Journal of Inequalities in Pure and Applied Mathematics

A NOTE ON WEIGHTED IDENTRIC AND LOGARITHMIC MEANS

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volume 7, issue 5, article 157, 2006.

Received 28 July, 2006; accepted 25 August, 2006.

Communicated by: P.S. Bullen

ISSN (electronic): 1443-5756
202-06

Abstract

Recently obtained inequalities [12] between the Gaussian hypergeometric function and the power mean are applied to establish new sharp inequalities involving the weighted identric, logarithmic, and power means.

2000 Mathematics Subject Classification: 26D07, 26D15, 33C05.
Key words: Identric mean, Logarithmic mean, Hypergeometric function.

Contents

1. Introduction

For $x, y > 0$, the *weighted power mean* of order λ is given by

$$\mathcal{M}_\lambda(\omega; x, y) \equiv \left[(1-\omega)\, x^\lambda + \omega\, y^\lambda\right]^{\frac{1}{\lambda}}$$

with $\omega \in (0,1)$ and $\mathcal{M}_0(\omega; x, y) \equiv \lim_{\lambda\to 0} \mathcal{M}_\lambda(\omega; x, y) = x^{1-\omega} y^\omega$. Since $\lambda \mapsto \mathcal{M}_\lambda$ is increasing, it follows that

$$\mathcal{G}(x,y) \leq \mathcal{M}_\lambda\left(\frac{1}{2}; x, y\right) \leq \mathcal{A}(x,y), \qquad \text{for } 0 \leq \lambda \leq 1,$$

where $\mathcal{G}(x,y) \equiv \mathcal{M}_0\left(\frac{1}{2}; x, y\right)$ and $\mathcal{A}(x,y) \equiv \mathcal{M}_1\left(\frac{1}{2}; x, y\right)$ are the well-known geometric and arithmetic means, respectively (e.g., see [4, p. 203]). Thus, $\mathcal{M}_\lambda$ provides a refinement of the classical inequality $\mathcal{G} \leq \mathcal{A}$. It is natural to seek other bivariate means that separate $\mathcal{G}$ and $\mathcal{A}$. Two such means are the *logarithmic mean* and the *identric mean*. For distinct $x, y > 0$, the logarithmic mean $\mathcal{L}$ is given by

$$\mathcal{L}(x,y) \equiv \frac{x-y}{\ln(x) - \ln(y)},$$

and $\mathcal{L}(x,x) \equiv x$. The integral representation

$$\mathcal{L}(1, 1-r) = \left(\int_0^1 (1-rt)^{-1}\, dt\right)^{-1}, \qquad r < 1 \tag{1.1}$$

is due to Carlson [6]. Similarly, the identric mean $\mathcal{I}$ is defined by

$$\mathcal{I}(x,y) \equiv \frac{1}{e}\left(\frac{x^x}{y^y}\right)^{\frac{1}{x-y}},$$

$\mathcal{I}(x, x) \equiv x$, and has the integral representation

$$\mathcal{I}(1, 1-r) = \exp\left(\int_0^1 \ln(1-rt)\, dt\right), \qquad r < 1. \tag{1.2}$$

The inequality $\mathcal{G} \leq \mathcal{L} \leq \mathcal{A}$ was refined by Carlson [6] who showed that $\mathcal{L}(x, y) \leq \mathcal{M}_{1/2}\left(\frac{1}{2}; x, y\right)$. Lin [8] then sharpened this by proving $\mathcal{L}(x, y) \leq \mathcal{M}_{1/3}\left(\frac{1}{2}; x, y\right)$. Shortly thereafter, Stolarsky [14] introduced the generalized logarithmic mean which has since come to bear his name. These and other efforts (e.g., [11, 15]) led to many interesting results, including the following well-known inequalities:

$$\mathcal{G} \leq \mathcal{L} \leq \mathcal{M}_{1/3} \leq \mathcal{M}_{2/3} \leq \mathcal{I} \leq \mathcal{A}, \tag{1.3}$$

where each is evaluated at (x, y), and the power means have equal weights $\omega = 1 - \omega = 1/2$. It also should be noted that the indicated orders of the power means in (1.3), namely 1/3 and 2/3, are sharp. Following the work of Leach and Sholander [7], Páles [10] gave a complete ordering of the general Stolarsky mean which provides an elegant generalization of (1.3). (For a more complete discussion of inequalites involving means, see [4].)

2. Main Results

Our main objective is to present a generalization of (1.3) using the *weighted* logarithmic and identric means. Moreover, sharp power mean bounds are provided. This can be accomplished using the Gaussian hypergeometric function ${}_2F_1$ which is given by

$$ {}_2F_1(\alpha,\beta;\gamma;r) \equiv \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} r^n, \qquad |r| < 1, $$

where $(\alpha)_n$ is the Pochhammer symbol defined by $(\alpha)_0 = 1$, $(\alpha)_1 = \alpha$, and $(\alpha)_{n+1} = (\alpha)_n(\alpha+n)$, for $n \in \mathbb{N}$. For $\gamma > \beta > 0$, ${}_2F_1$ has the following integral representation due to Euler (see [2]):

$$ {}_2F_1(\alpha,\beta;\gamma;r) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-rt)^{-\alpha}\,dt, $$

which, by continuation, extends the domain of ${}_2F_1$ to all $r < 1$. Here $\Gamma(z) \equiv \int_0^\infty t^{z-1}e^{-t}\,dt$ for $z > 0$; $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. Inequalities relating the Gaussian hypergeometric function to various means have been widely studied (see [1, 2, 3, 5, 12]). Of particular use here is the *hypergeometric mean* of order a discussed by Carlson in [5] and defined by

$$ \begin{aligned} \mathcal{H}_a(\omega;c;x,y) &\equiv \left[\frac{\Gamma(c)}{\Gamma(c\,\omega')\Gamma(c\,\omega)} \int_0^1 t^{c\,\omega-1}(1-t)^{c\,\omega'-1}(x(1-t)+yt)^a\,dt\right]^{\frac{1}{a}} \\ &= x\cdot\left[{}_2F_1\left(-a,c\,\omega;c;1-\frac{y}{x}\right)\right]^{\frac{1}{a}} \end{aligned} $$

with the parameter $c > 0$ and weights $\omega,\ \omega' > 0$ satisfying $\omega + \omega' = 1$. Clearly $\mathcal{H}_a(\omega; c; \rho x, \rho y) = \rho\,\mathcal{H}_a(\omega; c; x, y)$ for $\rho > 0$, so $\mathcal{H}_a$ is homogeneous. Euler's integral representation and (1.1) together yield

$$\mathcal{H}_{-1}\left(\frac{1}{2}; 2; 1, 1-r\right) = \left(\frac{\Gamma(2)}{\Gamma(1)^2}\int_0^1 (1-rt)^{-1}\,dt\right)^{-1} = \mathcal{L}(1, 1-r).$$

Multiplying by x, with $r = 1 - y/x$, and applying homogeneity yields $\mathcal{H}_{-1}\left(\frac{1}{2}; 2; x, y\right) = \mathcal{L}(x, y)$. This naturally leads to the *weighted logarithmic mean* $\hat{\mathcal{L}}$ which is defined as

$$\hat{\mathcal{L}}(\omega; c; x, y) \equiv \mathcal{H}_{-1}(\omega; c; x, y).$$

Weighted logarithmic means have been discussed by Pittenger [11] and Neuman [9], among others (see also [4, p. 391-392]). Similarly, the *weighted identric mean* $\hat{\mathcal{I}}$ is given by

$$\begin{aligned}\hat{\mathcal{I}}(\omega; c; x, y) &\equiv \mathcal{H}_0(\omega; c; x, y) \equiv \lim_{a\to 0}\mathcal{H}_a(\omega; c; x, y)\\ &= \exp\left(\frac{\Gamma(c)}{\Gamma(c\,\omega')\Gamma(c\,\omega)}\int_0^1 t^{c\,\omega-1}(1-t)^{c\,\omega'-1}\ln[x(1-t)+yt]\,dt\right)\end{aligned}$$

(see [5], [13]). Thus, $\hat{\mathcal{I}}\left(\frac{1}{2}; 2; x, y\right) = \mathcal{I}(x, y)$.

The following theorem establishes inequalities between the power means and the weighted identric and logarithmic means.

Theorem 2.1. *Suppose $x > y > 0$ and $c \geq 1$.*

If $0 < \omega \leq 1/2$, then the weighted identric mean $\hat{\mathcal{I}}$ satisfies

$$\mathcal{M}_{\frac{c}{c+1}}(\omega; x, y) \leq \hat{\mathcal{I}}(\omega; c; x, y). \tag{2.1}$$

If $1/2 \leq \omega < 1$ and $c \leq 3$, then the weighted logarithmic mean $\hat{\mathcal{L}}$ satisfies

$$\hat{\mathcal{L}}(\omega; c; x, y) \leq \mathcal{M}_{\frac{c-1}{c+1}}(\omega; x, y). \tag{2.2}$$

Moreover, the power mean orders $c/(c+1)$ and $(c-1)/(c+1)$ are sharp.

A key step in the proof will be an application of the following recently obtained result:

Proposition 2.2. *[12] Suppose $1 \geq a$ and $c > b > 0$. If $c \geq \max\{1 - 2a, 2b\}$, then*

$$\mathcal{M}_\lambda \left(\frac{b}{c}; 1, 1 - r\right) \leq [{}_2F_1(-a, b; c; r)]^{\frac{1}{a}} \quad \textit{for all } r \in (0, 1), \tag{2.3}$$

if and only if $\lambda \leq (a + c)/(1 + c)$. If $-a \leq c \leq \min\{1 - 2a, 2b\}$, then the inequality in (2.3) reverses if and only if $\lambda \geq (a + c)/(1 + c)$.

Proof of Theorem 2.1. Suppose $x > y > 0$, $c \geq 1$, $\omega \in (0, 1)$ and define $b \equiv c\omega$ with $r \equiv 1 - y/x \in (0, 1)$. If $\omega \leq 1/2$ and $a \in (0, 1)$, it follows that $c \geq \max\{1 - 2a, 2b\}$. Hence the previous proposition implies

$$\mathcal{M}_{\frac{a+c}{1+c}}(\omega; 1, 1 - r) \leq [{}_2F_1(-a, b; c; r)]^{\frac{1}{a}}. \tag{2.4}$$

Taking the limit of both sides of (2.4) as $a \to 0^+$ yields

$$\mathcal{M}_{\frac{c}{c+1}}(\omega; 1, 1-r) \le \mathcal{H}_0(\omega; c; 1, 1-r). \tag{2.5}$$

Now suppose $\omega \ge 1/2$ and $c \le 3$. Then $c \le 2b$ and $-a = 1 \le c \le 3 = 1 - 2a$ for $a = -1$. Thus

$$\mathcal{H}_{-1}(\omega; c; 1, 1-r) = \left[{}_2F_1(1, b; c; r)\right]^{-1} \le \mathcal{M}_{\frac{c-1}{c+1}}(\omega; 1, 1-r), \tag{2.6}$$

again by the above proposition. Multiplying both sides of the inequalities in (2.5) and (2.6) by x and applying homogeneity yields the desired results. $\square$

In the case that $\omega = 1/2$, we have

Corollary 2.3. *If $x, y > 0$, $1 \le c \le 3$, and $\omega = 1/2$ then*

$$\mathcal{H}_{-2} \le \mathcal{H}_{-1} \le \mathcal{M}_{\frac{c-1}{c+1}} \le \mathcal{M}_{\frac{c}{c+1}} \le \mathcal{H}_0 \le \mathcal{H}_1. \tag{2.7}$$

Moreover, $(c-1)/(c+1)$ and $c/(c+1)$ are sharp. If $c = 2$, then (2.7) reduces to (1.3).

Proof. Suppose $x > y > 0$, $1 \le c \le 3$, and $\omega = 1/2$. Hence (2.2) and (2.1), together with the fact that $\lambda \mapsto \mathcal{M}_\lambda$ is increasing, imply

$$\mathcal{H}_{-1}\left(\frac{1}{2}; c; x, y\right) \le \mathcal{M}_{\frac{c-1}{c+1}}\left(\frac{1}{2}; x, y\right) \le \mathcal{M}_{\frac{c}{c+1}}\left(\frac{1}{2}; x, y\right) \le \mathcal{H}_0\left(\frac{1}{2}; c; x, y\right).$$

The remaining inequalities follow directly from Carlson's observation [5] that $a \mapsto \mathcal{H}_a$ is increasing. The condition that $x > y$ can be relaxed by noting that $\mathcal{H}_a$ is symmetric in (x, y) when $\omega = 1/2$. This symmetry can be seen by making the substitution $s = 1 - t$ in Euler's integral representation:

$$\begin{aligned} \mathcal{H}_a\left(\frac{1}{2}; c; x, y\right)^a &= \frac{\Gamma(c)}{\Gamma(c/2)^2} \int_0^1 [t(1-t)]^{c/2-1}((1-t)x + ty)^a \, dt \\ &= \frac{\Gamma(c)}{\Gamma(c/2)^2} \int_0^1 [(1-s)s]^{c/2-1}(sx + (1-s)y)^a \, ds \\ &= \mathcal{H}_a\left(\frac{1}{2}; c; y, x\right)^a. \end{aligned}$$

Finally, note that $\mathcal{M}_{\frac{c-1}{c+1}} = \mathcal{M}_{\frac{1}{3}}$ and $\mathcal{M}_{\frac{c}{c+1}} = \mathcal{M}_{\frac{2}{3}}$ when $c = 2$. Also,

$$\mathcal{H}_{-2}\left(\frac{1}{2}; 2; 1, 1-r\right)^{-2} = {}_2F_1(2, 1; 2; r) = \frac{1}{1-r},$$

for $|r| < 1$. It follows that $\mathcal{H}_{-2}\left(\frac{1}{2}; 2; x, y\right) = (xy)^{\frac{1}{2}} = \mathcal{G}(x, y)$. Likewise, $\mathcal{H}_1\left(\frac{1}{2}; 2; x, y\right) = x(1 - (1 - y/x)/2) = \mathcal{A}(x, y)$. Thus (2.7) implies (1.3). $\square$

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