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## WALLIS INEQUALITY WITH A PARAMETER

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## Abstract

We introduce a parameter  $z$  for the well-known Wallis' inequality, and improve results on Wallis' inequality are proposed. Recent results by other authors are also improved.

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*Key words:* Wallis' inequality;  $\Gamma$ -function; Taylor formula.

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# 1. Introduction

Wallis' inequality is a well-known and important inequality, it has wide applications in mathematics formulae, in particular, in combinatorics (see [1, 2]). It can be defined by the following expression,

$$(1.1) \quad \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{\pi(n+1/2)}} < P_n < \frac{1}{\sqrt{\pi(n+1/4)}} < \frac{1}{\sqrt{3n+1}} < \frac{1}{\sqrt{2n+1}} < \frac{1}{\sqrt{2n}},$$

where

$$P_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(1-\frac{1}{2})(2-\frac{1}{2}) \cdot (n-\frac{1}{2})}{n!}.$$

Improvements of the lower and upper bounds of  $P_n$  in (1.1) and some generalizations can be found in [2] – [5]. The main results are

$$(1.2) \quad \frac{2(2n)!!}{\pi(2n+1)!!} < P_n < \frac{2(2n-2)!!}{\pi(2n-1)!!}.$$

$$(1.3) \quad \sqrt{\frac{8(n+1)}{(4n+3)(2n+1)\pi}} < P_n < \sqrt{\frac{4n+1}{(2n)(2n+1)\pi}}.$$

$$(1.4) \quad \frac{m-1}{m\sqrt[m]{n}} < \frac{(m-1)(2m-1) \cdot (nm-1)}{m(2m) \cdot (nm)} < \frac{m-1}{\sqrt[m]{n(m+1)+m-1}}.$$



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$$(1.5) \quad \frac{m}{(m+1)\sqrt[m]{n}} < \frac{m(2m) \cdot (nm)}{(m+1)(2m+1) \cdot (nm+1)} < \frac{m-1}{\sqrt[m]{n(m+1)+m+4}}.$$

The largest lower bound  $\frac{1}{\sqrt{\pi(n+1/2)}}$  and the smallest upper bound  $\frac{1}{\sqrt{\pi(n+1/4)}}$  of  $P_n$  in (1.1) were presented by Kazarinoff in 1956 (see [1]). The following improvement of the bound in (1.1) can be found in [7]

$$(1.6) \quad \frac{1}{\sqrt{\pi n \left(1 + \frac{1}{4n-1/2}\right)}} < P_n < \frac{1}{\sqrt{\pi n \left(1 + \frac{1}{4n-1/3}\right)}}, \quad (n \geq 1).$$

In [8] a new method of proof was proposed for the largest lower bound about Wallis' inequality, and in [9] the largest lower bound was improved. The result was

$$(1.7) \quad \frac{1}{\sqrt{\pi \left(n + \frac{4}{\pi} - 1\right)}} < P_n < \frac{1}{\sqrt{\pi(n+1/4)}}, \quad (n \geq 1).$$

In this paper, we introduce the parameter  $z$ , and use the  $\Gamma(z)$  formula (Euler) below,

$$(1.8) \quad \Gamma(z) = \lim_{n \rightarrow \infty} \frac{(n-1)!n^z}{z(z+1) \cdot (n-1+z)} \quad (\text{see [6]}).$$

An improvement and generalization of Wallis' inequality is proposed.



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For  $0 < z < 1$ ,  $n > 1$  and  $n$  a natural number,

$$\begin{aligned} \frac{1}{\Gamma(1-z)n^z\left(1+\frac{1-z}{2(n-1)}\right)^z} &< \frac{(1-z)(2-z)\cdots(n-z)}{n!} \\ &< \frac{1}{\Gamma(1-z)n^z\left(1+\frac{1-z}{2n+1-z}\right)^z} \\ &< \frac{1}{\Gamma(1-z)n^z\left(1+\frac{1-z}{2n+1}\right)^z}. \end{aligned}$$

When  $0 < z < 1$  and  $n \geq 22$ ,

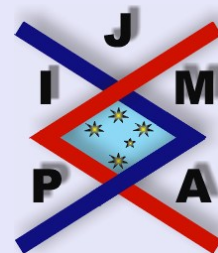
$$\begin{aligned} \frac{1}{\Gamma(1-z)n^z\left(1+\frac{1-z}{2(n-1)}\right)^z} &< \frac{(1-z)(2-z)\cdots(n-z)}{n!} \\ &< \frac{1}{\Gamma(1-z)n^z\left(1+\frac{1-z}{2n}\right)^z}. \end{aligned}$$

When  $z = \frac{1}{2}$ , we have Wallis' inequalities. The result in C.P. Chen and F. Qi [9] also is improved, and when  $z = \frac{1}{m}$  or  $z = 1 - \frac{1}{m}$ , the results of (1.4), (1.5) are improved. When  $n \geq 1$ ,  $z = \frac{1}{2}$ , and  $0 < \varepsilon < \frac{1}{2}$ , we also have,

$$\frac{1}{\sqrt{n\pi\left(1+\frac{1}{4n-1/2}\right)}} < P_n < \frac{1}{\sqrt{n\pi\left(1+\frac{1}{4n-1/2+\varepsilon}\right)}}.$$

The inequality on the right holds for  $n > n^*$ , where  $n^*$  is the maximal root of the equation  $32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0$ .

The result in [7] also is improved.



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## 2. Some Lemmas

**Lemma 2.1.** When  $0 < z < 1, n > k \geq 1$ ,

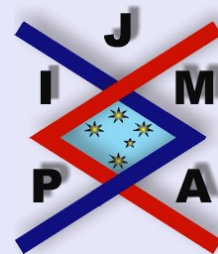
$$\int_0^1 \frac{t^{n-z} - t^n}{1-t} dt = \frac{z}{n} + \frac{z(z-1)}{2n(n-1)} + \dots + \frac{z(z-1)\cdots(z-k+1)}{kn(n-1)\cdots(n-k+1)} + \frac{z(z-1)\cdots(z-k)}{(k+1)n(n-1)\cdots(n-k+1)(n+\theta)},$$

where  $-k < \theta < 1 - z$ .

*Proof.* We easily get

$$\int_0^1 t^{n-k-1}(1-t)^k dt = \frac{k!}{n(n-1)\cdots(n-k)},$$

$$\begin{aligned} & \frac{1}{n(n-1)\cdots(n-k)} \\ & > \frac{1}{n(n-1)\cdots(n-k)} + \frac{z-k-1}{k!} \int_0^1 dx \int_0^x \frac{(1-x)^{k+1}}{1-t} x^{z-k-2} t^{n-z} dt \\ & > \frac{1}{n(n-1)\cdots(n-k)} + \frac{z-k-1}{k!} \int_0^1 dx \int_0^x (1-x)^k x^{z-k-2} t^{n-z} dt \\ & = \frac{1}{n(n-1)\cdots(n-k)} + \frac{z-k-1}{k!(n+1-z)} \int_0^1 x^{n-k-1} (1-x)^k dx \\ & = \frac{1}{n(n-1)\cdots(n-k+1)(n+1-z)}. \end{aligned}$$



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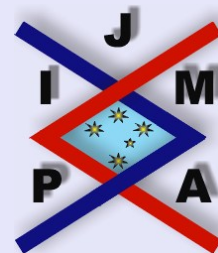
Hence,

$$\begin{aligned} & \frac{1}{n(n-1)\cdots(n-k)} \\ & + \frac{z-k-1}{k!} \int_0^1 dx \int_0^x \frac{(1-x)^{k+1}}{1-t} x^{z-k-2} t^{n-z} dt \\ & = \frac{1}{n(n-1)\cdots(n-k+1)(n+\theta)}, \end{aligned}$$

$(-k < \theta < 1 - z)$ .

Denote  $h(x) = x^z$ , and let  $0 < z < 1, n > k$ , then, by Taylor's formula we have

$$\begin{aligned} 1 - t^z &= h(1) - h(t) \\ &= h'(t)(1-t) + \frac{h''(t)}{2}(1-t)^2 + \cdots \\ &+ \frac{h^{(k+1)}(t)}{(k+1)!}(1-t)^{k+1} + \frac{1}{(k+1)!} \int_t^1 h^{(k+2)}(x)(1-x)^{k+1} dx \\ &= zt^{z-1}(1-t) + \frac{z(z-1)}{2!}t^{z-2}(1-t)^2 + \cdots \\ &+ \frac{z(z-1)\cdots(z-k)}{(k+1)!}t^{z-k-1}(1-t)^{k+1} \\ &+ \frac{z(z-1)\cdots(z-k-1)}{(k+1)!} \int_t^1 x^{z-k-2}(1-x)^{k+1} dx. \end{aligned}$$



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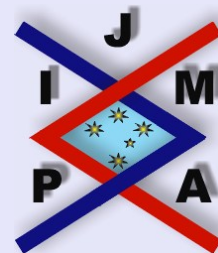
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Hence,

$$\begin{aligned}
 & \int_0^1 \frac{t^{n-z} - t^n}{1-t} dt \\
 &= \int_0^1 z t^{n-1} dt + \frac{z(z-1)}{2} \int_0^1 t^{n-2} (1-t) dt + \dots \\
 & \quad + \frac{z(z-1) \cdots (z-k+1)}{k!} \int_0^1 t^{n-k} (1-t)^{k-1} dt \\
 & \quad + \frac{z(z-1) \cdots (z-k)}{(k+1)!} \int_0^1 t^{n-k-1} (1-t)^k dt \\
 & \quad + \frac{z(z-1) \cdots (z-k-1)}{(k+1)!} \int_0^1 dt \int_t^1 \frac{(1-x)^{k+1}}{1-t} x^{z-k-2} t^{n-z} dx \\
 &= \frac{z}{n} + \frac{z(z-1)}{2n(n-1)} + \dots + \frac{z(z-1) \cdots (z-k+1)}{kn(n-1) \cdots (n-k+1)} \\
 & \quad + \frac{z(z-1) \cdots (z-k)}{(k+1)} \left[ \frac{1}{n(n-1) \cdots (n-k)} \right. \\
 & \quad \left. + \frac{z-k-1}{k!} \int_0^1 dx \int_0^x \frac{(1-x)^{k+1}}{1-t} x^{z-k-2} t^{n-z} dt \right] \\
 &= \frac{z}{n} + \frac{z(z-1)}{2n(n-1)} + \dots + \frac{z(z-1) \cdots (z-k+1)}{kn(n-1) \cdots (n-k+1)} \\
 & \quad + \frac{z(z-1) \cdots (z-k)}{(k+1)n(n-1) \cdots (n-k+1)(n+\theta)}.
 \end{aligned}$$

Hence, the proof of Lemma 2.1 is completed.  $\square$



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Let  $k = 1$  or  $k = 2$ , we then have:

$$\frac{z}{n} - \frac{z(1-z)}{2n(n-1)} < \int_0^1 \frac{t^{n-z} - t^n}{1-t} dt < \frac{z}{n} - \frac{z(1-z)}{2n(n+1-z)}$$

and

$$\frac{z}{n} - \frac{z(1-z)}{2n(n-1)} < \int_0^1 \frac{t^{n-z} - t^n}{1-t} dt < \frac{z}{n} - \frac{z(1-z)}{2n(n-1)} + \frac{z(z-1)(z-2)}{3n(n-1)(n-2)}.$$

**Lemma 2.2.** For  $0 < z < 1$  and  $n > 1$ , let

$$r_n(z) = \sum_{k=1}^{\infty} \left( \frac{1}{n+k-z} - \frac{1}{n+k} \right),$$

then

$$r_n(z) = \int_0^1 \frac{t^{n-z} - t^n}{1-t} dt.$$

Moreover,

$$(2.1) \quad \frac{z}{n} - \frac{z(1-z)}{2n(n-1)} < r_n(z) < \frac{z}{n} - \frac{z(1-z)}{2n(n+1-z)},$$

$$(2.2) \quad \frac{z}{n} - \frac{z(1-z)}{2n(n-1)} < r_n(z) < \frac{z}{n} - \frac{z(1-z)}{2n(n-1)} + \frac{z(z-1)(z-2)}{3n(n-1)(n-2)}.$$

*Proof.* Let  $g(t) = \sum_{k=1}^{\infty} \left( \frac{t^{n+k-z}}{n+k-z} - \frac{t^{n+k}}{n+k} \right)$ , then  $g(t)$  is convergent on  $[0, 1]$ .

Hence,  $g(t)$  is continuous on  $[0, 1]$ . Moreover, because

$$\sum_{k=1}^{\infty} \left( \frac{t^{n+k-z}}{n+k-z} - \frac{t^{n+k}}{n+k} \right)' = \sum_{k=1}^{\infty} (t^{n+k-z-1} - t^{n+k-1})$$



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is continuous on the closed region of  $[0, 1)$ , then, for  $0 < t < 1$ ,

$$g'(t) = \sum_{k=1}^{\infty} (t^{n+k-z-1} - t^{n+k-1}) = \frac{t^{n-z} - t^n}{1-t}.$$

Moreover,  $g(0) = 0$ . So  $g(x) = \int_0^x \frac{t^{n-z}-t^n}{1-t} dt$ ,

$$r_n(z) = \sum_{k=1}^{\infty} \left( \frac{1}{n+k-z} - \frac{1}{n+k} \right) = g(1),$$

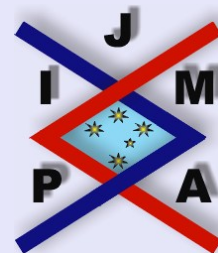
Hence,  $r_n(z) = \int_0^1 \frac{t^{n-z}-t^n}{1-t} dt$ . The proof of Lemma 2.2 is completed.  $\square$

### Lemma 2.3.

$$\frac{(1-z)(2-z)\cdots(n-z)n^z\Gamma(1-z)}{n!} = \frac{n-z}{n} \prod_{k=0}^{\infty} \left(1 - \frac{z}{n+k}\right)^{-1} \left(1 + \frac{1}{n+k}\right)^{-z}.$$

*Proof.* By (1.8), we know

$$\begin{aligned} \Gamma(1-z) &= \lim_{n \rightarrow \infty} \frac{(n-1)!n^{1-z}}{(1-z)(2-z)\cdots(n-z)} \\ &= \lim_{n \rightarrow \infty} \frac{n!}{(1-z)(2-z)\cdots(n-z)n^z} \\ &= \lim_{n \rightarrow \infty} \prod_{k=1}^n \left(1 - \frac{z}{k}\right)^{-1} \prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^{-z} \end{aligned}$$



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$$\begin{aligned}
&= \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right)^{-1} \left(1 + \frac{1}{k}\right)^{-z} \\
&= \prod_{k=1}^{n-1} \left(1 - \frac{z}{k}\right)^{-1} \left(1 + \frac{1}{k}\right)^{-z} \cdot \prod_{k=n}^{\infty} \left(1 - \frac{z}{k}\right)^{-1} \left(1 + \frac{1}{k}\right)^{-z} \\
&= \frac{(n-1)!}{(1-z)(2-z)\cdots(n-1-z)n^z} \prod_{k=0}^{\infty} \left(1 - \frac{z}{n+k}\right)^{-1} \left(1 + \frac{1}{n+k}\right)^{-z}.
\end{aligned}$$

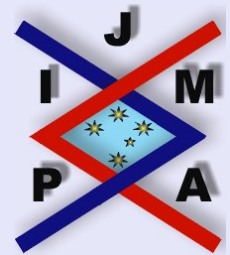
Hence,

$$\begin{aligned}
&\frac{(1-z)(2-z)\cdots(n-z)n^z\Gamma(1-z)}{n!} \\
&= \frac{n-z}{n} \prod_{k=0}^{\infty} \left(1 - \frac{z}{n+k}\right)^{-1} \left(1 + \frac{1}{n+k}\right)^{-z}.
\end{aligned}$$

The proof of Lemma 2.3 is completed.  $\square$

**Lemma 2.4.** When  $n \geq 1$ ,

$$\begin{aligned}
\frac{1}{2n} - \frac{1}{8n^2} &< r_n \left(\frac{1}{2}\right) \\
&= \int_0^1 \frac{t^{n-1/2} - t^n}{1-t} dt \\
&< \frac{1}{2n+1} + \frac{1}{2(2n+1)^2}.
\end{aligned}$$



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*Proof.* When  $k > 1$ ,

$$\begin{aligned}
 (2.3) \quad \int_0^1 \frac{t^k}{1+t} dt &= \frac{1}{k+1} \left( \frac{t^{k+1}}{1+t} \Big|_0^1 + \int_0^1 \frac{t^k \cdot t}{(1+t)^2} dt \right) \\
 &< \frac{1}{2(k+1)} + \int_0^1 \frac{t^k}{4(k+1)} dt \\
 &= \frac{1}{2(k+1)} + \frac{1}{4(k+1)^2}.
 \end{aligned}$$

By (2.3), we obtain

$$\int_0^1 \frac{t^k}{1+t} dt = \int_0^1 t^{k-1} dt - \int_0^1 \frac{t^{k-1}}{1+t} dt > \frac{1}{k} - \frac{1}{2k} - \frac{1}{4k^2} = \frac{1}{2k} - \frac{1}{4k^2}.$$

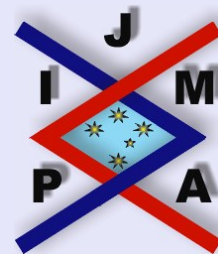
Moreover,

$$\int_0^1 \frac{t^{n-1/2} - t^n}{1-t} dt = \int_0^1 t^{n-1/2} \frac{1}{1+\sqrt{t}} dt = 2 \int_0^1 \frac{x^{2n}}{1+x} dx.$$

Hence,

$$\frac{1}{2n} - \frac{1}{8n^2} < r_n \left( \frac{1}{2} \right) = \int_0^1 \frac{t^{n-1/2} - t^n}{1-t} dt < \frac{1}{2n+1} + \frac{1}{2(2n+1)^2}.$$

The proof of Lemma 2.4 is completed. □




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### 3. Main Theorems

Denote  $P_n(z) = \frac{(1-z)(2-z)\cdots(n-z)}{n!}$ .

**Theorem 3.1.** When  $0 < z < 1$ ,  $n > 1$ ,

$$\begin{aligned} \frac{1}{n^z \Gamma(1-z) \left(1 + \frac{1-z}{2(n-1)}\right)^z} &< P_n(z) \\ &< \frac{1}{n^z \Gamma(1-z) \left(1 + \frac{1-z}{2n+1-z}\right)^z} \\ &< \frac{1}{n^z \Gamma(1-z) \left(1 + \frac{1-z}{2n+1}\right)^z}. \end{aligned}$$

*Proof.* Let

$$F(n, z, \alpha) = \frac{(1-z)(2-z)\cdots(n-z)n^z \Gamma(1-z)}{n!} \left(1 + \frac{1-z}{2n+\alpha}\right)^z.$$

By Lemma 2.3, we have, for  $0 < z < 1$ ,  $n > 1$ ,

$$\begin{aligned} &\ln F(n, z, \alpha) \\ &= \ln \frac{n-z}{n} - \sum_{k=0}^{\infty} \left[ \ln \left(1 - \frac{z}{n+k}\right) + z \ln \left(1 + \frac{1}{n+k}\right) \right] + z \ln \left(1 + \frac{1-z}{2n+\alpha}\right). \end{aligned}$$

We also know when  $k \geq 1$ ,

$$\left| \ln \left(1 - \frac{z}{n+k}\right) + z \ln \left(1 + \frac{1}{n+k}\right) \right| \sim \left| \frac{z+z^2}{2(n+k)^2} \right| \leq \frac{1}{k^2},$$



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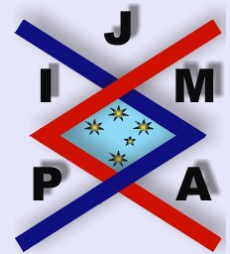
because  $\ln F(+\infty, z, \alpha) = 0$ ,

$$\begin{aligned}
 (3.1) \quad & \frac{\partial \ln F(n, z, \alpha)}{\partial n} \\
 &= \frac{1}{n-z} - \frac{1}{n} - \sum_{k=0}^{\infty} \left( \frac{1}{n+k-z} - \frac{1}{n+k} + \frac{z}{n+k+1} - \frac{z}{n+k} \right) \\
 &\quad + \frac{2z}{2n+\alpha+1-z} - \frac{2z}{2n+\alpha} \\
 &= - \sum_{k=1}^{\infty} \left( \frac{1}{n+k-z} - \frac{1}{n+k} \right) + \frac{z}{n} - \frac{2z(1-z)}{(2n+\alpha+1-z)(2n+\alpha)}.
 \end{aligned}$$

By Lemma 2.2 and (2.1), we can get

$$\begin{aligned}
 & \frac{\partial \ln F(n, z, 1-z)}{\partial n} \\
 &= - \sum_{k=1}^{\infty} \left( \frac{1}{n+k-z} - \frac{1}{n+k} \right) + \frac{z}{n} - \frac{z(1-z)}{(n+1-z)(2n+1-z)} \\
 &> \frac{z(1-z)}{2n(n+1-z)} - \frac{z(1-z)}{(n+1-z)(2n+1-z)} \\
 &= \frac{z(1-z)^2}{2n(n+1-z)(2n+1-z)} > 0.
 \end{aligned}$$

Hence,  $\ln F(n, z, 1-z) < 0$ . Moreover, we have  $F(n, z, 1-z) < 1$ , hence, the right inequality of Theorem 3.1 holds.



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Since

$$\begin{aligned} \frac{\partial \ln F(n, z, -2)}{\partial n} &= -\sum_{k=1}^{\infty} \left( \frac{1}{n+k-z} - \frac{1}{n+k} \right) + \frac{z}{n} - \frac{z(1-z)}{(2n-1-z)(n-1)} \\ &< \frac{z(1-z)}{2n(n-1)} - \frac{z(1-z)}{(2n-1-z)(n-1)} \\ &= -\frac{z(1-z)^2}{2n(n-1)(2n-1-z)} < 0. \end{aligned}$$

Moreover, we pay attention to  $\ln F(+\infty, z, \alpha) = 0$ , hence,  $\ln F(n, z, -2) > 0$ . Thus we have  $F(n, z, -2) > 1$  and the left inequality of Theorem 3.1 holds.

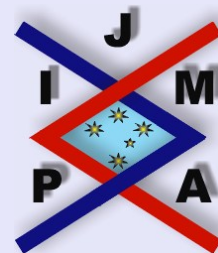
The proof of Theorem 3.1 is completed.  $\square$

**Theorem 3.2.** For  $0 < z < 1$  and  $n \geq 22$ ,

$$\frac{1}{\Gamma(1-z)n^z \left(1 + \frac{1-z}{2(n-1)}\right)^z} < P_n(z) < \frac{1}{\Gamma(1-z)n^z \left(1 + \frac{1-z}{2n}\right)^z}.$$

*Proof.* By (3.1), Lemma 2.2 and (2.2), we have

$$\begin{aligned} \frac{\partial \ln F(n, z, 0)}{\partial n} &= -\sum_{k=1}^{\infty} \left( \frac{1}{n+k-z} - \frac{1}{n+k} \right) + \frac{z}{n} - \frac{z(1-z)}{n(2n+1-z)} \\ &> \frac{z(1-z)}{2n(n-1)} - \frac{z(1-z)(2-z)}{3n(n-1)(n-2)} - \frac{z(1-z)}{n(2n+1-z)} \\ &= z(1-z) \left[ \frac{1}{2n(n-1)} - \frac{2-z}{3n(n-1)(n-2)} - \frac{1}{n(2n+1-z)} \right] \end{aligned}$$



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$$= z(1-z) \frac{n-22+nz+z(12-2z)}{6n(n-1)(n-2)(2n+1-z)} > 0 \quad (n \geq 22).$$

Since  $\ln F(+\infty, z, \alpha) = 0$ , then,  $\ln F(n, z, 0) < 0$ . Moreover,  $F(n, z, 0) < 1$ . So, the right inequality of Theorem 3.2 holds. The proof of Theorem 3.2 is completed.  $\square$

**Theorem 3.3.** When  $n \geq 1$ ,  $0 < \varepsilon < \frac{1}{2}$ ,

$$\frac{1}{\sqrt{n\pi \left(1 + \frac{1}{4n-1/2}\right)}} < \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} < \frac{1}{\sqrt{n\pi \left(1 + \frac{1}{4n-1/2+\varepsilon}\right)}}.$$

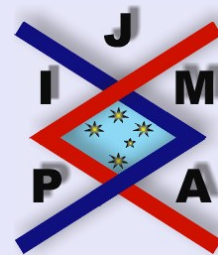
The right-hand inequality holds for  $n > n^*$ , where  $n^*$  is the maximal root on

$$32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0.$$

*Proof.* By Lemma 2.4 and (3.1)

$$\begin{aligned} \frac{\partial \ln F \left( n, \frac{1}{2}, -\frac{1}{4} \right)}{\partial n} &= -r_n \left( \frac{1}{2} \right) + \frac{1}{2n} - \frac{1}{2 \left( 2n - \frac{1}{4} \right) \left( 2n + \frac{1}{4} \right)} \\ &< -\frac{1}{2n} + \frac{1}{8n^2} + \frac{1}{2n} - \frac{1}{2 \left( 4n^2 - \frac{1}{16} \right)} < 0. \end{aligned}$$

Using a similar line of proof as in Theorem 3.1, we obtain the left-hand inequality.



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Now, for  $0 < \varepsilon < \frac{1}{2}$ , we have

$$\begin{aligned} & \frac{\partial \ln F\left(n, \frac{1}{2}, -\frac{1}{4} + \frac{\varepsilon}{2}\right)}{\partial n} \\ &= -r_n \left(\frac{1}{2}\right) + \frac{1}{2n} - \frac{1}{2\left(2n - \frac{1}{4} + \frac{\varepsilon}{2}\right)\left(2n + \frac{1}{4} + \frac{\varepsilon}{2}\right)} \\ &> -\frac{1}{2n+1} - \frac{1}{2(2n+1)^2} + \frac{1}{2n} - \frac{1}{2\left(2n - \frac{1}{4} + \frac{\varepsilon}{2}\right)\left(2n + \frac{1}{4} + \frac{\varepsilon}{2}\right)} \\ &= \frac{32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1}{32n(2n+1)^2\left(2n - \frac{1}{4} + \frac{\varepsilon}{2}\right)\left(2n + \frac{1}{4} + \frac{\varepsilon}{2}\right)} > 0 \quad (n > n^*). \end{aligned}$$

$n^*$  is the maximal root of the equation  $32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0$ . Using a similar line of proof as in Theorem 3.1, we obtain the right-hand inequality. The proof of Theorem 3.3 is completed.  $\square$

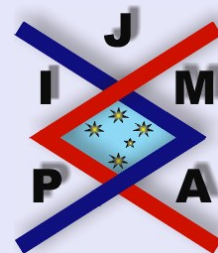
By Theorem 3.1 and Theorem 3.2, we obtain the following corollaries.

**Corollary 3.4.** When  $0 < z < 1$ ,

$$P_n(z) = \frac{1}{n^z \Gamma(1-z) \left(1 + \frac{1-z}{2(n-\theta)}\right)^z}, \quad -\frac{1}{2} < \theta < 1 \quad (n > 1),$$

$$P_n(z) = \frac{1}{n^z \Gamma(1-z) \left(1 + \frac{1-z}{2(n-\theta)}\right)^z}, \quad 0 < \theta < 1 \quad (n \geq 22).$$

**Remark 1.** Letting  $z = 1/m$  or  $z = 1 - 1/m$ , we can obtain better results than (1.4) and (1.5).



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When  $z = 1/2$  and  $n \geq 22$ , by Theorem 3.2, we have

$$\frac{1}{n^{1/2}\Gamma\left(\frac{1}{2}\right)\left(1 + \frac{1}{4(n-1)}\right)^{1/2}} < P_n\left(\frac{1}{2}\right) = P_n < \frac{1}{n^{1/2}\Gamma\left(\frac{1}{2}\right)\left(1 + \frac{1}{4n}\right)^{1/2}},$$

that is,

$$\frac{1}{\sqrt{n\pi}\left(1 + \frac{1}{4(n-1)}\right)} < P_n < \frac{1}{\sqrt{n\pi}\left(1 + \frac{1}{4n}\right)}.$$

It can also be shown that the inequality holds when  $n = 2, 3, \dots, 21$ .

**Corollary 3.5.** When  $n > 1$ ,

$$\frac{1}{\sqrt{n\pi}\left(1 + \frac{1}{4(n-1)}\right)} < P_n < \frac{1}{\sqrt{n\pi}\left(1 + \frac{1}{4n}\right)}.$$

**Remark 2.** Corollary 3.5 improves the result of C.P. Chen and F. Qi in [9].

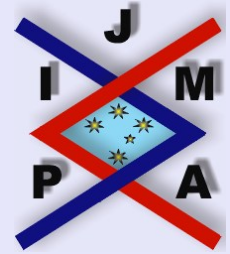
By Theorem 3.3, when  $\varepsilon = \frac{1}{6}$ ,

$$32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = \frac{8}{9}(6n^2 - 13n - 1).$$

Hence, when  $n > 3$ ,

$$\frac{1}{\sqrt{\pi n}\left(1 + \frac{1}{4n-1/2}\right)} < \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} = P_n\left(\frac{1}{2}\right) < \frac{1}{\sqrt{\pi n}\left(1 + \frac{1}{4n-1/3}\right)}.$$

It can also be shown that the inequality holds when  $n = 1, 2, 3$ .



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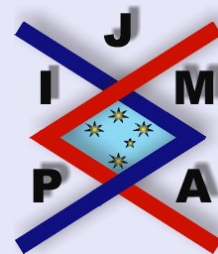
The following corollary holds.

**Corollary 3.6.** For  $n \geq 1$

$$\frac{1}{\sqrt{\pi n \left(1 + \frac{1}{4n-1/2}\right)}} < \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} = P_n \left(\frac{1}{2}\right) < \frac{1}{\sqrt{\pi n \left(1 + \frac{1}{4n-1/3}\right)}}.$$

**Remark 3.** When  $\varepsilon$  is a positive number, we obtain other inequalities. For example, when  $\varepsilon = 1/10$ , we have, for  $n > 1$ ,

$$\frac{1}{\sqrt{\pi n \left(1 + \frac{1}{4n-1/2}\right)}} < \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} = P_n \left(\frac{1}{2}\right) < \frac{1}{\sqrt{\pi n \left(1 + \frac{1}{4n-2/5}\right)}}.$$




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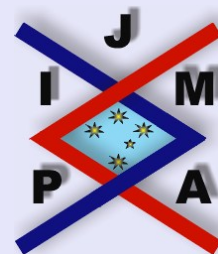
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