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THE PRESSURE OF FUNCTIONS OVER (G, τ)–EXTENSIONS

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ABSTRACT. Let $T\colon X \to X$ be a (free) (G,τ)–extension of $S\colon Y \to Y$. Moreover let $f_X, f_Y, f_G \geq 0$ be continuous functions defined on X, Y and G respectively. In this paper we obtain some inequalities for the pressure of f_X over the transformation T in relation to the pressure of f_Y over the transformation S and of f_G over τ.

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1. INTRODUCTION

Let $T\colon X \to X$ be a continuous map of a compact metric space X and $\tau\colon G \to G$ be an automorphism of a compact metric group G. Suppose G acts continuously and freely on the right of T so that the equation $T(xg) = T(x)\tau(g)$ holds true $\forall x \in X,\ g \in G$. Moreover let Y be the G–orbit space and S is the natural map on Y defined by $S(xG) = (Tx)G$, $\forall x \in X$. Then T is called a (G,τ)–extension of S.

Bowen [1] studied the topological entropy of the aforementioned extension system and, amongst other things, showed that the following formula holds:

$$h(T) = h(S) + h(\tau),$$

where $h(\cdot)$ is the topological entropy of the appropriate maps.

In this paper, we are interested in the pressure analogue of Bowen's formula, i.e., we consider the pressure of functions defined on the respective dynamical systems instead of topological entropy. Unfortunately the main result of this paper (i.e., the analogue of Bowen's formula, see Corollary 4.6) is somewhat short of an equality. Our examples indicate that equality holds but we are unable to prove this in general. The proofs of the results arrived at in this paper are of course modelled along the lines of Bowen.

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2. PRESSURE

We shall recall some elementary facts about pressure which are relevant to us in proving our results. The references for this section are [2] and [4].

As before let $T\colon X \to X$ be a continuous map of a compact metric space (X,d). Throughout this paper we shall assume that T have finite topological entropy. Let K be a compact subset of X. A subset F of X is said to be an (n,ϵ)-*spanning* set for K if for given $k \in K$ then there exists $x \in F$ such that $d(T^i(k), T^i(x)) \leq \epsilon$, $\forall 0 \leq i \leq n-1$. Now let f be a continuous real-valued function defined on X and consider the set defined by:

$$Q_n(T,f,\epsilon,K) = \inf\left\{\sum_{x\in F} e^{S_n f(x)} \colon F\ (n,\epsilon)\text{-}\textit{spans}\ K\right\}.$$

(Here we have used the standard notation: $S_n f(x) := f(x) + f(Tx) + \cdots + f(T^{n-1}x)$.) Then it is easy to see that $Q_n(T,f,\epsilon,K) \leq ||e^{S_n f(x)}|| r_n(T,\epsilon,K)$ where $r_n(T,\epsilon,K)$ is the cardinality of an (n,ϵ)-spanning set for K with a minimum number of elements. In particular, by virtue of compactness and continuity, we have $0 < Q_n(T,f,\epsilon,K) < \infty$. Now define:

$$Q(T,f,\epsilon,K) = \limsup_{n\to\infty} \frac{1}{n} \log Q_n(T,f,\epsilon,K)$$

Lemma 2.1. $Q(T,f,\epsilon,K) < \infty$

Proof. We know that

$$Q_n(T,f,\epsilon,K) \leq ||e^{S_n f(x)}|| r_n(T,\epsilon,K) < \infty.$$

Hence, since $||e^{S_n f(x)}|| \leq e^{n||f(x)||}$, we have

$$Q_n(T,f,\epsilon,K) \leq e^{n||f(x)||} r_n(T,\epsilon,K).$$

Thus

$$\frac{1}{n}\log Q_n(T,f,\epsilon,K) \leq ||f|| + \frac{1}{n}\log r_n(T,\epsilon,K).$$

In particular we have

$$Q(T,f,\epsilon,K) \leq ||f|| + \limsup_{n\to\infty} \frac{1}{n}\log r_n(T,\epsilon,K).$$

It is well known that $\limsup_{n\to\infty} \frac{1}{n}\log r_n(T,\epsilon,K) < \infty$. Hence this completes the proof. □

We are now ready to define the pressure: The pressure of f with respect to the subset K of X over the map $T\colon X \to X$ is defined by the quantity:

$$P(T,f,K) = \lim_{\epsilon\to 0} Q(T,f,\epsilon,K).$$

Remark 2.2.

(1) As is well-known, the metric on X can be arbitrarily chosen as long as it induces the same topology on X.
(2) When $K = X$ we obtain the usual definition of the pressure, $P(T,f)$, of the function f over the map $T\colon X \to X$.
(3) Recall that E is an (n,ϵ)-*separated* set of $K \subset X$ if for any two distinct points $x,y \in E$ there exists some $0 \leq i < n$ such that $d(T^i(x), T^i(y)) > \epsilon$. It can be checked that the above definition can also be arrived at by using separating sets. In this case we shall be concerned with the quantity

$$P_n(T,f,\epsilon,K) = \sup\left\{\sum_{x\in E} e^{S_n f(x)} \colon E\ (n,\epsilon)\text{-}\textit{separates}\ K\right\}.$$

As is well known, the next step is to define

$$P(T, f, K) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log P_n(T, f, \epsilon, K).$$

The following two results are straight forward consequences of the definition of pressure.

Proposition 2.3.

$$P(T, f, K) \leq P(T, f)$$

whenever $K \subset X$

Proof. Let F (n, ϵ)-span X. Then F also (n, ϵ)-spans $K \subset X$. Hence

$$\inf \left\{ \sum_{x \in F} e^{S_n f(x)} \colon F\ (n, \epsilon)\text{-spans } K \right\} \leq \inf \left\{ \sum_{x \in F} e^{S_n f(x)} \colon F\ (n, \epsilon)\text{-spans } X \right\}.$$

Thus $Q_n(T, f, \epsilon, K) \leq Q_n(T, f, \epsilon)$. The result follows by taking the appropriate logarithms and limits. □

Lemma 2.4. *Let* $s_n(T, 4\epsilon, X)$ *denote the cardinality of a* $(n, 4\epsilon)$*-separated set of* X *with maximum number of elements. Then*

$$P_n(T, f, 8\epsilon, X) \leq e^{n\|f\|} s_n(T, 4\epsilon, X).$$

Proof. For any $\epsilon > 0$, it is not difficult to check that $P_n(T, f, 2\epsilon, X) \leq Q_n(T, f, \epsilon, X)$ and $r_n(T, \epsilon, X) \leq s_n(T, \epsilon, X)$. Hence since $Q_n(T, f, \epsilon, X) \leq e^{\|S_n f\|} r_n(T, \epsilon, X)$ we have

$$\begin{aligned} P_n(T, f, 8\epsilon, X) &\leq Q_n(T, f, 4\epsilon, X) \\ &\leq e^{\|S_n f\|} r_n(T, 4\epsilon, X) \\ &\leq e^{n\|f\|} s_n(T, 4\epsilon, X). \end{aligned}$$

□

3. GENERAL EXTENSIONS

In this section we shall start off with a straightforward modification of a crucial estimate of Bowen and later show how this estimate is used when dealing with pressure. But first recall the following definition:

Let $T \colon X \to X$ and $S \colon Y \to Y$ be continuous maps of compact metric spaces X and Y. Moreover let π be a continuous surjective map from X to Y such that $\pi \circ T = S \circ \pi$. Then as is well-known T is called an extension of S.

With respect to this extension system, we have the following result which is essentially due to Bowen [1].

Lemma 3.1. *Let* $\epsilon > 0$, $\alpha > 0$ *and integer* $n > 0$ *be arbitrary. Also let* f_X *be a continuous positive function defined on* X. *Then there exists some* $\delta > 0$ *such that if* Y_n *is an* (n, δ)*-spanning set for* Y *with minimum cardinality then for any* $(n, 4\epsilon)$*-separating set* F *of* X *we have*

$$\textit{Card. } F \leq \textit{Card. } Y_n \cdot e^{(a+\alpha)(n+M)},$$

where $a = \sup_{y \in Y} P(T, f_X, \pi^{-1}(y))$ *and* M *is some finite positive real number.*

Proof. Let $a = \sup_{y\in Y} P(T, f_X, \pi^{-1}(y))$. For each $y \in Y$, choose an integer $m(y) > 0$ such that

$$\begin{aligned} a+\alpha &\geq P(T,f_X,\pi^{-1}y)+\alpha \\ &\geq \frac{1}{m(y)}\log Q_{m(y)}(T,f_X,\epsilon,\pi^{-1}(y)). \end{aligned} \tag{*}$$

Also for each $y \in Y$ let E_y be a $(m(y), \epsilon)$-spanning set of $\pi^{-1}(y)$. Now consider the open neighborhood of $\pi^{-1}(y)$

$$U_y = \bigcup_{z\in E_y} \bigcap_{k=0}^{m(y)-1} T^{-k}B_{2\epsilon}(T^k z).$$

Then it is clear that

$$(X/U_y) \cap \bigcap_{\gamma>0} \pi^{-1}(\overline{B_\gamma(y)}) = \emptyset,$$

where $B_\gamma(y)$ is the open-ball centered at y with radius γ. Since X is compact, the finite intersection property for compact sets then implies there exists some $\gamma = \gamma(y) > 0$ such that $\pi^{-1}(B_\gamma(y)) \subset U_y$. In particular, since X is compact, there exists $y_1, y_2, \dots, y_r \in Y$ such that Y is covered by the open balls $B_\gamma(y_i)$, $i = 1, 2, \dots, r$ and

$$\pi^{-1}\left(B_\gamma(y_i)\right) \subset U_{y_i},$$

where

$$U_{y_i} = \bigcup_{z\in E_{y_i}} \bigcap_{k=0}^{m(y_i)-1} T^{-k}B_{2\epsilon}(T^k z)$$

and E_{y_i} is an $(m(y_i), \epsilon)$-spanning set of $\pi^{-1}(y_i)$.

Now let $\delta > 0$ be the Lebesgue number of this cover of Y and let Y_n be a (n, δ)-spanning set for Y with minimum cardinality. Hence for each $\overline{y} \in Y_n$ we can define $c_j(\overline{y})$ as the element in $\{y_1, y_2, \dots, y_r\}$ satisfying $B_\delta(S^j(\overline{y})) \subset B_\gamma(c_i(\overline{y}))$ for each $j = 0, 1, \dots, n-1$.

Next define recursively the positive integers

$$\begin{aligned} t_0(\overline{y}) &= 0, \\ t_s(\overline{y}) &= \sum_{r=0}^{s-1} m\left(c_{t_r(\overline{y})}(\overline{y})\right) \end{aligned}$$

for each $s = 1, 2, \dots, q$, where $q = q(\overline{y})$ having the property $t_{q+1}(\overline{y}) \geq n$. Now for each $q+1$-ple

$$(x_0, x_1, \dots, x_q) \in E_{c_{t_0(\overline{y})}(\overline{y})} \times E_{c_{t_1(\overline{y})}(\overline{y})} \times \cdots \times E_{c_{t_q(\overline{y})}(\overline{y})}$$

define the set

$$\begin{aligned} V(\overline{y};(x_0,x_1,\dots,x_q)) = \big\{x\in X\colon d\left(T^{t+t_s(\overline{y})}(x),T^t(x_s)\right) < 2\epsilon \\ \forall 0\leq t< m\left(c_{t_s(\overline{y})}(\overline{y})\right) \,\&\, 0\leq s\leq q\big\}. \end{aligned}$$

Then it is easy to check that

$$\bigcup_{(\overline{y};(x_0,x_1,\dots,x_q))} V(\overline{y};(x_0,x_1,\dots,x_q)) = X$$

and if F is an $(n, 4\epsilon)$-separated subset of X, then

$$\text{Card.}(F \cap V(\overline{y};(x_0,x_1,\dots,x_q))) = 0 \text{ or } 1 \tag{3.1}$$

for each $q+2$-ple $(\overline{y};(x_0,x_1,\dots,x_q))$, where $\overline{y} \in Y_n$ and $x_s \in E_{c_{t_s(\overline{y})}(\overline{y})}$, $s = 0, 1, \dots, q$.

To complete the proof of this lemma, we shall now obtain an estimate for the number $N_{\overline{y}}$ of the $q+1$-ple's $(x_0, x_1, \dots, x_{q(\overline{y})})$. Since

$$N_{\overline{y}} = \prod_{s=0}^{q(\overline{y})} r_{m\left(c_{t_s(\overline{y})}(\overline{y})\right)} \left(T, \epsilon, \pi^{-1}\left(c_{t_s(\overline{y})}(\overline{y})\right)\right)$$

we have by virtue of $(*)$ and $f_X \geq 0$

$$\begin{aligned}
\log N_{\overline{y}} &\leq \sum_{s=0}^{q(\overline{y})} \log r_{m\left(c_{t_s(\overline{y})}(\overline{y})\right)} \left(T, \epsilon, \pi^{-1}\left(c_{t_s(\overline{y})}(\overline{y})\right)\right) \\
&\leq \sum_{s=0}^{q(\overline{y})} \log Q_{m\left(c_{t_s(\overline{y})}\right)}(T, f_X, \epsilon, \pi^{-1}(c_{t_s(\overline{y})})) \\
&\leq \sum_{s=0}^{q(\overline{y})} m\left(c_{t_s(\overline{y})}(\overline{y})\right)(a+\alpha).
\end{aligned}$$

Recall that

$$t_{q+1}(\overline{y}) = \sum_{r=0}^{q(\overline{y})} m\left(c_{t_s(\overline{y})}(\overline{y})\right).$$

Also $t_{q+1}(\overline{y}) = t_q(\overline{y}) + m\left(c_{t_q(\overline{y})}(\overline{y})\right)$. Therefore

$$\begin{aligned}
t_{q+1}(\overline{y}) &\leq n - 1 + m\left(c_{t_q(\overline{y})}(\overline{y})\right) \\
&\leq n + M,
\end{aligned}$$

where $M = \max\{m(y_1), m(y_2), \dots, m(y_r)\}$.
Hence

$$\log N_{\overline{y}} \leq (n+M)(a+\alpha)$$

so that $N_{\overline{y}} \leq e^{(n+M)(a+\alpha)}$. In particular, (3.1) now implies

$$\begin{aligned}
\text{Card. } F &\leq \text{Card. } Y_n \cdot N_{\overline{y}} \\
&\leq \text{Card. } Y_n \cdot e^{(n+M)(a+\alpha)}
\end{aligned}$$

and this completes the proof of this lemma. □

Some remarks are in order:

(1) Apart from the choices of integers $m(y)$, the rest of the proof of the above lemma is an exact copy of Bowen's theorem [1, Theorem 17].
(2) When F is a $(n, 4\epsilon)$-separating set of X with maximum cardinality, then

$$s_n(T, 4\epsilon, X) \leq r_n(S, \delta, Y) e^{(n+M)(a+\alpha)}$$

where as before $s_n(T, 4\epsilon, X)$ is the cardinality of such F and $r_n(S, \delta, Y)$ is the cardinality of Y_n.

The following theorem is the pressure analogue of Bowen's Theorem [1, Theorem 17] (see also [3]).

Theorem 3.2. *Let f_X and f_Y be continuous real-valued functions defined on X and Y respectively such that $f_X \geq 0$ and $f_Y \geq 0$. Then*

$$P(T, f_X) \leq ||f_X|| + P(S, f_Y) + \sup_{y \in Y} P(T, f_X, \pi^{-1} y).$$

Proof. Let $\epsilon > 0$, $\alpha > 0$ and integer $n > 0$ be arbitrary and $a = \sup_y P(T, f_X, \pi^{-1}y)$. Also let $s_n(T, 4\epsilon, X)$ denote the cardinality of a $(n, 4\epsilon)$-separated set of X with maximum cardinality. Then Lemmas 2.4 and 3.1 give us

$$\begin{aligned} P_n(T, f_X, 8\epsilon, X) &\leq e^{n\|f_X\|} s_n(T, 4\epsilon, X) \\ &\leq e^{n\|f_X\|} r_n(S, \delta, Y) e^{(n+M)(a+\alpha)} \\ &\leq e^{n\|f_X\|} Q_n(S, f_Y, \delta, Y) e^{(n+M)(a+\alpha)}, \end{aligned}$$

where the last line follows from the positivity of f_Y.
Hence

$$\log P_n(T, f_X, 8\epsilon, X) \leq n\|f_X\| + (n+M)(a+\alpha) + \log Q_n(S, f_Y, \delta, Y).$$

In particular, on dividing by n and taking limit superior, we have as $\epsilon \to 0$

$$P(T, f_X) \leq \|f_X\| + P(S, f_Y) + a + \alpha.$$

The result then follows since α is arbitrary. □

Corollary 3.3. *Let X and Y be compact metric spaces and $T\colon X \to X$ and $\pi\colon X \to Y$ be continuous such that $\pi \circ T = \pi$. Moreover let $f \geq 0$ be continuous on X.Then*

$$\sup_{y\in Y} P(T, f, \pi^{-1}y) \leq P(T, f) \leq ||f|| + \sup_{y\in Y} P(T, f, \pi^{-1}y).$$

Proof. The first inequality follows from Prop. 2.3. Then in the above theorem take $S = Id$, $f_Y = 0$ and $f_X = f$. □

By taking $f = 0$ in the above corollary, we have

Corollary 3.4. *Let X and Y be compact metric spaces and $T\colon X \to X$ and $\pi\colon X \to Y$ be continuous such that $\pi \circ T = \pi$. Then*

$$h(T) = \sup_{y\in Y} h(T, \pi^{-1}y).$$

The last corollary is contained in [1, Corollary 18].

4. $(G, \tau)-$EXTENSIONS

As in the introduction let $T\colon X \to X$ be a (G, τ) - extension of $S\colon Y \to Y$. For the rest of this paper let f_Y and f_G be positive real-valued functions defined on Y and G respectively. Now define the function $f\colon X \to \mathcal{R}$ as $f(xg) = f_Y(xG) + f_G(g)$ so that f is also positive and continuous.

Lemma 4.1. *The function f is well-defined.*

Proof. Let $xg = zg'$ where $x, z \in X$ and $g, g' \in G$. Consider the projection map $\pi\colon X \to Y$. Then $\pi(xg) = \pi(zg')$ implies $xG = zG$ so that $x = z\overline{g}$ for some $\overline{g} \in G$. Hence $f(xg) = f(z\overline{g}g) = f_Y(zG) + f_G(\overline{g}g)$. Also $f(zg') = f_Y(zG) + f_G(g')$. And this implies $f(xg) = f(zg')$ if $f_G(\overline{g}g) = f_G(g')$. But this is true since $x = z\overline{g}$ implies $z\overline{g}g = zg'$ and in turn by virtue of free acting this implies $\overline{g}g = g'$. In other words $f_G(\overline{g}g) = f_G(g')$ and this completes the proof. □

Let $y \in Y$. Then recall that $Q_n(T, f, \epsilon, \pi^{-1}y)$ is defined as

$$Q_n(T, f, \epsilon, \pi^{-1}y) = \inf\left\{\sum_{x\in F} e^{S_n f(x)} \colon F\ (n, \epsilon) - \textit{spans}\ \pi^{-1}y\right\}.$$

We have

Proposition 4.2. *Given $y \in Y$, integer $n \geq 1$ and $\epsilon > 0$,*

$$Q_n(T, f, \epsilon, \pi^{-1}y) \leq e^{n||f_Y||} Q_n(\tau, f_G, \delta)$$

for some $\delta > 0$.

Proof. Let d and d' be the metrics associated with X and G respectively. Then since G acts continuously on X, we have by uniform continuity, there exists $\delta > 0$ such that $d(xg, xg') \leq \epsilon$ whenever $d'(g, g') \leq \delta$. Now let $x \in \pi^{-1}y$ and E_n be a (n, δ)-spanning set for G. Then it is easy to check that xE_n is a (n, ϵ)-spanning set for $\pi^{-1}y$. Observe that by commutativity of T and S (via π) and the relation $T(xg) = T(x)\tau(g)$ we have

$$e^{S_n f(xg)} = e^{S_n f_Y(xG)} e^{S_n f_G(g)}$$

with respect to the appropriate maps T, S and τ, so that

$$\sum_{g \in E_n} e^{S_n f(xg)} = e^{S_n f_Y(xG)} \sum_{g \in E_n} e^{S_n f_G(g)}$$

or

$$\sum_{xg \in zE_n} e^{S_n f(xg)} = e^{S_n f_Y(zG)} \sum_{g \in E_n} e^{S_n f_G(g)}.$$

Therefore

$$Q_n(T, f, \epsilon, \pi^{-1}y) \leq e^{S_n f_Y(xG)} Q_n(\tau, f_G, \delta).$$

Note that the above manipulation is independent of $x \in \pi^{-1}y$ since if $x' \in \pi^{-1}y$ then $x' = xg$ for some $g \in G$ so that $x'G = xG$ which in turn implies $S_n f_Y(x'G) = S_n f_Y(xG)$. The result follows since $e^{S_n f_Y(xG)} \leq e^{n\|f_Y\|}$. □

Theorem 4.3. *Given $y \in Y$, we have*

$$P(T, f, \pi^{-1}y) \leq \|f_Y\| + P(\tau, f_G).$$

Proof. From Proposition 4.2 we have

$$Q_n(T, f, \epsilon, \pi^{-1}y) \leq e^{n||f_Y||} Q_n(\tau, f_G, \delta).$$

Therefore

$$\limsup \frac{1}{n} \log Q_n(T, f, \epsilon, \pi^{-1}y) \leq \|f_Y\| + \limsup \frac{1}{n} \log Q_n(\tau, f_G, \delta),$$

that is

$$Q(T, f, \epsilon, \pi^{-1}y) \leq ||f_Y|| + Q(\tau, f_G, \delta).$$

The result follows by taking $\epsilon \to 0$ and $\delta \to 0$. □

Combining Theorem 3.2 and Theorem 4.3 and taking $f = f_X$, we have

Proposition 4.4.

$$P(T, f) \leq \|f\| + \|f_Y\| + P(S, f_Y) + P(\tau, f_G)$$

We also have:

Proposition 4.5.

$$P(T, f) \geq P(S, f_Y) + P(\tau, f_G)$$

Proof. Let $\epsilon > 0$ and let d'' be the metric on Y. Then by the uniform continuity of π and the fact that G-acts freely on X there exists some $\delta > 0$ such that
a. $d''(\pi(x), \pi(z)) \leq \epsilon$ when $d(x, z) \leq \delta$
b. $d(xg, xg') > \delta$ when $x \in X$ and $d(g, g') > \epsilon$.
Now let $G_n \subset G$ be (n, ϵ)-separated and $Y_n \subset Y$ be (n, ϵ)-separated and choose $X_n \subset X$ so that $\pi_{|X_n} : X_n \to Y_n$ is a bijection. Then $X_n G_n$ is (n, δ)-separated. Thus

$$P_n(T, f, \delta) \geq \sum_{xg \in X_n G_n} e^{(S_n f)(xg)} = P_n(S, f_Y, \epsilon) \cdot P(\tau, f_G, \epsilon).$$

In particular

$$P(T, f, \delta) \geq P(S, f_Y, \epsilon) + P(\tau, f_G, \epsilon).$$

The result follows by taking $\epsilon \to \infty$ and $\delta \to \infty$. □

Corollary 4.6.

$$P(S, f_Y) + P(\tau, f_G) \leq P(T, f) \leq \|f\| + \|f_Y\| + P(S, f_Y) + P(\tau, f_G).$$

And by taking $f_Y \equiv 0 \equiv f_G$, we recover Bowen's formula

Corollary 4.7.

$$h(T) = h(S) + h(\tau).$$

5. FINAL REMARKS

When f_Y and f_G are both constant, by using the variational formula for pressure and Bowen's formula, it is easy to deduce that $P(T, f) = P(S, f_Y) + P(\tau, f_G)$, i.e., equality holds in this trivial case.

Perhaps, a non-trivial example supporting the equality is as follows:

Example 5.1. Let $Y = \{-1, 1\}^{\mathbb{Z}}$ and $\sigma \colon Y \to Y$ be the full two-shift. Consider the group extension given by

$$\hat{\sigma} \colon Y \times \mathbb{Z}_3 \to Y \times \mathbb{Z}_3$$
$$(y, g) \mapsto (\sigma y, (g + 2y_0) \text{ mod } 3).$$

Of course, in this case $\tau = \text{Id}$. Also, let $f(y, g) = f_Y(y) + f_G(g)$, where $f_Y(y) = 0$ if $y_0 = -1$, $f_Y(y) = 1$ if $y_0 = 1$ and $f_G = 2$, constant. Then one can easily check that $P(\sigma, f_Y) = \log(1+e)$ and $P(\text{Id}, f_G) = 2$. Moreover it is not difficult to see that $P(\hat{\sigma}, f) = \log(e^2 + e^3)$. In particular, we have $P(\hat{\sigma}, f) = P(\sigma, f_Y) + P(\text{Id}, f_G)$.

We end with the following conjecture:

Conjecture 5.1. *Let $T \colon X \to X$ be a (free) (G, τ)–extension of $S \colon Y \to Y$ such that T has finite topological entropy. Also let $f \colon G \to \mathbb{R}$ be defined as $f(xg) = f_Y(xG) + f_G(g)$ where f_Y, f_G are positive real-valued functions on Y and G respectively. Then*

$$P(T, f) = P(S, f_Y) + P(\tau, f_G).$$

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