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SHARP INEQUALITIES BETWEEN CENTERED MOMENTS

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ABSTRACT. Inspired by a result of Chuprunov and Fazekas, we prove sharp inequalities between centered moments of the same order, but with respect to different probability measures.

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1. INTRODUCTION

The following inequality was proved by A. Chuprunov and I. Fazekas [3].

Consider the probability measure $\mathbb{P}$ and the conditional probability measure $\mathbb{P}^A$ with respect to the fixed event A. Let $\mathbb{E}^A$ denote the expectation with respect to $\mathbb{P}^A$. Then

$$\mathbb{E}^A|S - \mathbb{E}^A S|^p \le 2^{2p-1}\,\frac{\mathbb{E}|S - \mathbb{E}S|^p}{\mathbb{P}(A)}\,. \tag{1.1}$$

There are several inequalities involving centered moments known in the literature. Most of them are between different moments of the same random variable, like Lyapunov's classical result

$$\left(\mathbb{E}|S|^q\right)^{r-p} \le \left(\mathbb{E}|S|^p\right)^{r-q}\left(\mathbb{E}|S|^r\right)^{q-p}$$

for $0 < p < q < r$. A new inequality of the same taste for centered-like moments is presented in [6], and generalized in [1]. There exist moment inequalities in particular cases, where additional conditions, such as unimodality or boundedness, are imposed on the distributions, see e.g. [2] and [4], also the monograph [5].

In the Chuprunov–Fazekas inequality the order of the moment is the same on both sides. What differs is the underlying probability measure. In that case centering cannot be considered as a special case of the general (uncentered) problem; it needs further attention.

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In this note we extend, generalize and sharpen inequality (1.1). We start from the observation that $\mathbb{P}^A \ll \mathbb{P}$, and $\frac{d\mathbb{P}^A}{d\mathbb{P}} = \frac{I_A}{\mathbb{P}(A)}$, where I_A stands for the indicator of event A. First we extend inequality (1.1), with a rather simple proof.

Theorem 1.1. *Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be probability measures defined on the same measurable space. Let $\mathbb{E}_1$ and $\mathbb{E}_2$, resp., denote the corresponding expectations. Assume $\mathbb{P}_1 \ll \mathbb{P}_2$, and $\sup \frac{d\mathbb{P}_1}{d\mathbb{P}_2} = C < \infty$. Let $p \geq 1$ and suppose that $\mathbb{E}_1|S|^p < \infty$, then*

$$\mathbb{E}_1|S - \mathbb{E}_1 S|^p \leq C\, 2^p\, \mathbb{E}_2|S - \mathbb{E}_2 S|^p. \tag{1.2}$$

Proof. Let $S' = S - \mathbb{E}_2 S$, then

$$\begin{aligned} \mathbb{E}_1|S - \mathbb{E}_1 S|^p &= \mathbb{E}_1|S' - \mathbb{E}_1 S'|^p \\ &\leq 2^{p-1}\left(\mathbb{E}_1|S'|^p + |\mathbb{E}_1 S'|^p\right) \\ &\leq 2^p\, \mathbb{E}_1|S'|^p \leq C\, 2^p\, \mathbb{E}_2|S'|^p = C\, 2^p\, \mathbb{E}_2|S - \mathbb{E}_2 S|^p. \end{aligned}$$

□

In particular, when $\mathbb{P}_2 = \mathbb{P}$ and $\mathbb{P}_1 = \mathbb{P}^A$, we obtain the Chuprunov–Fazekas inequality with 2^p in place of 2^{2p-1} on the right-hand side.

In Section 2 we derive sharp inequalities between centered p th moments of the same random variable with respect to different probability measures. In Section 3 we return to the original problem of Chuprunov and Fazekas, comparing conditional and unconditional moments.

2. COMPARISON OF CENTERED MOMENTS WITH RESPECT TO DIFFERENT PROBABILITY MEASURES

In this section we investigate to what extent the constant 2^p can be decreased in inequality (1.2). From the proof of Theorem 1.1 it is clear that we are looking for the minimal positive number C_p with which the inequality $\mathbb{E}|S - \mathbb{E}S|^p \leq C_p\, \mathbb{E}|S|^p$ holds for every random variable S having finite p th moment. That is,

$$C_p = \max_S \frac{\mathbb{E}|S - \mathbb{E}S|^p}{\mathbb{E}|S|^p}. \tag{2.1}$$

First we determine C_p, then we set bounds for it, and analyze its asymptotic behaviour as $p \to \infty$.

Theorem 2.1. $C_1 = 2$, *and for* $p > 1$

$$C_p = \max_{0<\alpha<1} \left(\alpha^{p-1} + (1-\alpha)^{p-1}\right)\left(\alpha^{\frac{1}{p-1}} + (1-\alpha)^{\frac{1}{p-1}}\right)^{p-1}. \tag{2.2}$$

Proof. For the sake of convenience introduce $q = p - 1$.

Suppose first that $p > 1$, that is, $q > 0$.

Let the distribution of S be the following: $\mathbb{P}(S = -x) = 1 - \alpha$, $\mathbb{P}(S = 1 - x) = \alpha$, where

$$x = \frac{\alpha^{1/q}}{\alpha^{1/q} + (1-\alpha)^{1/q}}\,. \tag{2.3}$$

It follows that

$$\mathbb{E}S = \frac{\alpha(1-\alpha)^{1/q} - (1-\alpha)\alpha^{1/q}}{\alpha^{1/q} + (1-\alpha)^{1/q}}\,, \quad \mathbb{E}|S|^p = \frac{\alpha(1-\alpha)}{\left(\alpha^{1/q} + (1-\alpha)^{1/q}\right)^q}\,. \tag{2.4}$$

In addition, $\mathbb{P}(S - \mathbb{E}S = -\alpha) = 1 - \alpha$ and $\mathbb{P}(S - \mathbb{E}S = 1 - \alpha) = \alpha$, hence

$$\mathbb{E}|S - \mathbb{E}S|^p = \alpha(1-\alpha)\left(\alpha^q + (1-\alpha)^q\right). \tag{2.5}$$

By (2.4) and (2.5) it follows that C_p is not less than the maximum on the right-hand side of (2.2).

On the other hand, if $\mathbb{E}S = c$ and $Y = S - c$, then

$$C_p = \max\left\{\frac{\mathbb{E}|Y|^p}{\mathbb{E}|Y+c|^p} : c \in \mathbb{R},\ \mathbb{E}Y = 0,\ \mathbb{P}(Y+c=0) < 1\right\}. \tag{2.6}$$

Every zero mean probability distribution is a mixture of distributions concentrated on two points and having zero mean. Thus the maximum does not change if we only consider random variables with not more than two possible values. We can also assume that these two values are $-\alpha$ and $1-\alpha$, where $0 < \alpha < 1$; in that case $\mathbb{P}(Y = -\alpha) = 1 - \alpha$ and $\mathbb{P}(Y = 1-\alpha) = \alpha$.

We now fix α and find a c that maximizes the fraction in (2.6). To do this one has to minimize the expression

$$\mathbb{E}|Y+c|^p = (1-\alpha)\,|-\alpha+c|^p + \alpha\,|1-\alpha+c|^p$$

in c. This is increasing for $c \geq \alpha$, and decreasing for $c \leq \alpha - 1$. We can, therefore, suppose that $\alpha - 1 \leq c \leq \alpha$, so

$$\mathbb{E}|Y+c|^p = (1-\alpha)\,(\alpha-c)^p + \alpha\,(c+1-\alpha)^p. \tag{2.7}$$

Differentiating this with respect to c we get $p\big(-(1-\alpha)\,(\alpha-c)^q + \alpha\,(c+1-\alpha)^q\big)$, from which

$$c = \alpha - \frac{\alpha^{1/q}}{\alpha^{1/q} + (1-\alpha)^{1/q}} = \alpha - x,$$

with x defined in (2.3). Thus $\mathbb{P}(Y+c=-x) = 1-\alpha,\ \mathbb{P}(Y+c=1-x) = \alpha$, and, following the calculations that led to (2.4), we arrive at

$$\frac{\mathbb{E}|Y|^p}{\mathbb{E}|Y+c|^p} = \Big(\alpha^q + (1-\alpha)^q\Big)\Big(\alpha^{1/q} + (1-\alpha)^{1/q}\Big)^q.$$

This proves (2.2).

If we next let $p = 1$ then, on the one hand, $\mathbb{E}|S - \mathbb{E}S| \leq \mathbb{E}|S| + |\mathbb{E}S| \leq 2\mathbb{E}|S|$, thus $C_1 \leq 2$. On the other hand, if $\mathbb{P}(S=1) = \alpha$, $\mathbb{P}(S=0) = 1-\alpha$, then $\mathbb{E}|S| = \alpha$, $\mathbb{E}|S - \mathbb{E}S| = 2\alpha(1-\alpha)$, implying that $C_1 \geq 2(1-\alpha)$ for arbitrary $0 < \alpha < 1$. □

Theorem 2.2.

$$C_{3/2} = \sqrt{\frac{17+7\sqrt{7}}{27}} = 1{,}1469\ldots, \quad C_2 = 1, \quad C_3 = \frac{17+7\sqrt{7}}{27} = 1{,}3155\ldots, \tag{2.8}$$

$$1 \leq C_p \leq 2^{|p-2|}, \tag{2.9}$$

and if $\frac{1}{p} + \frac{1}{r} = 1$, *then*

$$C_r = C_p^{r-1}. \tag{2.10}$$

Proof. The value of C_2 follows obviously from Theorem 2.1, while getting C_3 requires more extensive but straightforward calculations. From this the value of $C_{3/2}$ follows by (2.10), since 3 and $3/2$ are conjugate numbers. Equation (2.10) itself is an obvious corollary to (2.2), because $(p-1)(r-1) = 1$.

For an arbitrary positive exponent s one can write $\alpha^s + (1-\alpha)^s \leq 2^{(1-s)^+}$, therefore we have

$$C_p \leq 2^{(2-p)^+}\ 2^{(p-2)^+} = 2^{|p-2|}.$$

The lower bound in (2.9) is obvious. □

Since $2 \leq p < \infty \Leftrightarrow 1 < r \leq 2$, by (2.10) it suffices to focus on the case $p \geq 2$.

Theorem 2.3. *If $p \geq 2$ then $C_p \geq \frac{2^{p-1}}{\sqrt{2ep}}$, and $C_p \sim \frac{2^{p-1}}{\sqrt{2ep}}$, as $p \to \infty$.*

Proof. Introduce the notation $f(\alpha) = \big(\alpha^q + (1-\alpha)^q\big)\big(\alpha^{1/q} + (1-\alpha)^{1/q}\big)^q$. First we show that

$$C_p \geq f\left(\frac{1}{2p}\right) \geq \frac{2^{p-1}}{\sqrt{2ep}}. \tag{2.11}$$

Indeed, since

$$\left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{2q}} + \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{2q}} \geq 2,$$

therefore $\alpha^{1/q} + (1-\alpha)^{1/q} \geq 2\big(\alpha(1-\alpha)\big)^{1/2q}$, from which

$$f(\alpha) \geq 2^q (1-\alpha)^{q+\frac{1}{2}}\sqrt{\alpha}.$$

By substituting $\alpha = \frac{1}{2p}$, and using the fact that

$$\left(1 - \frac{1}{2p}\right)^{q+\frac{1}{2}} = \left(\frac{2p-1}{2p}\right)^{\frac{2p-1}{2}} \geq e^{-1/2},$$

we immediately obtain (2.11), as needed.

We now turn our attention to the upper estimation. By symmetry we may suppose that $0 < \alpha \leq 1/2$. We will show that $\alpha \sim \frac{1}{2q}$ holds for the argument of the maximum.

First, let $\alpha \leq (cq)^{-1}$, where c is sufficiently large (specified later), we then have

$$\begin{aligned} f(\alpha) &\leq \big(1 + \alpha^{1/q}\big)^q \\ &= 2^q\Big(1 - \frac{1}{2}\big(1 - \alpha^{1/q}\big)\Big)^q \\ &< 2^q \exp\Big(-\frac{q}{2}\big(1 - \alpha^{1/q}\big)\Big) \\ &\leq 2^q \exp\Big(-\frac{q}{2}\Big(1 - \Big(\frac{1}{cq}\Big)^{1/q}\Big)\Big). \end{aligned}$$

Here the Taylor expansion gives

$$\alpha^{1/q} = \exp\Big(\frac{1}{q}\log\alpha\Big) = 1 + \frac{1}{q}\log\alpha + \frac{\theta}{2}\Big(\frac{\log\alpha}{q}\Big)^2, \tag{2.12}$$

where $0 \leq \theta \leq 1$. Thus, if $q \geq c$,

$$f(\alpha) \leq 2^q \exp\left(-\frac{1}{2}\log(cq) + \frac{\log^2(cq)}{4q}\right) \leq \frac{2^q}{\sqrt{cq}} \exp\left(\frac{\log^2 c}{c}\right);$$

this is still less than the lower estimation we derived for the maximum in (2.11); for instance, when $c = 16$.

Secondly, let $\alpha > \frac{1}{q}\log q$, then, applying the trivial estimation 2^q to the second term of $f(\alpha)$ we get

$$f(\alpha) \leq 2^q\Big(2^{-q} + \Big(1 - \frac{1}{q}\log q\Big)^q\Big) \leq 1 + \frac{2^q}{q};$$

which is still less than the lower bound if $p \geq 8$.

Finally, let $\frac{1}{cq} < \alpha < \frac{1}{q}\log q$, then, by (2.12),

$$\alpha^{1/q} = 1 + \frac{\log\alpha}{q} + O\Big(\frac{(\log q)^2}{q^2}\Big),$$

uniformly in α. Moreover,

$$1-(1-\alpha)^{1/q} \le \frac{\alpha}{q(1-\alpha)} = O\left(\frac{\log q}{q^2}\right),$$

hence

$$\begin{aligned}\frac{\alpha^{1/q}+(1-\alpha)^{1/q}}{2} &= 1+\frac{\log\alpha}{2q}+O\left(\frac{(\log q)^2}{q^2}\right)\\ &= \exp\left(\frac{\log\alpha}{2q}+O\left(\frac{(\log q)^2}{q^2}\right)\right).\end{aligned}$$

Consequently,

$$\left(\alpha^{1/q}+(1-\alpha)^{1/q}\right)^q = 2^q\sqrt{\alpha}\left(1+O\left(\frac{(\log q)^2}{q}\right)\right), \tag{2.13}$$

uniformly in α.

The first term of $f(\alpha)$ can be estimated in the following way. The function $e^\alpha(1-\alpha)$ is decreasing, hence in the considered domain we have $1 \ge e^\alpha(1-\alpha) = 1+O(q^{-2})$. Thus $1-\alpha = e^{-\alpha}\left(1+O(q^{-2})\right)$, therefore $(1-\alpha)^q = e^{-q\alpha}\left(1+O(q^{-1})\right)$. In the end we obtain that

$$\alpha^q+(1-\alpha)^q = e^{-q\alpha}\left(1+O(q^{-1})\right). \tag{2.14}$$

Considering both (2.13) and (2.14) we conclude that, uniformly in the domain under consideration,

$$f(\alpha) = 2^q\sqrt{\alpha}\,e^{-q\alpha}\left(1+O(q^{-1})\right).$$

Let $\arg\max f(\alpha) = \frac{x_q}{q}$. For every q large enough we have $1/c \le x_q \le \log q$, hence

$$\max f(\alpha) = \frac{2^q}{\sqrt{q}}\cdot x_q^{1/2}e^{-x_q}\left(1+O(q^{-1})\right).$$

By virtue of all these it is clear that $x_q \to \arg\max x^{1/2}e^{-x} = 1/2$, and $\max f(\alpha) \sim \frac{2^q}{\sqrt{2eq}}$, as stated. □

By applying (2.10) to Theorem 2.3 we can derive similar results for the case $1 < p \le 2$.

Corollary 2.4. *Let* $0 < \varepsilon \le 1$. *Then* $C_{1+\varepsilon} \ge 2\left(\frac{\varepsilon}{2e(1+\varepsilon)}\right)^{\varepsilon/2}$, *and*

$$C_{1+\varepsilon} = 2-\varepsilon\log(1/\varepsilon)-\varepsilon(1+\log 2)+o(\varepsilon),$$

as $\varepsilon \to 0$.

3. COMPARISON OF CONDITIONAL AND UNCONDITIONAL MOMENTS

Returning to the special case of Chuprunov and Fazekas, we fix $\mathbb{P}(A) > 0$, and look for the minimal positive constant $K = K(p, \mathbb{P}(A))$, for which the inequality

$$\mathbb{E}^A|S-\mathbb{E}^AS|^p \le \frac{K}{\mathbb{P}(A)}\,\mathbb{E}|S-\mathbb{E}S|^p \tag{3.1}$$

holds for every random variable S having finite p th moment. Then it follows that

$$1 \le K \le C_p. \tag{3.2}$$

The upper bound is obvious, while the lower bound can be seen from the example where $S = 0$ on the complement of A, and $\mathbb{E}S = 0$.

How much can this be improved, however?

Theorem 3.1. *Representing* $\mathbb{P}(A)/(1-\mathbb{P}(A))$ *by* R*,*

$$K = \sup_{x,y>0} \frac{yx^p + xy^p}{y(x+1)^p + x|y-1|^p + R^{p-1}(x+y)}\,. \tag{3.3}$$

If $p=1$ *or* $p=2$*, then* $K=1$*.*

Suppose that $p<2$*, then*

$$K \le \frac{2}{1+\mathbb{P}(A)^{p-1}}\,. \tag{3.4}$$

Suppose that $p>2$*, then*

$$K \le \begin{cases} \dfrac{\big(1-\mathbb{P}(A)\big)^{p-1} C_p}{\big(1-\mathbb{P}(A)\big)^{p-1} + \mathbb{P}(A)^{p-1}\big(C_p^{1/p}-1\big)^p}\,, & \textit{if } \mathbb{P}(A) \le C_p^{-1/p}\,, \\ \dfrac{1}{\mathbb{P}(A)^{p-1}} \le C_p^{1-\frac{1}{p}}\,, & \textit{if } \mathbb{P}(A) > C_p^{-1/p}\,. \end{cases} \tag{3.5}$$

Remark 1. For arbitrary $\mathbb{P}(A)$ we have

$$\frac{C_p}{\mathbb{P}(A)} \cdot \frac{\big(1-\mathbb{P}(A)\big)^{p-1}}{\big(1-\mathbb{P}(A)\big)^{p-1} + \mathbb{P}(A)^{p-1}\big(C_p^{1/p}-1\big)^p} \le \frac{1}{\mathbb{P}(A)^p}, \tag{3.6}$$

and equality holds if and only if $\mathbb{P}(A) = C_p^{-1/p}$.

In order to show this, let $\big(C_p^{1/p}-1\big)$ be denoted by x, then the left-hand side of (3.6) can be rewritten in the form

$$\frac{(x+1)^p}{1+R^{p-1}x^p}.$$

By differentiating, one can easily verify that

$$\max_{x\ge 0} \frac{(x+1)^p}{1+R^{p-1}x^p} = \left(\frac{R+1}{R}\right)^{p-1} = \frac{1}{\mathbb{P}(A)^{p-1}}\,,$$

and the maximum is attained at $x = 1/R$.

Remark 2. When C_p is not explicitly known, we can substitute C_p by its upper estimate 2^{p-2} everywhere in (3.5), including the conditions of the cases. This is justified, because

$$\frac{(x+1)^p}{1+R^{p-1}x^p}$$

is an increasing function of x for $x \le 1/R$, that is, whenever $\mathbb{P}(A) \le C_p^{-1/p}$.

Proof of Theorem 3.1. From (3.1) it follows that

$$K = \sup_S \frac{\mathbb{P}(A)\,\mathbb{E}^A\left|S-\mathbb{E}^A S\right|^p}{\mathbb{E}\,|S-\mathbb{E}S|^p}. \tag{3.7}$$

We may assume that $\mathbb{E}^A S = 0$. Let B denote the complement of event A. In the denominator of (3.7) $\mathbb{E}S = \mathbb{P}(B)\,\mathbb{E}^B S$, and

$$\begin{aligned} \mathbb{E}\,|S-\mathbb{E}S|^p &= \mathbb{P}(A)\,\mathbb{E}^A\left|S-\mathbb{P}(B)\,\mathbb{E}^B S\right|^p + \mathbb{P}(B)\,\mathbb{E}^B\left|S-\mathbb{P}(B)\,\mathbb{E}^B S\right|^p \\ &\ge \mathbb{P}(A)\,\mathbb{E}^A\left|S-\mathbb{P}(B)\,\mathbb{E}^B S\right|^p + \mathbb{P}(B)\,\left|\mathbb{E}^B S-\mathbb{P}(B)\,\mathbb{E}^B S\right|^p \\ &= \mathbb{P}(A)\,\mathbb{E}^A\left|S-\mathbb{P}(B)\,\mathbb{E}^B S\right|^p + \mathbb{P}(A)\,R^{p-1}\left|\mathbb{P}(B)\,\mathbb{E}^B S\right|^p. \end{aligned}$$

Equality holds, for example, if S is constant on the event B. At this point we remark that the conditional distributions of S given A, or B, resp., can be prescribed arbitrarily, provided that

$\mathbb{E}^A S = 0$, and $\mathbb{E}^A|S|^p < \infty$, $\mathbb{E}^B|S|^p < \infty$. In a sufficiently rich probability space one can construct a random variable S and an event A in such a way that $\mathbb{P}(A)$ and the conditional distributions of S given A and its complement B meet the specifications. If X and Y are arbitrary random variables and the event A is independent of them, then the conditional distribution of $S = I_A X + I_B Y$ given A, resp. B, is equal to the distribution of X, resp. Y. Hence we can suppose that S is constant on B and focus on the conditional distribution given A.

If $\mathbb{E}^B S = 0$, then $\mathbb{E}\,|S - \mathbb{E}S|^p = \mathbb{P}(A)\mathbb{E}^A|S|^p$. In what follows we assume $\mathbb{E}^B S \neq 0$. The right-hand side of (3.7) is homogeneous in S, thus we may also suppose that $\mathbb{P}(B)\,\mathbb{E}^B S = 1$. Consequently, we have to find

$$K = \sup\left\{ \frac{\mathbb{E}^A|S|^p}{\mathbb{E}^A|S-1|^p + R^{p-1}} : \mathbb{E}^A S = 0 \right\}. \tag{3.8}$$

From (2.1) it follows that

$$\sup\left\{ \frac{\mathbb{E}^A|S|^p}{\mathbb{E}^A|S-1|^p} : \mathbb{E}^A S = 0 \right\} = C_p. \tag{3.9}$$

As in the proof of Theorem 2.1, it suffices to deal with random variables with just two possible values. Let these be $-x$ and y, with positive x and y, then

$$\mathbb{P}^A(S = -x) = \frac{y}{x+y}\,, \quad \mathbb{P}^A(S = y) = \frac{x}{x+y}\,,$$

because of the vanishing conditional expectation. Thus we have

$$\frac{\mathbb{E}^A|S|^p}{\mathbb{E}^A|S-1|^p + R^{p-1}} = \frac{yx^p + xy^p}{y(x+1)^p + x|y-1|^p + R^{p-1}(x+y)}\,, \tag{3.10}$$

which, together with (3.8), imply (3.3).

Suppose $K > 1$. If either x or y tends to infinity, the right-hand side of (3.10) converges to 1, thus the supremum is attained at some x and y.

First we show that $1 \leq y$. Suppose, to the contrary, that $0 < y < 1$. Let $z = 2 - y$, then $z > y$, $|z-1| = |y-1|$, and

$$\begin{aligned}\frac{zx^p + xz^p}{z(x+1)^p + x|z-1|^p + R^{p-1}(x+z)} &= \frac{x^p + xz^{p-1}}{(x+1)^p + x|y-1|^p/z + R^{p-1}(1+x/z)} \\ &> \frac{x^p + xy^{p-1}}{(x+1)^p + x|y-1|^p/y + R^{p-1}(1+x/y)} \\ &= \frac{yx^p + xy^p}{y(x+1)^p + x|y-1|^p + R^{p-1}(x+y)}\,.\end{aligned}$$

At this point, the case of $p = 1$ follows immediately, since the right-hand side of (3.10) is always less than 1,

$$K = \sup_{x>0,\, y\geq 1} \frac{yx + xy}{y(x+1) + x(y-1) + (x+y)} = \sup_{x>0} \frac{x}{x+1} = 1.$$

The case of $p = 2$ is implied by (3.2), since $C_2 = 2$.

Next we show that $x \leq y$ or $x \geq y$, according to whether $p > 2$ or $p < 2$.

Indeed, since $yx^p + xy^p > y(x+1)^p + x(y-1)^p$ must hold, we have

$$\begin{aligned}0 &< yx^p + xy^p - y(x+1)^p + x(y-1)^p \\ &< -ypx^{p-1} + xpy^{p-1} = pxy\big(y^{p-2} - x^{p-2}\big).\end{aligned}$$

Let $p < 2$. Then

$$1 < K = \frac{xy^p + yx^p}{y(x+1)^p + x(y-1)^p + R^{p-1}(x+y)} \tag{3.11}$$
$$\leq \frac{xy^p + yx^p}{yx^p + x(y-1)^p + R^{p-1}x}$$
$$= \frac{y^p + yx^{p-1}}{yx^{p-1} + (y-1)^p + R^{p-1}}\,.$$

If x is increased, the same positive quantity is added to the numerator and the denominator of the fraction on the right-hand side. As a result, the value of the fraction decreases. Thus we can obtain an upper estimate by changing x to y, namely,

$$K \leq \frac{2y^p}{y^p + (y-1)^p + R^{p-1}}\,.$$

One can easily verify that the maximum of the right-hand side is attained at $y = R+1$. Thus

$$K \leq \frac{2(R+1)^p}{(R+1)^p + R^p + R^{p-1}} = \frac{2}{1 + \mathbb{P}(A)^{p-1}}\,.$$

Finally, let us turn to the case $p > 2$. Applying the trivial inequality

$$\frac{a+b}{c+d} \leq \max\left\{\frac{a}{c}\,,\ \frac{b}{d}\right\}, \quad a, b, c, d \geq 0,$$

to the right-hand side of (3.10) we get

$$K \leq \max\left\{\frac{x^p}{(x+1)^p + R^{p-1}}\,,\ \frac{y^p}{(y-1)^p + R^{p-1}}\right\} \tag{3.12}$$
$$= \frac{y^p}{(y-1)^p + R^{p-1}}\,,$$

which has to be greater than 1.

Another estimate can be obtained by applying the inequality

$$yx^p + xy^p \leq C_p\big(y(x+1)^p + x|y-1|^p\big),$$

which comes from (3.9), to the denominator on the right-hand side of (3.10). It follows that

$$K \leq \frac{C_p}{1 + C_p R^{p-1}\,\frac{x+y}{xy^p+yx^p}}\,.$$

The right-hand side is an increasing function of both x and y. Hence we can increase x to its upper bound y, obtaining

$$K \leq \frac{C_p\, y^p}{y^p + C_p\, R^{p-1}}\,. \tag{3.13}$$

From (3.12) and (3.13) it follows that

$$K \leq \max_{y\geq 1}\ \min\left\{\frac{y^p}{(y-1)^p + R^{p-1}}\,,\ \frac{C_p\, y^p}{y^p + C_p\, R^{p-1}}\right\}. \tag{3.14}$$

The second function on the right-hand side is increasing; the first one is increasing at the beginning, then decreasing. Its maximum is at $y = R+1$. At $y = 1$ the first function is greater than the second one, while the converse is true for every sufficiently large y. The two functions are equal at

$$y_0 = \frac{C_p^{1/p}}{C_p^{1/p} - 1},$$

thus for $y < y_0$ the first one, and for $y > y_0$ the second one is greater. Therefore, in (3.14) the maximum is equal to the maximum of the first function if $R + 1 \leq y_0$, while in the complementary case it is the common value at y_0. Elementary calculations lead to (3.5). □

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