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NEW SUBCLASSES OF MEROMORPHIC $p-$VALENT FUNCTIONS

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ABSTRACT. In this paper, we introduce two subclasses $\Omega_p^*(\alpha)$ and $\Lambda_p^*(\alpha)$ of meromorphic p-valent functions in the punctured disk $\mathcal{D} = \{z : 0 < |z| < 1\}$. Coefficient inequalities, distortion theorems, the radii of starlikeness and convexity, closure theorems and Hadamard product (or convolution) of functions belonging to these classes are obtained.

Key words and phrases: Meromorphic $p-$valent functions, Meromorphically starlike and convex functions.

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1. INTRODUCTION AND DEFINITIONS

Let Σ_p denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1} z^{p+n-1} \qquad (p \in \mathbb{N}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk $\mathcal{D} = \{z : 0 < |z| < 1\}$. A function $f \in \Sigma_p$ is said to be in the class $\Omega_p(\alpha)$ of meromorphic p-valently starlike functions of order α in $\mathcal{D}$ if and only if

$$\text{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \mathcal{D};\ 0 \le \alpha < p;\ p \in \mathbb{N}). \tag{1.2}$$

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Furthermore, a function $f \in \Sigma_p$ is said to be in the class $\Lambda_p(\alpha)$ of meromorphic p-valently convex functions of order α in $\mathcal{D}$ if and only if

$$\text{Re}\left\{-1-\frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in \mathcal{D};\ 0 \le \alpha < p;\ p \in \mathbb{N}). \tag{1.3}$$

The classes $\Omega_p(\alpha)$, $\Lambda_p(\alpha)$ and various other subclasses of Σ_p have been studied rather extensively by Aouf *et.al.* [1] – [3], Joshi and Srivastava [4], Kulkarni *et. al.* [5], Mogra [6], Owa *et. al.* [7], Srivastava and Owa [8], Uralegaddi and Somantha [9], and Yang [10].

In the next section we derive sufficient conditions for $f(z)$ to be in the classes $\Omega_p(\alpha)$ and $\Lambda_p(\alpha)$, which are obtained by using coefficient inequalities.

2. Coefficient Inequalities

Theorem 2.1. *Let* $\sigma_n(p,k,\alpha) = (p+n+k-1) + |p+n+2\alpha-k-1|$. *If* $f(z) \in \Sigma_p$ *satisfies*

$$\sum_{n=1}^{\infty} \sigma_n(p,k,\alpha)\,|a_{p+n-1}| < 2(p-\alpha) \tag{2.1}$$

for some α $(0 \le \alpha < p)$ *and some* k $(k \ge p)$, *then* $f(z) \in \Omega_p(\alpha)$.

Proof. Suppose that (2.1) holds true for α $(0 \le \alpha < p)$ and k $(k \ge p)$. For $f(z) \in \Sigma_p$, it suffices to show that

$$\left|\frac{\frac{zf'(z)}{f(z)}+k}{\frac{zf'(z)}{f(z)}+(2\alpha-k)}\right| < 1 \qquad (z \in \mathcal{D}).$$

We note that

$$\begin{aligned}\left|\frac{\frac{zf'(z)}{f(z)}+k}{\frac{zf'(z)}{f(z)}+(2\alpha-k)}\right| &= \left|\frac{k-p+\sum_{n=1}^{\infty}(p+n+k-1)a_{p+n-1}z^{2p+n-1}}{2\alpha-k-p+\sum_{n=1}^{\infty}(p+n+2\alpha-k-1)a_{p+n-1}z^{2p+n-1}}\right| \\ &\le \frac{k-p+\sum_{n=1}^{\infty}(p+n+k-1)\,|a_{p+n-1}|\,|z|^{2p+n-1}}{p+k-2\alpha-\sum_{n=1}^{\infty}|p+n+2\alpha-k-1|\,|a_{p+n-1}|\,|z|^{2p+n-1}} \\ &< \frac{k-p+\sum_{n=1}^{\infty}(p+n+k-1)\,|a_{p+n-1}|}{p+k-2\alpha-\sum_{n=1}^{\infty}|p+n+2\alpha-k-1|\,|a_{p+n-1}|}.\end{aligned}$$

The last expression is bounded above by 1 if

$$k-p+\sum_{n=1}^{\infty}(p+n+k-1)\,|a_{p+n-1}| < p+k-2\alpha-\sum_{n=1}^{\infty}|p+n+2\alpha-k-1|\,|a_{p+n-1}|$$

which is equivalent to our condition (2.1) of the theorem. □

Example 2.1. The function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{4(p-\alpha)}{n(n+1)\sigma_n(p,k,\alpha)} z^{p+n-1} \qquad (p \in \mathbb{N}) \tag{2.2}$$

belongs to the class $\Omega_p(\alpha)$.

Since $f(z) \in \Omega_p(\alpha)$ if and only if $zf'(z) \in \Lambda_p(\alpha)$, we can prove:

Theorem 2.2. *If* $f(z) \in \Sigma_p$ *satisfies*

$$\sum_{n=1}^{\infty}(p+n-1)\sigma_n(p,k,\alpha)\,|a_{p+n-1}| < 2(p-\alpha) \tag{2.3}$$

for some $\alpha(0 \le \alpha < p)$ *and some* $k(k \ge p)$, *then* $f(z) \in \Lambda_p(\alpha)$.

Example 2.2. The function $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{4(p-\alpha)}{n(n+1)(p+n-1)\sigma_n(p,k,\alpha)} z^{p+n-1} \tag{2.4}$$

belongs to the class $\Lambda_p(\alpha)$.

In view of Theorem 2.1 and Theorem 2.2, we now define the subclasses:

$$\Omega_p^*(\alpha) \subset \Omega_p(\alpha) \text{ and } \Lambda_p^*(\alpha) \subset \Lambda_p(\alpha),$$

which consist of functions $f(z) \in \Sigma_p$ satisfying the conditions (2.1) and (2.3), respectively.

Letting $p = 1$, $1 \leq k \leq n + 2\alpha$, where $0 \leq \alpha < 1$ in Theorem 2.1 and Theorem 2.2, we have the following corollaries:

Corollary 2.3. *If $f(z) \in \Sigma_1$ satisfies*

$$\sum_{n=1}^{\infty} (n+\alpha) |a_n| < 1 - \alpha$$

then $f(z) \in \Omega_1(\alpha) = \Sigma^*(\alpha)$ the class of meromorphically starlike functions of order α in $\mathcal{D}$.

Corollary 2.4. *If $f(z) \in \Sigma_1$ satisfies*

$$\sum_{n=1}^{\infty} n(n+\alpha) |a_n| < 1 - \alpha$$

then $f(z) \in \Lambda_1(\alpha) = \Sigma_{\mathcal{K}}^*(\alpha)$ the class of meromorphically convex functions of order α in $\mathcal{D}$.

3. DISTORTION THEOREMS

A distortion property for functions in the class $\Omega_p^*(\alpha)$ is contained in

Theorem 3.1. *If the function $f(z)$ defined by (1.1) is in the class $\Omega_p^*(\alpha)$, then for $0 < |z| = r < 1$, we have*

$$\frac{1}{r^p} - \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} r^p \leq |f(z)| \leq \frac{1}{r^p} + \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} r^p, \tag{3.1}$$

and

$$\frac{p}{r^{p+1}} - \frac{2p(p-\alpha)}{p+k+|p+2\alpha-k|} r^{p-1} \leq |f'(z)| \leq \frac{p}{r^{p+1}} + \frac{2p(p-\alpha)}{p+k+|p+2\alpha-k|} r^{p-1}. \tag{3.2}$$

The bounds in (3.1) and (3.2) are attained for the functions $f(z)$ given by

$$f(z) = \frac{1}{z^p} + \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} z^p \qquad (p \in \mathbb{N};\ z \in \mathcal{D}). \tag{3.3}$$

Proof. Since $f \in \Omega_p^*(\alpha)$, from the inequality (2.1), we have

$$\sum_{n=1}^{\infty} |a_{p+n-1}| \leq \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|}. \tag{3.4}$$

Thus, for $0 < |z| = r < 1,$ and making use of (3.4) we have

$$|f(z)| \leq \left|\frac{1}{z^p}\right| + \sum_{n=1}^{\infty} |a_{p+n-1}|\,|z|^{p+n-1} \tag{3.5}$$
$$\leq \frac{1}{r^p} + r^p \sum_{n=1}^{\infty} |a_{p+n-1}|$$
$$\leq \frac{1}{r^p} + \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} r^p$$

and

$$|f(z)| \geq \left|\frac{1}{z^p}\right| - \sum_{n=1}^{\infty} |a_{p+n-1}|\,|z|^{p+n-1} \tag{3.6}$$
$$\geq \frac{1}{r^p} - r^p \sum_{n=1}^{\infty} |a_{p+n-1}|$$
$$\geq \frac{1}{r^p} - \frac{2(p-\alpha)}{p+k+|p+2\alpha-k|} r^p.$$

We also observe that

$$\frac{p+k+|p+2\alpha-k|}{p} \sum_{n=1}^{\infty} (p+n-1)\,|a_{p+n-1}| \leq 2(p-\alpha) \tag{3.7}$$

which readily yields the following distortion inequalities:

$$|f'(z)| \leq \frac{p}{|z|^{p+1}} + \sum_{n=1}^{\infty} (p+n-1)\,|a_{p+n-1}|\,|z|^{p+n-2} \tag{3.8}$$
$$\leq \frac{p}{r^{p+1}} + r^{p-1} \sum_{n=1}^{\infty} (p+n-1)\,|a_{p+n-1}|$$
$$\leq \frac{p}{r^{p+1}} + \frac{2p(p-\alpha)}{p+k+|p+2\alpha-k|} r^{p-1}$$

and

$$|f'(z)| \geq \frac{p}{|z|^{p+1}} - \sum_{n=1}^{\infty} (p+n-1)\,|a_{p+n-1}|\,|z|^{p+n-2} \tag{3.9}$$
$$\geq \frac{p}{r^{p+1}} - r^{p-1} \sum_{n=1}^{\infty} (p+n-1)\,|a_{p+n-1}|$$
$$\geq \frac{p}{r^{p+1}} - \frac{2p(p-\alpha)}{p+k+|p+2\alpha-k|} r^{p-1}.$$

This completes the proof of Theorem 3.1. □

Similarly, for function $f(z) \in \Lambda_p^*(\alpha)$, and making use of (2.3), we can prove

Theorem 3.2. *If the function $f(z)$ defined by (1.1) is in the class $\Lambda_p^*(\alpha)$, then for $0<|z|=r<1$, we have*

$$\begin{aligned}\frac{1}{r^p}-\frac{2(p-\alpha)}{p[p+k+|p+2\alpha-k|]}r^p &\leq |f(z)| \\ &\leq \frac{1}{r^p}+\frac{2(p-\alpha)}{p[p+k+|p+2\alpha-k|]}r^p,\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\frac{p}{r^{p+1}}-\frac{2(p-\alpha)}{p+k+|p+2\alpha-k|}r^{p-1} &\leq |f'(z)| \\ &\leq \frac{p}{r^{p+1}}+\frac{2(p-\alpha)}{p+k+|p+2\alpha-k|}r^{p-1}.\end{aligned} \tag{3.11}$$

The bounds in (3.10) and (3.11) are attained for the functions $f(z)$ given by

$$g(z)=\frac{1}{z^p}+\frac{2(p-\alpha)}{p[p+k-1+|p+2\alpha-k|]}z^p \qquad (p\in\mathbb{N};\ z\in\mathcal{D}). \tag{3.12}$$

4. RADII OF STARLIKENESS AND CONVEXITY

The radii of starlikeness and convexity for the classes $\Omega_p^*(\alpha)$ is given by

Theorem 4.1. *If the function $f(z)$ be defined by (1.1) is in the class $\Omega_p^*(\alpha)$, then $f(z)$ is meromorphically p-valently starlike of order $\delta(0\leq\delta<p)$ in $|z|<r_1$, where*

$$r_1=\inf_{n\geq 1}\left\{\frac{(p-\delta)\sigma_n(p,k,\alpha)}{2(3p+n+1-\delta)(p-\alpha)}\right\}^{\frac{1}{2p+n-1}} \qquad (p\in\mathbb{N}). \tag{4.1}$$

Furthermore, $f(z)$ is meromorphically p-valently convex of order $\delta(0\leq\delta<p)$ in $|z|<r_2$, where

$$r_2=\inf_{n\geq 1}\left\{\frac{p(p-\delta)\sigma_n(p,k,\alpha)}{2[(p+n-1)[3p+n-1-\delta](p-\alpha)}\right\}^{\frac{1}{2p+n-1}} \qquad (p\in\mathbb{N}). \tag{4.2}$$

The results (4.1) and (4.2) are sharp for the function $f(z)$ given by

$$f(z)=\frac{1}{z^p}+\frac{2(p-\alpha)}{\sigma_n(p,k,\alpha)}z^{p+n-1} \qquad (p\in\mathbb{N};\ z\in\mathcal{D}). \tag{4.3}$$

Proof. It suffices to prove that

$$\left|\frac{zf'(z)}{f(z)}+p\right|\leq p-\delta, \tag{4.4}$$

for $|z|\leq r_1$. We have

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}+p\right| &= \left|\frac{\sum_{n=1}^{\infty}(2p+n-1)a_{p+n-1}z^{p+n-1}}{\frac{1}{z^p}+\sum_{n=1}^{\infty}a_{p+n-1}z^{p+n-1}}\right| \\ &\leq \frac{\sum_{n=1}^{\infty}(2p+n-1)\left|a_{p+n-1}\right|\left|z\right|^{2p+n-1}}{1-\sum_{n=1}^{\infty}\left|a_{p+n-1}\right|\left|z\right|^{2p+n-1}}.\end{aligned} \tag{4.5}$$

Hence (4.5) holds true if

$$\sum_{n=1}^{\infty}(2p+n-1)\left|a_{p+n-1}\right|\left|z\right|^{2p+n-1}\leq(p-\delta)\left(1-\sum_{n=1}^{\infty}\left|a_{p+n-1}\right|\left|z\right|^{2p+n-1}\right), \tag{4.6}$$

or

$$\sum_{n=1}^{\infty} \frac{3p+n-1-\delta}{(p-\delta)} |a_{p+n-1}| \, |z|^{2p+n-1} \leq 1, \tag{4.7}$$

with the aid of (2.1), (4.7) is true if

$$\frac{3p+n-1-\delta}{(p-\delta)} |z|^{2p+n-1} \leq \frac{\sigma_n(p,k,\alpha)}{2(p-\alpha)} \qquad (n \geq 1). \tag{4.8}$$

Solving (4.8) for $|z|$, we obtain

$$|z| < \left\{ \frac{(p-\delta)\sigma_n(p,k,\alpha)}{2(3p+n+1-\delta)(p-\alpha)} \right\}^{\frac{1}{2p+n-1}} \qquad (n \geq 1). \tag{4.9}$$

In precisely the same manner, we can find the radius of convexity asserted by (4.2), by requiring that

$$\left| \frac{zf''(z)}{f'(z)} + p + 1 \right| \leq p - \delta, \tag{4.10}$$

in view of (2.1). This completes the proof of Theorem 4.1. □

Similarly, we can get the radii of starlikeness and convexity for functions in the class $\Lambda_p^*(\alpha)$.

Theorem 4.2. *If the function $f(z)$ be defined by (1.1) is in the class $\Lambda_p^*(\alpha)$, then $f(z)$ is meromorphically p-valently starlike of order $\delta(0 \leq \delta < p)$ in $|z| < r_3$, where*

$$r_3 = \inf_{n \geq 1} \left\{ \frac{(p-\delta)(p+n-1)\sigma_n(p,k,\alpha)}{2(3p+n+1-\delta)(p-\alpha)} \right\}^{\frac{1}{2p+n-1}} \qquad (p \in \mathbb{N}). \tag{4.11}$$

Furthermore, $f(z)$ is meromorphically p-valently convex of order $\delta(0 \leq \delta < p)$ in $|z| < r_4$, where

$$r_4 = \inf_{n \geq 1} \left\{ \frac{p(p-\delta)(p+n-1)\sigma_n(p,k,\alpha)}{2[(p+n-1)[3p+n-1-\delta](p-\alpha)} \right\}^{\frac{1}{2p+n-1}} \qquad (p \in \mathbb{N}). \tag{4.12}$$

The results (4.11) and (4.12) are sharp for the function $g(z)$ given by

$$g(z) = \frac{1}{z^p} + \frac{2(p-\alpha)}{(p+n-1)\sigma_n(p,k,\alpha)} z^{p+n-1} \qquad (p \in \mathbb{N};\ z \in \mathcal{D}). \tag{4.13}$$

5. CLOSURE THEOREMS

Let the functions $f_j(z)$ be defined, for $j \in \{1, 2, \ldots, m\}$,by

$$f_j(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1,j} z^{p+n-1}, \qquad (z \in \mathcal{D}). \tag{5.1}$$

Now, we shall prove the following results for the closure of functions in the classes $\Omega_p^*(\alpha)$ and $\Lambda_p^*(\alpha)$.

Theorem 5.1. *Let the functions $f_j(z), j \in \{1, 2, \ldots, m\}$, defined by (5.1) be in the class $\Omega_p^*(\alpha)$. Then the function $h(z) \in \Omega_p^*(\alpha)$ where*

$$h(z) = \sum_{j=1}^{m} b_j \, f_j(z), \qquad b_j \geq 0 \quad \textit{and} \ \sum_{j=1}^{m} b_j = 1). \tag{5.2}$$

Proof. From (5.2), we can write $h(z)$ as

$$h(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} c_{p+n-1}\, z^{p+n-1}, \tag{5.3}$$

where

$$c_{p+n-1} = \sum_{j=1}^{m} b_j a_{p+n-1,j}, \qquad j \in \{1, 2, \dots, m\}. \tag{5.4}$$

Since $f_j(z) \in \Omega_p^*(\alpha)$, ($j \in \{1, 2, \dots, m\}$), from (2.1) , we have

$$\begin{aligned}\sum_{n=1}^{\infty} \left[\frac{\sigma_n(p,k,\alpha)}{2(p-\alpha)}\right] \left(\sum_{j=1}^{m} b_j\, |a_{p+n-1,j}|\right) \\ &= \sum_{j=1}^{m} b_j \left(\sum_{n=1}^{\infty} \frac{\sigma_n(p,k,\alpha)}{2(p-\alpha)}\, |a_{p+n-1,j}|\right) \\ &\le \sum_{j=1}^{m} b_j = 1,\end{aligned}$$

which shows that $h(z) \in \Omega_p^*(\alpha)$. This completes the proof of Theorem 5.1. □

Using the same technique as in the proof of Theorem 5.1, we have

Theorem 5.2. *Let the functions $f_j(z), j \in \{1, 2, \dots, m\}$, defined by (5.1) be in the class $\Lambda_p^*(\alpha)$. Then the function $h(z) \in \Lambda_p^*(\alpha)$, where $h(z)$ defined by (5.2).*

Theorem 5.3. *Let*

$$f_{p-1}(z) = \frac{1}{z^p} \qquad (z \in \mathcal{D}) \tag{5.5}$$

and

$$f_{p+n-1}(z) = \frac{1}{z^p} + \frac{2(p-\alpha)}{\sigma_n(p,k,\alpha)} z^{p+n-1}, \tag{5.6}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $z \in \mathcal{D}$. Then $f(z) \in \Omega_p^*(\alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z) \tag{5.7}$$

where $\lambda_{p+n-1} \ge 0$, $(n \in \mathbb{N}_0)$ and $\sum_{n=0}^{\infty} \lambda_{p+n-1} = 1$.

Proof. From (5.5), (5.6) and (5.7), it is easily seen that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \lambda_{p+n-1} f_{n+p-1}(z) \\ &= \frac{1}{z^p} + \frac{2(p-\alpha)}{\sigma_n(p,k,\alpha)} \lambda_{p+n-1} z^{p+n-1}. \end{aligned} \tag{5.8}$$

Since

$$\sum_{n=1}^{\infty} \frac{\sigma_n(p,k,\alpha)}{2(p-\alpha)} . \frac{2(p-\alpha)}{\sigma_n(p,k,\alpha)} \lambda_{p+n-1} = \sum_{n=1}^{\infty} \lambda_{p+n-1} = 1 - \lambda_{p-1} \le 1,$$

it follows from Theorem 2.1 that the function $f(z)$ given by (5.6) is in the class $\Omega_p^*(\alpha)$.

Conversely, let us suppose that $f(z) \in \Omega_p^*(\alpha)$. Since

$$|a_{p+n-1}| \leq \frac{2(p-\alpha)}{\sigma_n(p,k,\alpha)} \qquad (n \geq 1),$$

setting

$$\lambda_{p+n-1} = \frac{\sigma_n(p,k,\alpha)}{2(p-\alpha)} |a_{p+n-1}|, \quad (n \geq 1)$$

and

$$\lambda_{p-1} = 1 - \sum_{n=1}^{\infty} \lambda_{p+n-1},$$

it follows that

$$f(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1} f_{p+n-1}(z).$$

This completes the proof of the theorem. □

Similarly, we can prove the same result for the class $\Lambda_p^*(\alpha)$.

Theorem 5.4. *Let*

$$g_{p-1}(z) = \frac{1}{z^p} \qquad (z \in \mathcal{D}) \tag{5.9}$$

and

$$g_{p+n-1}(z) = \frac{1}{z^p} + \frac{2(p-\alpha)}{(p+n-1)\sigma_n(p,k,\alpha)} z^{p+n-1} \tag{5.10}$$

where $n \in \mathbb{N}_0$ and $z \in \mathcal{D}$. Then $g(z) \in \Lambda_p^*(\alpha)$ if and only if it can be expressed in the form

$$g(z) = \sum_{n=0}^{\infty} \lambda_{p+n-1} g_{p+n-1}(z) \tag{5.11}$$

where $\lambda_{p+n-1} \geq 0$, $(n \in \mathbb{N}_0)$ and $\sum_{n=0}^{\infty} \lambda_{p+n-1} = 1$.

Next, we state a theorem which exhibits the fact that the classes $\Omega^*(\alpha)$ and $\Lambda_p^*(\alpha)$ are closed under convex linear combinations. The proof is fairly straightforward so we omit it.

Theorem 5.5. *Suppose that $f(z)$ and $g(z)$ are in the class $\Omega^*(\alpha)$ (or in $\Lambda_p^*(\alpha)$). Then the function $h(z)$ defined by*

$$h(z) = tf(z) + (1-t)g(z), \qquad (0 \leq t \leq 1) \tag{5.12}$$

is also in the class $\Omega^*(\alpha)$ (or in $\Lambda_p^*(\alpha)$).

6. Convolution Properties

For functions

$$f_j(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1,j} z^{p+n-1}, \qquad (j = 1,2) \tag{6.1}$$

belonging to the class Σ_p, we denote by $(f_1 * f_2)(z)$ the Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{p+n-1,1} a_{p+n-1,2} z^{p+n-1}. \tag{6.2}$$

Finally, we prove the following.

Theorem 6.1. *Let each of the functions* $f_j(z)$ $(\ j=1,2)$ *defined by (6.1) be in the class* $\Omega^*(\alpha)$. *Then* $(f_1 * f_2)(z) \in \Omega^*(\eta)$, *where*

$$\frac{1}{2}(k+1-p-n) \leq \eta = \frac{p\left([p+k+|p+2\alpha-k|]^2-4(p-\alpha)^2\right)}{4(p-\alpha)^2+[p+k+|p+2\alpha-k|]^2},\ (k \geq p;\ p,n \in \mathbb{N}). \tag{6.3}$$

The result is sharp.

Proof. For $f_j(z) \in \Omega^*(\alpha)$ $(j=1,2)$, we need to find the largest η such that

$$\sum_{n=1}^{\infty} \frac{\sigma_n(p,k,\eta)}{2(p-\eta)} |a_{p+n-1,1}|\,|a_{p+n-1,2}| \leq 1. \tag{6.4}$$

From (2.1), we have

$$\sum_{n=1}^{\infty} \frac{\sigma_n(p,k,\alpha)}{2(p-\alpha)} |a_{p+n-1,1}| \leq 1 \tag{6.5}$$

and

$$\sum_{n=1}^{\infty} \frac{\sigma_n(p,k,\alpha)}{2(p-\alpha)} |a_{p+n-1,2}| \leq 1. \tag{6.6}$$

Therefore, by the Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{\sigma_n(p,k,\alpha)}{2(p-\alpha)} \sqrt{|a_{p+n-1,1}|\,|a_{p+n-1,2}|} \leq 1. \tag{6.7}$$

Thus it is sufficient to show that

$$\frac{\sigma_n(p,k,\eta)}{2(p-\eta)} |a_{p+n-1,1}|\,|a_{p+n-1,2}| \leq \frac{\sigma_n(p,k,\alpha)}{2(p-\alpha)} \sqrt{|a_{p+n-1,1}|\,|a_{p+n-1,2}|}, \qquad (n \geq 1) \tag{6.8}$$

that is, that

$$\sqrt{|a_{p+n-1,1}|\,|a_{p+n-1,2}|} \leq \frac{(p-\eta)\sigma_n(p,k,\alpha)}{(p-\alpha)\sigma_n(p,k,\eta)}, \qquad (n \geq 1). \tag{6.9}$$

From (6.7), we have

$$\sqrt{|a_{p+n-1,1}|\,|a_{p+n-1,2}|} \leq \frac{2(p-\alpha)}{\sigma_n(p,k,\alpha)}.$$

Consequently, we need only to prove that

$$\frac{2(p-\alpha)}{\sigma_n(p,k,\alpha)} \leq \frac{(p-\eta)\sigma_n(p,k,\alpha)}{(p-\alpha)\sigma_n(p,k,\eta)}, \qquad (n \geq 1). \tag{6.10}$$

Let $\eta \geq \frac{1}{2}(k+1-p-n)$, where $k \geq p$ and $p,n \in \mathbb{N}$. It follows from (6.10) that

$$\eta \leq \frac{p[\sigma_n(p,k,\alpha)]^2 - 4(p-\alpha)^2(p+n-1)}{4(p-\alpha)^2+[\sigma_n(p,k,\alpha)]^2} = \Psi(n). \tag{6.11}$$

Since $\Psi(k)$ is an increasing function of n $(n \geq 1)$, letting $n=1$ in (6.11), we obtain

$$\eta \leq \Psi(1) = \frac{p\left([p+k+|p+2\alpha-k|]^2-4(p-\alpha)^2\right)}{4(p-\alpha)^2+[p+k+|p+2\alpha-k|]^2}, \tag{6.12}$$

which proves the main assertion of Theorem 6.1.

Finally, by taking the functions

$$f_j(z) = \frac{1}{z^p} + \frac{2(p-\alpha)}{\sigma_n(p,k,\alpha)} z^{p+n-1}, \qquad (j=1,2) \tag{6.13}$$

we can see the result is sharp. □

Similarly, and as the above proof, we can prove the following.

Theorem 6.2. *Let each of the functions $f_j(z)$ $(j = 1, 2)$ defined by (6.1) be in the class $\Lambda_p^*(\alpha)$. Then $(f_1 * f_2)(z) \in \Lambda_p^*(\xi)$, where*

$$\frac{1}{2}(k+1-p-n) \leq \xi = \frac{p\,(p[p+k+|p+2\alpha-k|]^2 - 4(p-\alpha)^2)}{4(p-\alpha)^2 + p[p+k+|p+2\alpha-k|]^2}, \ (k \geq p;\ p, n \in \mathbb{N}). \tag{6.14}$$

The result is sharp for the functions

$$f_j(z) = \frac{1}{z^p} + \frac{2(p-\alpha)}{(p+n-1)\sigma_n(p,k,\alpha)} z^{p+n-1}, \qquad (j = 1, 2). \tag{6.15}$$

REFERENCES

[1] M.K. AOUF, New criteria for multivalent meromorphic starlike functions of order alpha, *Proc. Japan. Acad. Ser. A. Math. Sci.*, **69** (1993), 66–70.

[2] M.K. AOUF AND H.M. HOSSEN, New criteria for meromorphic p-valent starlike functions, *Tsukuba J. Math.*, **17** (1993) 481–486.

[3] M.K. AOUF AND H.M. SRIVASTAVA, A new criteria for meromorphic p-valent convex functions of order alpha, *Math. Sci. Res. Hot-line*, **1**(8) (1997), 7–12.

[4] S.B. JOSHI AND H.M. SRIVASTAVA, A certain family of meromorphically multivalent functions, *Computers Math. Appl.*, **38** (1999), 201–211.

[5] S.R. KUKARNI, U.H. NAIK AND H.M. SRIVASTAVA, A certain class of meromorphically p-valent quasi-convex functions, *Pan Amer. Math. J.*, **8**(1) (1998), 57–64.

[6] M.L. MOGRA, Meromorphic multivalent functions with positive coefficients I and II, *Math. Japon.*, **35** (1990), 1–11 and 1089–1098.

[7] S. OWA, H.E. DARWISH AND M.K. AOUF, Meromorphic multivalent functions with positive and fixed second coefficients, *Math. Japon.*, **46** (1997), 231–236.

[8] H.M. SRIVASTAVA AND S. OWA (Eds.), Current Topics in Analytic Function Theory, *World Scientific, Singapore/New Jersey/London/Hong Kong*, (1992).

[9] B.A. URALEGADDI AND C. SOMANATHA, Certain classes of meromorphic multivalent functions, *Tamkang J. Math.*, **23** (1992), 223–231.

[10] D.G. YANG, On new subclasses of meromorphic p-valent functions, *J. Math. Res. Exposition*, **15** (1995) 7–13.