = q = 2$, $s = 18$ and $m = 1$, the left-hand side of (1.3) is greater than the right-hand side of (1.3). Therefore, we refer to a paper of Krnić and Pečarić, [4], where the next inequality is given:

$$(1.4)\quad \sum_{n=1}^\infty \frac{1}{(m+n)^s}\frac{m^{\alpha_1}}{n^{\alpha_2}} < m^{1-s+\alpha_1-\alpha_2} B(1-\alpha_2, s+\alpha_2-1),$$

where $0 < s \le 14$, $1-s < \alpha_2 < 1$ for $s \le 2$ and $-1 \le \alpha_2 < 1$ for $s > 2$. By using this result, here we shall obtain a generalization of Theorem B but with the condition $2-\min\{p,q\} < s \le 2+\min\{p,q\}$. Also, the following result is given in [9]:

$$(1.5)\quad \sum_{n=1}^\infty \frac{1}{m^s+n^s}\left(\frac{m}{n}\right)^{\frac{2-s}{q}} < \frac{1}{s} B\left(\frac{q+s-2}{sq}, \frac{p+s-2}{sp}\right) m^{1-s},$$

for $m \in \{1,2,\dots\}$ and $s > 2-\min\{p,q\}$. Similarly as before, for $p = q = 2$, $s = 6$ and $m = 1$, the left-hand side of (1.5) is greater than the right-hand side of (1.5). The case of nontrivial weights is essential in Theorem A and Theorem B, since for $s = 1$ only the trivial functions and sequences satisfy the assumptions.

In 1951, Bonsall established the following conditions for non-conjugate exponents (see [1]). Let p and q be real parameters, such that

$$p>1,\ q>1,\ \frac{1}{p}+\frac{1}{q}\geq 1, \tag{1.6}$$

and let p' and q' respectively be their conjugate exponents, that is, $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Further, define

$$\lambda=\frac{1}{p'}+\frac{1}{q'} \tag{1.7}$$

and note that $0<\lambda\leq 1$ for all p and q as in (1.6). In particular, $\lambda=1$ holds if and only if $q=p'$, that is, only when p and q are mutually conjugate. Otherwise, we have $0<\lambda<1$, and in such cases p and q will be referred to as non-conjugate exponents. Also, in this paper we shall obtain some generalizations of (1.1). It will be done in simplier way than in [9]. Our results will be based on the following results of Pečarić et al., [2], for the non-conjugate and conjugate exponents.

Theorem C. *Let p, p', q, q' and λ be as in (1.6) and (1.7). If K, φ, ψ, f and g are non-negative measurable functions, then the following inequalities hold and are equivalent*

$$\int_{\Omega^2} K^{\lambda}(x,y)f(x)g(y)dxdy \leq \left(\int_\Omega (\varphi F f)^p(x)dx\right)^{\frac{1}{p}}\left(\int_\Omega (\psi G g)^q(y)dy\right)^{\frac{1}{q}} \tag{1.8}$$

and

$$\left(\int_\Omega \left(\frac{1}{\psi G(y)}\int_\Omega K^{\lambda}(x,y)f(x)dx\right)^{q'}dy\right)^{\frac{1}{q'}} \leq \left(\int_\Omega (\varphi F f)^p(x)dx\right)^{\frac{1}{p}}, \tag{1.9}$$

where the functions F, G are defined by

$$F(x)=\left(\int_\Omega \frac{K(x,y)}{\psi^{q'}(y)}dy\right)^{\frac{1}{q'}} \quad \textit{and} \quad G(y)=\left(\int_\Omega \frac{K(x,y)}{\varphi^{p'}(x)}dx\right)^{\frac{1}{p'}}.$$

The next inequalities from [5] can be seen as a special case of (1.8) and (1.9) respectively for the conjugate exponents:

$$\int_{\Omega^2} K(x,y)f(x)g(y)dxdy \leq \left(\int_\Omega \varphi^p(x)F(x)f^p(x)dx\right)^{\frac{1}{p}}\left(\int_\Omega \psi^q(y)G(y)g^q(y)dy\right)^{\frac{1}{q}} \tag{1.10}$$

and

$$\int_\Omega G^{1-p}(y)\psi^{-p}(y)\left(\int_\Omega K(x,y)f(x)dx\right)^p dy \leq \int_\Omega \varphi^p(x)F(x)f^p(x)dx, \tag{1.11}$$

where

$$F(x)=\int_\Omega \frac{K(x,y)}{\psi^p(y)}dy \quad \text{and} \quad G(y)=\int_\Omega \frac{K(x,y)}{\varphi^q(x)}dx. \tag{1.12}$$

In particular, inequalities (1.10) and (1.11) are equivalent.

On the other hand, here we also refer to a paper of Brnetić et al., [8], where a general Hilbert-type inequality was obtained for $n\geq 2$ conjugate exponents, that is, real parameters $p_1,\dots,p_n>1$, such that $\sum_{i=1}^n \frac{1}{p_i}=1$. Namely, we let $K:\Omega^n\to\mathbb{R}$ and $\phi_{ij}:\Omega\to\mathbb{R}$,

$i, j = 1, \dots, n$, be non-negative measurable functions. If $\prod_{i,j=1}^{n} \phi_{ij}(x_j) = 1$, then the inequality

$$\int_{\Omega^n} K(x_1, \dots, x_n) \prod_{i=1}^{n} f_i(x_i)\, dx_1 \dots dx_n \le \prod_{i=1}^{n} \left(\int_{\Omega} F_i(x_i)(\phi_{ii} f_i)^{p_i}(x_i)\, dx_i \right)^{\frac{1}{p_i}}, \tag{1.13}$$

holds for all non-negative measurable functions $f_1, \dots, f_n : \Omega \to \mathbb{R}$, where

$$F_i(x_i) = \int_{\Omega^{n-1}} K(x_1, \dots, x_n) \prod_{j=1, j\neq i}^{n} \phi_{ij}^{p_i}(x_j) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n, \tag{1.14}$$

for $i = 1, \dots, n$.

2. INTEGRAL CASE

In this section we shall state our main results. We suppose that all integrals converge and shall omit these types of conditions. Thus, we have the following.

Theorem 2.1. *Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. If $K(x, y)$, $\varphi(x)$, $\psi(y)$ are non-negative functions and $f(x)$, $g(y)$ are absolutely continuous functions such that $f(0) = g(0) = 0$, then the following inequalities hold*

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{K(x,y)|f(x)|\,|g(y)|}{qx^{p-1} + py^{q-1}} dxdy \\ &\le \int_0^\infty \int_0^\infty K(x,y)|f(x)|\,|g(y)| d\left(x^{\frac{1}{p}}\right) d\left(y^{\frac{1}{q}}\right) \\ &\le \frac{1}{pq} \left(\int_0^\infty \int_0^x \varphi^p(x) F(x) |f'(\tau)|^p d\tau dx \right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^y \psi^q(y) G(y) |g'(\delta)|^q d\delta dy \right)^{\frac{1}{q}}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\int_0^\infty G^{1-p}(y) \psi^{-p}(y) \left(\int_0^\infty K(x,y)|f(x)| d\left(x^{\frac{1}{p}}\right) \right)^p dy \\ &\qquad \le \frac{1}{p^p} \int_0^\infty \int_0^x \varphi^p(x) F(x) |f'(\tau)|^p d\tau dx, \end{aligned} \tag{2.2}$$

where $F(x)$ and $G(y)$ are defined as in (1.12).

Proof. By using Hölder's inequality, (see also [9]), we have

$$|f(x)|\,|g(y)| \le x^{\frac{1}{q}} y^{\frac{1}{p}} \left(\int_0^x |f'(\tau)|^p d\tau \right)^{\frac{1}{p}} \left(\int_0^y |g'(\delta)|^q d\delta \right)^{\frac{1}{q}}. \tag{2.3}$$

From (2.3) and using the elementary inequality

$$xy \le \frac{x^p}{p} + \frac{y^q}{q}, \ \ x \ge 0, \ \ y \ge 0, \ \ \frac{1}{p} + \frac{1}{q} = 1, \ \ p > 1, \tag{2.4}$$

we observe that

$$\frac{pq|f(x)|\,|g(y)|}{qx^{p-1} + py^{q-1}} \le \frac{|f(x)|\,|g(y)|}{x^{\frac{1}{q}} y^{\frac{1}{p}}} \le \left(\int_0^x |f'(\tau)|^p d\tau \right)^{\frac{1}{p}} \left(\int_0^y |g'(\delta)|^q d\delta \right)^{\frac{1}{q}} \tag{2.5}$$

and therefore

$$pq\int_0^\infty\int_0^\infty \frac{K(x,y)|f(x)|\,|g(y)|}{qx^{p-1}+py^{q-1}}dxdy \tag{2.6}$$
$$\leq \int_0^\infty\int_0^\infty \frac{K(x,y)}{x^{\frac{1}{q}}y^{\frac{1}{p}}}|f(x)||g(y)|dxdy$$
$$\leq \int_0^\infty\int_0^\infty K(x,y)\left(\int_0^x |f'(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^y |g'(\delta)|^q d\delta\right)^{\frac{1}{q}} dxdy.$$

Applying the substitutions

$$f_1(x)=\left(\int_0^x |f'(\tau)|^p d\tau\right)^{\frac{1}{p}},\quad g_1(y)=\left(\int_0^y |g'(\delta)|^q d\delta\right)^{\frac{1}{q}}$$

and (1.10), we obtain

$$\int_0^\infty\int_0^\infty K(x,y)f_1(x)g_1(y)dxdy \tag{2.7}$$
$$\leq \left(\int_0^\infty \varphi^p(x)F(x)f_1^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty \psi^q(y)G(y)g_1^q(y)dy\right)^{\frac{1}{q}}$$
$$=\left(\int_0^\infty\int_0^x \varphi^p(x)F(x)|f'(\tau)|^p d\tau dx\right)^{\frac{1}{p}}\left(\int_0^\infty\int_0^y \psi^q(y)G(y)|g'(\delta)|^q d\delta dy\right)^{\frac{1}{q}}.$$

By using (2.6) and (2.7) we obtain (2.1). The second inequality (2.2) can be proved by applying (1.11) and the inequality

$$|f(x)|\leq x^{\frac{1}{q}}\left(\int_0^x |f'(t)|^p dt\right)^{\frac{1}{p}}.$$

□

Now we can apply our main result to non-negative homogeneous functions. Recall that for a homogeneous function of degree $-s$, $s>0$, the equality $K(tx,ty)=t^{-s}K(x,y)$ is satisfied. Further, we define

$$k(\alpha):=\int_0^\infty K(1,u)u^{-\alpha}du$$

and suppose that $k(\alpha)<\infty$ for $1-s<\alpha<1$. To prove first application of our main results we need the following lemma.

Lemma 2.2. *If $s>0$, $1-s<\alpha<1$ and $K(x,y)$ is a non-negative homogeneous function of degree $-s$, then*

$$\int_0^\infty K(x,y)\left(\frac{x}{y}\right)^\alpha dy = x^{1-s}k(\alpha). \tag{2.8}$$

Proof. By using the substitution $u=\frac{y}{x}$ and the fact that $K(x,y)$ is homogeneous function, the equation (2.8) follows easily. □

Corollary 2.3. *Let $s>0$, $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$. If $f(x)$, $g(y)$ are absolutely continuous functions such that $f(0)=g(0)=0$, and $K(x,y)$ is a non-negative symmetrical and homogeneous* *function of degree* $-s$, *then the following inequalities hold*

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{K(x,y)|f(x)|\,|g(y)|}{qx^{p-1}+py^{q-1}}dxdy &\le \int_0^\infty\int_0^\infty K(x,y)|f(x)|\,|g(y)|d\left(x^{\frac{1}{p}}\right)d\left(y^{\frac{1}{q}}\right)\\ &\le \frac{L}{pq}\left(\int_0^\infty\int_0^x x^{1-s+p(A_1-A_2)}|f'(\tau)|^p d\tau dx\right)^{\frac{1}{p}}\\ &\quad\times\left(\int_0^\infty\int_0^y y^{1-s+q(A_2-A_1)}|g'(\delta)|^q d\delta dy\right)^{\frac{1}{q}} \end{aligned}\tag{2.9}$$

and

$$\begin{aligned}\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)}\left(\int_0^\infty K(x,y)|f(x)|d(x^{\frac{1}{p}})\right)^p dy\\ \le \left(\frac{L}{p}\right)^p\int_0^\infty\int_0^x x^{1-s+p(A_1-A_2)}|f'(\tau)|^p d\tau dx,\end{aligned}\tag{2.10}$$

where $A_1 \in (\frac{1-s}{q},\frac{1}{q})$, $A_2\in(\frac{1-s}{p},\frac{1}{p})$ *and* $L = k(pA_2)^{\frac{1}{p}}k(qA_1)^{\frac{1}{q}}$.

Proof. Let $F(x)$, $G(y)$ be the functions defined as in (1.12). Setting $\varphi(x)=x^{A_1}$ and $\psi(y)=y^{A_2}$, by Lemma 2.2 we obtain

$$\begin{aligned}\int_0^\infty\int_0^x \varphi^p(x)F(x)|f'|^p d\tau dx &= \int_0^\infty\int_0^x |f'(\tau)|^p\left(\int_0^\infty K(x,y)\left(\frac{x}{y}\right)^{pA_2}dy\right)x^{p(A_1-A_2)}d\tau dx\\ &= k(pA_2)\int_0^\infty\int_0^x x^{1-s+p(A_1-A_2)}|f'(\tau)|^p d\tau dx,\end{aligned}\tag{2.11}$$

and similarly

$$\int_0^\infty\int_0^y \psi^q(y)G(y)|g'(\delta)|^q d\delta dy = k(qA_1)\int_0^\infty\int_0^y y^{1-s+q(A_2-A_1)}|g'(\delta)|^q d\delta dy.\tag{2.12}$$

From (2.1), (2.11) and (2.12), we get (2.9). Similarly, the inequality (2.10) follows from (2.2). □

We proceed with some special homogeneous functions. First, by putting $K(x,y)=\frac{1}{(x+y)^s}$ in Corollary 2.3, we get the following.

Corollary 2.4. *Let* $s>0$, $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$. *If* $f(x)$, $g(y)$ *are absolutely continuous functions such that* $f(0)=g(0)=0$, *then the following inequalities hold*

$$\begin{aligned}\int_0^\infty\int_0^\infty \frac{|f(x)|\,|g(y)|}{(qx^{p-1}+py^{q-1})(x+y)^s}dxdy &\le \int_0^\infty\int_0^\infty \frac{|f(x)|\,|g(y)|}{(x+y)^s}d\left(x^{\frac{1}{p}}\right)d\left(y^{\frac{1}{q}}\right)\\ &\le \frac{L_1}{pq}\left(\int_0^\infty\int_0^x x^{1-s+p(A_1-A_2)}|f'(\tau)|^p d\tau dx\right)^{\frac{1}{p}}\left(\int_0^\infty\int_0^y y^{1-s+q(A_2-A_1)}|g'(\delta)|^q d\delta dy\right)^{\frac{1}{q}}\end{aligned}$$

and

$$\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)} \left(\int_0^\infty \frac{|f(x)|}{(x+y)^s} d\left(x^{\frac{1}{p}}\right) \right)^p dy \leq \left(\frac{L_1}{p}\right)^p \int_0^\infty \int_0^x x^{1-s+p(A_1-A_2)} |f'(\tau)|^p d\tau dx,$$

where $A_1 \in (\frac{1-s}{q}, \frac{1}{q})$, $A_2 \in (\frac{1-s}{p}, \frac{1}{p})$ *and*

$$L_1 = [B(1-pA_2, pA_2+s-1)]^{\frac{1}{p}} [B(1-qA_1, qA_1+s-1)]^{\frac{1}{q}}.$$

Remark 2.5. By putting $A_1 = A_2 = \frac{2-s}{pq}$ in Corollary 2.4, with the condition $s > 2 - \min\{p, q\}$, we obtain Theorem A from the introduction.

Since the function $K(x,y) = \frac{\ln \frac{y}{x}}{y-x}$ is symmetrical and homogeneous of degree -1, by using Corollary 2.3 we obtain:

Corollary 2.6. *Let* $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$. *If* $f(x)$, $g(y)$ *are absolutely continuous functions such that* $f(0) = g(0) = 0$, *then the following inequalities hold*

$$\begin{aligned}\int_0^\infty \int_0^\infty & \frac{\ln \frac{y}{x} |f(x)|\, |g(y)|}{(qx^{p-1} + py^{q-1})(y-x)} dxdy \\ &\leq \int_0^\infty \int_0^\infty \frac{\ln \frac{y}{x} |f(x)|\, |g(y)|}{y-x} d\left(x^{\frac{1}{p}}\right) d\left(y^{\frac{1}{q}}\right) \\ &\leq \frac{L_2}{pq} \left(\int_0^\infty \int_0^x x^{p(A_1-A_2)} |f'(\tau)|^p d\tau dx \right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^y y^{q(A_2-A_1)} |g'(\delta)|^q d\delta dy \right)^{\frac{1}{q}}\end{aligned}$$

and

$$\int_0^\infty y^{p(A_1-A_2)} \left(\int_0^\infty \frac{|f(x)| \ln \frac{y}{x}}{y-x} d\left(x^{\frac{1}{p}}\right) \right)^p dy \leq \left(\frac{L_2}{p}\right)^p \int_0^\infty \int_0^x x^{p(A_1-A_2)} |f'(\tau)|^p d\tau dx,$$

where $A_1 \in (0, \frac{1}{q})$, $A_2 \in (0, \frac{1}{p})$ *and*

$$L_2 = \pi^2 (\sin pA_2\pi)^{-\frac{2}{p}} (\sin qA_1\pi)^{-\frac{2}{q}}.$$

Similarly, for the symmetrical homogeneous function of degree $-s$, $K(x,y) = \frac{1}{\max\{x,y\}^s}$, we have:

Corollary 2.7. *Let* $s > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$. *If* $f(x)$, $g(y)$ *are absolutely continuous functions such that* $f(0) = g(0) = 0$, *then the following inequalities hold*

$$\begin{aligned}\int_0^\infty \int_0^\infty & \frac{|f(x)|\, |g(y)|}{(qx^{p-1} + py^{q-1}) \max\{x,y\}^s} dxdy \\ &\leq \int_0^\infty \int_0^\infty \frac{|f(x)|\, |g(y)|}{\max\{x,y\}^s} d\left(x^{\frac{1}{p}}\right) d\left(y^{\frac{1}{q}}\right) \\ &\leq \frac{L_3}{pq} \left(\int_0^\infty \int_0^x x^{1-s+p(A_1-A_2)} |f'(\tau)|^p d\tau dx \right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^y y^{1-s+q(A_2-A_1)} |g'(\delta)|^q d\delta dy \right)^{\frac{1}{q}}\end{aligned}$$

and

$$\int_0^\infty y^{(p-1)(s-1)+p(A_1-A_2)} \left(\int_0^\infty \frac{|f(x)|}{\max\{x,y\}^s} d\left(x^{\frac{1}{p}}\right) \right)^p dy \leq \left(\frac{L_3}{p}\right)^p \int_0^\infty \int_0^x x^{1-s+p(A_1-A_2)} |f'(\tau)|^p d\tau dx,$$

where $A_1 \in \left(\frac{1-s}{q}, \frac{1}{q}\right)$, $A_2 \in \left(\frac{1-s}{p}, \frac{1}{p}\right)$ *and* $L_3 = k(pA_2)^{\frac{1}{p}} k(qA_1)^{\frac{1}{q}}$, *where* $k(\alpha) = \frac{s}{(1-\alpha)(s+\alpha-1)}$.

At the end of this section we give a generalization of the inequality (2.1) from Theorem 2.1. In the proof we used a general Hilbert-type inequality (1.13) of Brnetić et al., [8].

Theorem 2.8. *Let* $n \in \mathbb{N}$, $n \geq 2$, $\sum_{i=1}^n \frac{1}{p_i} = 1$ *with* $p_i > 1$, $i = 1, \dots, n$. *Let* q_i, α_i, $i = 1, \dots, n$, *are defined with* $\frac{1}{q_i} = 1 - \frac{1}{p_i}$ *and* $\alpha_i = \prod_{j=1, j\neq i}^n p_j$. *If* $K(x_1, \dots, x_n)$, $\phi_{ij}(x_j)$, $i, j = 1, \dots, n$, *are non-negative functions such that* $\prod_{i,j=1}^n \phi_{ij}(x_j) = 1$, *and* $f_i(x_i)$, $i = 1, \dots, n$, *are absolutely continuous functions such that* $f_i(0) = 0$, $i = 1, \dots, n$, *then the following inequality holds*

$$\int_0^\infty \dots \int_0^\infty \frac{K(x_1, \dots, x_n) \prod_{i=1}^n |f_i(x_i)|}{\sum_{i=1}^n \alpha_i x_i^{p_i - 1}} dx_1 \dots dx_n$$
$$\leq \int_0^\infty \dots \int_0^\infty K(x_1, \dots, x_n) \prod_{i=1}^n |f_i(x_i)| d\left(x_1^{\frac{1}{p_1}}\right) \dots d\left(x_n^{\frac{1}{p_n}}\right)$$
$$\leq \frac{1}{p_1 \dots p_n} \prod_{i=1}^n \left(\int_0^\infty \int_0^{x_i} \phi_{ii}^{p_i}(x_i) F_i(x_i) |f_i'(\tau_i)|^{p_i} d\tau_i dx_i\right)^{\frac{1}{p_i}},$$

where $F_i(x_i)$ *are defined as in (1.14) for* $i = 1, \dots, n$.

3. DISCRETE CASE

We also give results for the discrete case. For that, we apply the following result from [5].

Theorem 3.1. *If* $\{a(m)\}$ *and* $\{b(n)\}$ *are non-negative real sequences,* $K(x, y)$ *is non-negative homogeneous function of degree* $-s$ *strictly decreasing in both parameters* x *and* y, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, A, B, α, $\beta > 0$, *then the following inequalities hold and are equivalent*

$$\sum_{m=1}^\infty \sum_{n=1}^\infty K(Am^\alpha, Bn^\beta) a_m b_n$$
$$< N \left(\sum_{m=1}^\infty m^{\alpha(1-s)+\alpha p(A_1 - A_2)+(p-1)(1-\alpha)} a_m^p\right)^{\frac{1}{p}}$$
$$\cdot \left(\sum_{n=1}^\infty n^{\beta(1-s)+\beta q(A_2 - A_1)+(q-1)(1-\beta)} b_n^q\right)^{\frac{1}{q}}, \tag{3.1}$$

and

$$\sum_{n=1}^\infty n^{\beta(s-1)(p-1)+p\beta(A_1 - A_2)+\beta-1} \left(\sum_{m=1}^\infty K(Am^\alpha, Bn^\beta) a_m\right)^p$$
$$< N^p \sum_{m=1}^\infty m^{\alpha(1-s)+\alpha p(A_1 - A_2)+(p-1)(1-\alpha)} a_m^p, \tag{3.2}$$

where $A_1 \in (\max\{\frac{1-s}{q}, \frac{\alpha-1}{\alpha q}\}, \frac{1}{q})$, $A_2 \in \left(\max\left\{\frac{1-s}{p}, \frac{\beta-1}{\beta p}\right\}, \frac{1}{p}\right)$ *and*

$$N = \alpha^{-\frac{1}{q}} \beta^{-\frac{1}{p}} A^{\frac{2-s}{p} + A_1 - A_2 - 1} B^{\frac{2-s}{q} + A_2 - A_1 - 1} k(pA_2)^{\frac{1}{p}} k(2 - s - qA_1)^{\frac{1}{q}}. \tag{3.3}$$

Applying Theorem 3.1, we obtain the following.

Corollary 3.2. *Let* $s > 0$, $\frac{1}{p}+\frac{1}{q} = 1$ *with* $p > 1$. *Let* $\{a(m)\}$ *and* $\{b(n)\}$ *be two sequences of real numbers where* $m, n \in \mathbb{N}_0$, *and* $a(0) = b(0) = 0$. *If* $K(x, y)$ *is a non-negative homogeneous function of degree* $-s$ *strictly decreasing in both parameters* x *and* y, $A, B, \alpha, \beta > 0$, *then the following inequalities hold*

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}&\frac{K(Am^{\alpha}, Bn^{\beta})|a_m|\,|b_n|}{qm^{p-1}+pn^{q-1}} \\ &\le \frac{1}{pq}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{K(Am^{\alpha}, Bn^{\beta})|a_m|\,|b_n|}{m^{\frac{1}{q}}n^{\frac{1}{p}}} \\ &< \frac{N}{pq}\left(\sum_{m=1}^{\infty}\sum_{\tau=1}^{m} m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}|\nabla a(\tau)|^p\right)^{\frac{1}{p}} \\ &\quad\cdot\left(\sum_{n=1}^{\infty}\sum_{\delta=1}^{n} n^{\beta(1-s)+\beta q(A_2-A_1)+(q-1)(1-\beta)}|\nabla b(\delta)|^q\right)^{\frac{1}{q}},\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\beta(s-1)(p-1)+p\beta(A_1-A_2)+\beta-1}&\left(\sum_{m=1}^{\infty}K(Am^{\alpha}, Bn^{\beta})\frac{|a_m|}{m^{\frac{1}{q}}}\right)^p \\ &< N^p\sum_{m=1}^{\infty}\sum_{\tau=1}^{m} m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}|\nabla a(\tau)|^p,\end{aligned} \tag{3.5}$$

where $A_1 \in \left(\max\left\{\frac{1-s}{q}, \frac{\alpha-1}{\alpha q}\right\}, \frac{1}{q}\right)$, $A_2 \in \left(\max\left\{\frac{1-s}{p}, \frac{\beta-1}{\beta p}\right\}, \frac{1}{p}\right)$ *and the constant* N *is defined as in (3.3).*

Proof. By using Hölder's inequality, (see also [9]), we have

$$|a_m|\,|b_n| \le m^{\frac{1}{q}}n^{\frac{1}{p}}\left(\sum_{\tau=1}^{m}|\nabla a(\tau)|^p\right)^{\frac{1}{p}}\left(\sum_{\delta=1}^{n}|\nabla b(\delta)|^q\right)^{\frac{1}{q}}. \tag{3.6}$$

From (2.4) and (3.6), we get

$$\frac{pq|a_m|\,|b_n|}{qm^{p-1}+pn^{q-1}} \le \frac{|a_m|\,|b_n|}{m^{\frac{1}{q}}n^{\frac{1}{p}}} \le \left(\sum_{\tau=1}^{m}|\nabla a(\tau)|^p\right)^{\frac{1}{p}}\left(\sum_{\delta=1}^{n}|\nabla b(\delta)|^q\right)^{\frac{1}{q}}, \tag{3.7}$$

and therefore

$$\begin{aligned}pq\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}&\frac{K(Am^{\alpha}, Bn^{\beta})|a_m|\,|b_n|}{qm^{p-1}+pn^{q-1}} \\ &\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{K(Am^{\alpha}, Bn^{\beta})|a_m|\,|b_n|}{m^{\frac{1}{q}}n^{\frac{1}{p}}} \\ &\le \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}K(Am^{\alpha}, Bn^{\beta})\left(\sum_{\tau=1}^{m}|\nabla a(\tau)|^p\right)^{\frac{1}{p}}\left(\sum_{\delta=1}^{n}|\nabla b(\delta)|^q\right)^{\frac{1}{q}}.\end{aligned} \tag{3.8}$$

Applying the substitutions $\widetilde{a}_m = \left(\sum_{\tau=1}^{m} |\nabla a(\tau)|^p\right)^{\frac{1}{p}}$, $\widetilde{b}_n = \left(\sum_{\delta=1}^{n} |\nabla b(\delta)|^q\right)^{\frac{1}{q}}$ and (3.1), we have

$$\begin{aligned}(3.9)\qquad &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(Am^{\alpha}, Bn^{\beta})\widetilde{a}_m\widetilde{b}_n \\ &< N\left(\sum_{m=1}^{\infty} m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}\widetilde{a}_m^p\right)^{\frac{1}{p}} \\ &\quad\cdot\left(\sum_{n=1}^{\infty} n^{\beta(1-s)+\beta q(A_2-A_1)+(q-1)(1-\beta)}\widetilde{b}_n^q\right)^{\frac{1}{q}}, \\ &= \left(\sum_{m=1}^{\infty}\sum_{\tau=1}^{m} m^{\alpha(1-s)+\alpha p(A_1-A_2)+(p-1)(1-\alpha)}|\nabla a(\tau)|^p\right)^{\frac{1}{p}} \\ &\quad\cdot\left(\sum_{n=1}^{\infty}\sum_{\delta=1}^{n} n^{\beta(1-s)+\beta q(A_2-A_1)+(q-1)(1-\beta)}|\nabla b(\delta)|^q\right)^{\frac{1}{q}},\end{aligned}$$

where $A_1 \in (\max\{\frac{1-s}{q}, \frac{\alpha-1}{\alpha q}\}, \frac{1}{q})$, $A_2 \in (\max\{\frac{1-s}{p}, \frac{\beta-1}{\beta p}\}, \frac{1}{p})$ and the constant N is defined as in (3.3). Now, by applying (3.8) and (3.9) we obtain (3.4). The second inequality (3.5) can be proved by using (3.2) and the inequality

$$|a_m| \le m^{\frac{1}{q}}\left(\sum_{\tau=1}^{m}|\nabla a(\tau)|^p\right)^{\frac{1}{p}}.$$

□

Remark 3.3. If the function $K(x,y)$ from the previous corollary is symmetrical, then $k(2-s-qA_1) = k(qA_1)$. So, if $K(x,y) = \frac{1}{(x+y)^s}$, then we can put $A_1 = A_2 = \frac{2-s}{pq}$, $A = B = \alpha = \beta = 1$ in Corollary 3.2 and obtain Theorem B from the introduction but with the condition $2 - \min\{p,q\} < s < 2$.

By using (1.4), see [4], we will obtain a larger interval for the parameter s. More precisely, we have:

Corollary 3.4. *Let $\frac{1}{p}+\frac{1}{q} = 1$ with $p > 1$ and $2-\min\{p,q\} < s \le 2+\min\{p,q\}$. Let $\{a(m)\}$ and $\{b(n)\}$ be two sequences of real numbers where $m, n \in \mathbb{N}_0$, and $a(0) = b(0) = 0$. Then the following inequalities hold*

$$\begin{aligned}(3.10)\qquad &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m|\,|b_n|}{(qm^{p-1}+pn^{q-1})(m+n)^s} \\ &\le \frac{1}{pq}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m|\,|b_n|}{m^{\frac{1}{q}}n^{\frac{1}{p}}(m+n)^s} \\ &< \frac{N_1}{pq}\left(\sum_{m=1}^{\infty}\sum_{\tau=1}^{m} m^{1-s}|\nabla a(\tau)|^p\right)^{\frac{1}{p}}\cdot\left(\sum_{n=1}^{\infty}\sum_{\delta=1}^{n} n^{1-s}|\nabla b(\delta)|^q\right)^{\frac{1}{q}},\end{aligned}$$

and

$$(3.11)\qquad \sum_{n=1}^{\infty} n^{(s-1)(p-1)}\left(\sum_{m=1}^{\infty}\frac{|a_m|}{m^{\frac{1}{q}}(m+n)^s}\right)^p < N_1^p\sum_{m=1}^{\infty}\sum_{\tau=1}^{m} m^{1-s}|\nabla a(\tau)|^p,$$

where $N_1 = B(\frac{s+q-2}{q}, \frac{s+p-2}{p})$.

Proof. As in the proof of Corollary 3.2, by using Hölder's inequality we obtain

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m|\,|b_n|}{(qm^{p-1}+pn^{q-1})(m+n)^s}\\
&\le \frac{1}{pq}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{|a_m|\,|b_n|}{m^{\frac{1}{q}}n^{\frac{1}{p}}(m+n)^s}\\
&\le \frac{1}{pq}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+n)^s}\left(\sum_{\tau=1}^{m}|\nabla a(\tau)|^p\right)^{\frac{1}{p}}\left(\sum_{\delta=1}^{n}|\nabla b(\delta)|^q\right)^{\frac{1}{q}}\left(\frac{m}{n}\right)^{\frac{2-s}{pq}}\left(\frac{n}{m}\right)^{\frac{2-s}{pq}}\\
&\le \frac{1}{pq}\left(\sum_{m=1}^{\infty}\sum_{\tau=1}^{m}\left(\sum_{n=1}^{\infty}\frac{1}{(m+n)^s}\left(\frac{m}{n}\right)^{\frac{2-s}{q}}\right)|\nabla a(\tau)|^p\right)^{\frac{1}{p}}\\
&\qquad\cdot\left(\sum_{n=1}^{\infty}\sum_{\delta=1}^{n}\left(\sum_{m=1}^{\infty}\frac{1}{(m+n)^s}\left(\frac{n}{m}\right)^{\frac{2-s}{p}}\right)|\nabla b(\delta)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Now, the inequality (3.10) follows from (1.4). Let us show that the inequality (3.11) is valid. For this purpose we use the following inequality from [4]

$$\sum_{n=1}^{\infty} n^{(s-1)(p-1)}\left(\sum_{m=1}^{\infty}\frac{a_m}{(m+n)^s}\right)^p < L_1\sum_{m=1}^{\infty} m^{1-s}a_m^p, \tag{3.12}$$

where $2-\min\{p,q\} < s \le 2+\min\{p,q\}$ and $L_1 = B(\frac{s+p-2}{p}, \frac{s+q-2}{q})$. Setting

$$a_m = \left(\sum_{\tau=1}^{m}|\nabla a(\tau)|^p\right)^{\frac{1}{p}}$$

in (3.12) and using

$$|a_m| \le m^{\frac{1}{q}}\left(\sum_{\tau=1}^{m}|\nabla a(\tau)|^p\right)^{\frac{1}{p}},$$

the inequality (3.11) follows easily. □

4. NON-CONJUGATE EXPONENTS

Let $p,\ p',\ q,\ q'$ and λ be as in (1.6) and (1.7). To obtain an analogous result for the case of non-conjugate exponents, we introduce real parameters $r',\ r$ such that $p \le r' \le q'$ and $\frac{1}{r'}+\frac{1}{r}=1$. For example, we can define $\frac{1}{r'} = \frac{1}{q'}+\frac{1-\lambda}{2}$ or $r' = (2-\lambda)p$.

It is easy to see that

$$x^{\frac{1}{p'}}y^{\frac{1}{q'}} \le \frac{1}{rr'}\left(rx^{\frac{r'}{p'}}+r'y^{\frac{r}{q'}}\right), \qquad x\ge 0,\ y\ge 0, \tag{4.1}$$

and

$$|f(x)|\,|g(y)| \le x^{\frac{1}{p'}}y^{\frac{1}{q'}}\left(\int_0^x|f'(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^y|g'(\delta)|^q d\delta\right)^{\frac{1}{q}}, \tag{4.2}$$

hold, where $f(x)$, $g(y)$ are absolutely continuous functions on $(0,\infty)$.

Applying Theorem C, (4.1) and (4.2) in the same way as in the proof of Theorem 2.1, we obtain the following result for non-conjugate exponents.

Theorem 4.1. *Let p, p', q, q' and λ be as in (1.6) and (1.7). Let r', r be real parameters such that $p \le r' \le q'$ and $\frac{1}{r'} + \frac{1}{r} = 1$. If $K(x,y)$, $\varphi(x)$, $\psi(y)$ are non-negative functions and $f(x)$, $g(y)$ are absolutely continuous functions such that $f(0) = g(0) = 0$, then the following inequalities hold*

$$\begin{aligned}(4.3)\qquad &\int_0^\infty \int_0^\infty \frac{K^\lambda(x,y)|f(x)|\,|g(y)|}{rx^{\frac{r'}{p'}} + r'y^{\frac{r}{q'}}}dxdy \\ &\le \frac{pq}{rr'} \int_0^\infty \int_0^\infty K^\lambda(x,y)|f(x)|\,|g(y)|d\left(x^{\frac{1}{p}}\right) d\left(y^{\frac{1}{q}}\right) \\ &\le \frac{1}{rr'} \left(\int_0^\infty \int_0^x (\varphi F)^p(x)|f'(\tau)|^p d\tau dx\right)^{\frac{1}{p}} \left(\int_0^\infty \int_0^y (\psi G)^q(y)|g'(\delta)|^q d\delta dy\right)^{\frac{1}{q}},\end{aligned}$$

and

$$\begin{aligned}(4.4)\qquad &\left(\int_0^\infty \left(\frac{1}{\psi G(y)} \int_0^\infty K^\lambda(x,y)|f(x)|d(x^{\frac{1}{p}})\right)^{q'} dy\right)^{\frac{1}{q'}} \\ &\qquad\le \frac{1}{p}\left(\int_0^\infty \int_0^x (\varphi F)^p(x)|f'(\tau)|^p d\tau dx\right)^{\frac{1}{p}},\end{aligned}$$

where $F(x)$ and $G(y)$ are defined as in Theorem C.

Obviously, Theorem 4.1 is the generalization of Theorem 2.1. Namely, if $\lambda = 1$, $r' = p$ and $r = q$, then the inequalities (4.3) and (4.4) become respectively the inequalities (2.1) and (2.2). If $K(x,y)$ is a non-negative symmetrical and homogeneous function of degree $-s$, $s > 0$, then we obtain:

Corollary 4.2. *Let $s > 0$, p, p', q, q' and λ be as in (1.6) and (1.7). If $f(x)$, $g(y)$ are absolutely continuous functions such that $f(0) = g(0) = 0$, and $K(x,y)$ is a non-negative symmetrical and homogeneous function of degree $-s$, then the following inequalities hold*

$$\begin{aligned}(4.5)\qquad &\int_0^\infty \int_0^\infty \frac{K^\lambda(x,y)|f(x)|\,|g(y)|}{qx^{(p-1)(2-\lambda)} + py^{(q-1)(2-\lambda)}}dxdy \\ &\le \frac{1}{2-\lambda} \int_0^\infty \int_0^\infty K^\lambda(x,y)|f(x)|\,|g(y)|d\left(x^{\frac{1}{p}}\right) d\left(y^{\frac{1}{q}}\right) \\ &\le \frac{M}{pq(2-\lambda)} \left(\int_0^\infty \int_0^x x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}|f'(\tau)|^p d\tau dx\right)^{\frac{1}{p}} \\ &\qquad \cdot \left(\int_0^\infty \int_0^y y^{\frac{q}{p'}(1-s)+q(A_2-A_1)}|g'(\delta)|^q d\delta dy\right)^{\frac{1}{q}}\end{aligned}$$

and

$$\begin{aligned}(4.6)\qquad &\left(\int_0^\infty y^{\frac{q'}{p'}(s-1)+q'(A_1-A_2)} \left(\int_0^\infty K^\lambda(x,y)|f(x)|d\left(x^{\frac{1}{p}}\right)\right)^{q'} dy\right)^{\frac{1}{q'}} \\ &\qquad\le \frac{M}{p}\left(\int_0^\infty \int_0^x x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}|f'(\tau)|^p d\tau dx\right)^{\frac{1}{p}},\end{aligned}$$

where $A_1 \in \left(\frac{1-s}{p'}, \frac{1}{p'}\right)$, $A_2 \in \left(\frac{1-s}{q'}, \frac{1}{q'}\right)$ and $M = k(p'A_1)^{\frac{1}{p'}} k(q'A_2)^{\frac{1}{q'}}$.

Proof. The proof follows directly from Theorem 4.1 setting $r' = (2-\lambda)p$, $r = (2-\lambda)q$, $\varphi(x) = x^{A_1}$ and $\psi(y) = y^{A_2}$ in the inequalities (4.3) and (4.4). Namely, if $F(x)$ and $G(y)$ are the functions defined by

$$F(x) = \left(\int_0^\infty \frac{K(x,y)}{\psi^{q'}(y)}dy\right)^{\frac{1}{q'}} \quad \text{and} \quad G(y) = \left(\int_0^\infty \frac{K(x,y)}{\varphi^{p'}(x)}dx\right)^{\frac{1}{p'}},$$

then applying Lemma 2.2 we have

$$(\varphi F)^p(x) = x^{pA_1}\left(\int_0^\infty K(x,y)y^{-q'A_2}dy\right)^{\frac{p}{q'}} \tag{4.7}$$
$$= x^{pA_1 - pA_2}\left(\int_0^\infty K(x,y)\left(\frac{x}{y}\right)^{q'A_2}dy\right)^{\frac{p}{q'}}$$
$$= x^{\frac{p}{q'}(1-s)+p(A_1-A_2)}k^{\frac{p}{q'}}(q'A_2),$$

and similarly

$$(\psi G)^q(y) = y^{\frac{q}{p'}(1-s)+q(A_2-A_1)}k^{\frac{q}{p'}}(p'A_1). \tag{4.8}$$

Now, by using (4.3), (4.7) and (4.8) we obtain (4.5).

The second inequality (4.6) follows directly from (4.4). □

Remark 4.3. Setting $K(x,y) = \frac{1}{(x+y)^s}$ in Corollary 4.2 we obtain that the constant M is equal to $M = B(1-p'A_1, p'A_1+s-1)^{\frac{1}{p'}}B(1-q'A_2, q'A_2+s-1)^{\frac{1}{q'}}$.

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