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INEQUALITIES IN THE COMPLEX PLANE

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ABSTRACT. A differential inequality is generalised and improved. Several other differential inequalities are considered.

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1. INTRODUCTION

Let $\mathcal{H}(U)$ be the set of holomorfic functions defined on the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

In [2, pp. 38 Example 2.4. d] and [3, pp. 192 Example 9.3.4] the authors have proved, as an application of the developed theory, the implication:

If $f \in \mathcal{H}(U)$, $f(0) = 1$ and

$$\mathrm{Re}(f(z) + zf'(z) + z^2 f''(z)) > 0,\ z \in U \text{ then } \mathrm{Re}\, f(z) > 0,\ z \in U.$$

The aim of this paper is to generalise this inequality and to determine the biggest $\alpha \in R$ for which the implication

$$f(0) = 1,\ \mathrm{Re}(f(z) + zf'(z) + z^2 f''(z)) > 0,\ (\forall)\ z \in U \Rightarrow \mathrm{Re}\, f(z) > \alpha,\ (\forall)\ z \in U$$

holds true.

In this paper each many-valued function is taken with the principal value.

2. PRELIMINARIES

In our study we need the following definitions and lemmas:

Let X be a locally convex linear topological space. For a subset $U \subset X$ the closed convex hull of U is defined as the intersection of all closed convex sets containing U and will be denoted by $co(U)$. If $U \subset V \subset X$ then U is called an extremal subset of V provided that whenever $u = tx + (1-t)y$ where $u \in U$, $x, y \in V$ and $t \in (0, 1)$ then $x, y \in U$.

An extremal subset of U consisting of just one point is called an extreme point of U.

The set of the extreme points of U will be denoted by EU.

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Lemma 2.1 ([1, pp. 45]). *If $J : \mathcal{H}(U) \to \mathbb{R}$ is a real-valued, continuous convex functional and $\mathcal{F}$ is a compact subset of $\mathcal{H}(U)$, then*

$$\max\{J(f) : f \in co(\mathcal{F})\} = \max\{J(f) : f \in \mathcal{F}\} = \max\{J(f) : f \in E(co(\mathcal{F}))\}.$$

In the particular case if J is a linear map then we also have:

$$\min\{J(f) : f \in co(\mathcal{F})\} = \min\{J(f) : f \in \mathcal{F}\} = \min\{J(f) : f \in E(co(\mathcal{F}))\}.$$

Suppose that $f, g \in \mathcal{H}(U)$. The function f is subordinate to g if there exists a function $\theta \in \mathcal{H}(U)$ such that $\theta(0) = 0$, $|\theta(z)| < 1$, $z \in U$ and $f(z) = g(\theta(z))$, $z \in U$.

The subordination will be denoted by $f \prec g$.

Observation 1. *Suppose that $f, g \in \mathcal{H}(U)$ and g is univalent. If $f(0) = g(0)$ and $f(U) \subset g(U)$ then $f \prec g$.*

When $F \in \mathcal{H}(U)$ we use the notation

$$s(F) = \{f \in \mathcal{H}(U) : f \prec F\}.$$

Lemma 2.2 ([1, pp. 51]). *Suppose that F_α is defined by the equality*

$$F_\alpha(z) = \left(\frac{1+cz}{1-z}\right)^\alpha, \qquad |c| \le 1,\ c \ne -1.$$

If $\alpha \ge 1$ then $co(s(F_\alpha))$ consists of all functions in $\mathcal{H}(U)$ represented by

$$f(z) = \int_0^{2\pi} \left(\frac{1+cze^{-it}}{1-ze^{-it}}\right)^\alpha d\mu(t)$$

where μ is a positive measure on $[0, 2\pi]$ having the property $\mu([0, 2\pi]) = 1$ and

$$E(co(s(F_\alpha))) = \left\{ \frac{1+cze^{-it}}{1-ze^{-it}} \middle| t \in [0, 2\pi] \right\}.$$

Observation 2. *If $L : \mathcal{H}(U) \to \mathcal{H}(U)$ is an invertible linear map and $\mathcal{F} \subset \mathcal{H}(U)$ is a compact subset, then $L(co(\mathcal{F})) = co(L(\mathcal{F}))$ and the set $E(co(\mathcal{F}))$ is in one-to-one correspondence with $EL(co(\mathcal{F}))$.*

3. THE MAIN RESULT

Theorem 3.1. *If $f \in \mathcal{H}(U)$, $f(0) = 1$; $m, p \in \mathbb{N}^*$; $a_k \in \mathbb{R}$, $k = \overline{1, p}$ and*

$$\operatorname{Re} \sqrt[m]{f(z) + a_1 z f'(z) + \cdots + a_p z^p f^{(p)}(z)} > 0, \qquad z \in U \tag{3.1}$$

then

$$1 + \inf_{z \in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{\sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1}}{P(n)} z^n\right) < \operatorname{Re} f(z) < 1 + \sup_{z\in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{\sum_{k=0}^{m} C_m^k C_{m+n-k-1}^{m-1}}{P(n)} z^n\right), \qquad z \in U \tag{3.2}$$

where $P(x) = 1 + a_1 x + a_2 x(x-1) + \cdots + a_p x(x-1)\cdots(x-p+1)$.

Proof. The condition of the theorem can be rewritten in the form

$$\sqrt[m]{f(z) + a_1 z f'(z) + \cdots + a_p z^p f^{(p)}(z)} \prec \frac{1+z}{1-z},$$

which is equivalent to

$$f(z)+a_1zf'(z)+\cdots+a_pz^pf^{(p)}(z)\prec\left(\frac{1+z}{1-z}\right)^m.$$

According to the results of Lemma 2.2,

$$f(z)+a_1zf'(z)+\cdots+a_pz^pf^{(p)}(z)=\int_0^{2\pi}\left(\frac{1+ze^{-it}}{1-ze^{-it}}\right)^m d\mu(t)=h(z),$$

where $\mu([0,2\pi])=1$.

If

$$f(z)=1+\sum_{n=1}^{\infty}b_nz^n,\qquad z\in U$$

then

$$f(z)+a_1zf'(z)+\cdots+a_pz^pf^{(p)}(z)=1+\sum_{n=1}^{\infty}b_nP(n)z^n.$$

On the other hand

$$\int_0^{2\pi}\left(\frac{1+ze^{-it}}{1-ze^{-it}}\right)^m d\mu(t)=1+\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}C_m^kC_{m+n-k-1}^{m-1}\right)z^n\int_0^{2\pi}e^{-int}d\mu(t),$$

with $C_p^q=0$ if $q>p$. The equalities $C_p^q=0$ if $q>p$ imply also that:

$$\sum_{k=0}^{n}C_m^kC_{m+n-k-1}^{m-1}=\sum_{k=0}^{m}C_m^kC_{m+n-k-1}^{m-1}.$$

The above two developments in power series imply that:

$$1+\sum_{n=0}^{\infty}b_nP(n)z^n=1+\sum_{n=1}^{\infty}\left(\sum_{k=0}^{m}C_m^kC_{m+n-k-1}^{m-1}\right)z^n\int_0^{2\pi}e^{-int}d\mu(t)$$

and

$$b_n=\frac{1}{P(n)}\left(\sum_{k=0}^{m}C_m^kC_{m+n-k-1}^{m-1}\right)\int_0^{2\pi}e^{-int}d\mu(t).$$

Consequently,

$$f(z)=1+\sum_{n=1}^{\infty}\frac{1}{P(n)}\left(\sum_{k=0}^{m}C_m^kC_{m+n-k-1}^{m-1}\right)z^n\int_0^{2\pi}e^{-int}d\mu(t).$$

If

$$\mathcal{B}=\left\{h\in\mathcal{H}(U)\,\middle|\,h(z)=\int_0^{2\pi}\left(\frac{1+ze^{-it}}{1-ze^{-it}}\right)^m d\mu(t),\ z\in U,\ \mu([0,2\pi])=1\right\},$$

$$\mathcal{C}=\left\{f\in\mathcal{H}(U)\,\middle|\,\mathrm{Re}\left(\sqrt[m]{f(z)+a_1zf'(z)+\cdots+a_pz^pf^{(p)}(z)}\right)>0,\ z\in U\right\}$$

then the correspondence $L:\mathcal{B}\to\mathcal{C}$, $L(h)=f$ defines an invertible linear map and according to Observation 2 the extreme points of the class $\mathcal{C}$ are

$$f_t(z)=1+\sum_{n=1}^{\infty}\frac{1}{P(n)}\left(\sum_{k=0}^{m}C_m^kC_{m+n-k-1}^{m-1}\right)z^ne^{-int},\qquad z\in U,\ t\in[0,2\pi).$$

This result and Lemma 2.1 implies the assertion of Theorem 3.1. □

4. Particular Cases

If we put $p = 2, a_1 = a_2 = m = 1$ in Theorem 3.1 then we get:

Corollary 4.1. *If* $f \in \mathcal{H}(U)$, $f(0) = 1$ *and*

$$\operatorname{Re}(f(z) + zf'(z) + z^2 f''(z)) > 0, \quad z \in U, \tag{4.1}$$

then

$$\frac{\pi(e^{2\pi}+1)}{e^{2\pi}-1} > \operatorname{Re} f(z) > \frac{2\pi e^{\pi}}{e^{2\pi}-1}, \quad z \in U \tag{4.2}$$

and these results are sharp in the sense that

$$\sup_{\substack{z\in U\\ f\in\mathcal{C}}} \operatorname{Re} f(z) = \frac{\pi(e^{2\pi}+1)}{e^{2\pi}-1} \quad \textit{and}$$

$$\inf_{\substack{z\in U\\ f\in\mathcal{C}}} \operatorname{Re} f(z) = \frac{2\pi e^{\pi}}{e^{2\pi}-1}.$$

Proof. Theorem 3.1 implies the following inequalities:

$$1 + \inf_{z\in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2+1} z^n\right) < \operatorname{Re} f(z) < 1 + \sup_{z\in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2+1} z^n\right).$$

The minimum and maximum principle for harmonic functions imply that

$$\sup_{z\in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2+1} z^n\right) = \sup_{t\in[0,2\pi]} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2+1} e^{int}\right)$$

$$\inf_{z\in U} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2+1} z^n\right) = \inf_{t\in[0,2\pi]} \operatorname{Re}\left(\sum_{n=1}^{\infty} \frac{2}{n^2+1} e^{int}\right).$$

By considering the integral

$$I_n = \int_{|z|=n+\frac{1}{2}} \frac{e^{izt}}{(z^2+1)(e^{2\pi iz}-1)} dz, \quad t \in [0, 2\pi),$$

using the equality $\lim_{n\to\infty} I_n = 0$ and residue theory we deduce that

$$1 + \operatorname{Re}\left(\sum_{k=1}^{\infty} \frac{2}{k^2+1} e^{ikt}\right) = \frac{\pi(e^t + e^{2\pi - t})}{e^{2\pi}-1}, \quad t \in [0, 2\pi)$$

and so we get

$$\frac{\pi(e^{2\pi}+1)}{e^{2\pi}-1} > \operatorname{Re}(f(z)) > \frac{2\pi e^{\pi}}{e^{2\pi}-1}, \quad z \in U.$$

□

If we put $m = 2$, $a_1 = 0$, $a_2 = 4$, Theorem 3.1 implies

Corollary 4.2. *If* $f \in \mathcal{H}(U)$, $f(0) = 1$ *and*

$$\operatorname{Re} \sqrt{f(z) + 4z^2 f''(z)} > 0, \quad z \in U, \tag{4.3}$$

then

$$\operatorname{Re} f(z) > 1 + 4\sum_{k=1}^{\infty} \frac{(-1)^k \cdot k}{(2k-1)^2}, \quad z \in U \tag{4.4}$$

and this result is sharp.

Proof. Theorem 3.1 and the minimum principle imply that

$$\operatorname{Re} f(z) > 1 + 4 \inf_{t\in[0,2\pi]} \operatorname{Re}\left(\sum_{k=1}^{\infty} \frac{e^{ikt}\cdot k}{(2k-1)^2}\right).$$

It is easy to observe that

$$\begin{aligned} 2\sum_{k=1}^{\infty}\frac{ke^{ikt}}{(2k-1)^2} &= \sum_{n=1}^{\infty}\frac{e^{ikt}}{2k-1} + \sum_{n=1}^{\infty}\frac{e^{ikt}}{(2k-1)^2} \\ &= \int_0^1 \sum_{k=1}^{\infty}(x^2)^{k-1}e^{ikt}dx + \int_0^1\int_0^1 \sum_{k=1}^{\infty}(x^2y^2)^{k-1}e^{ikt}dxdy \\ &= \int_0^1 \frac{e^{it}}{1-x^2e^{it}}dx + \int_0^1\int_0^1 \frac{e^{it}}{1-x^2y^2e^{it}}dxdy, \qquad t\in[0,2\pi). \end{aligned} \tag{4.5}$$

Since

$$\operatorname{Re}\frac{e^{it}}{1-x^2e^{it}} \geq \frac{-1}{1+x^2}, \qquad x\in[0,1],\ t\in[0,2\pi)$$

and

$$\operatorname{Re}\frac{e^{it}}{1-x^2y^2e^{it}} \geq \frac{-1}{1+x^2\cdot y^2}, \qquad x,y\in[0,1],\ t\in[0,2\pi),$$

by integrating we get that

$$\operatorname{Re}\int_0^1 \frac{e^{it}}{1-x^2e^{it}}dx \geq -\int_0^1 \frac{1}{1+x^2}dx$$

and

$$\operatorname{Re}\int_0^1 \frac{e^{it}}{1-x^2y^2e^{it}}dxdy \geq -\int_0^1 \frac{1}{1+x^2y^2}dxdy.$$

In the derived inequalities, equality occurs if $t=\pi$, this means that

$$\inf_{t\in[0,2\pi)} \operatorname{Re}\sum_{k=1}^{\infty}\frac{ke^{ikt}}{(2k-1)^2} = \sum_{k=1}^{\infty}\frac{(-1)^k\cdot k}{(2k-1)^2}$$

and the inequality (4.4) holds true. □

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