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# ON ESTIMATES OF THE GENERALIZED JORDAN-VON NEUMANN CONSTANT OF BANACH SPACES

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## Abstract

In this paper, we study the generalized Jordan-von Neumann constant and obtain its estimates for the normal structure coefficient $N(X)$, improving the known results of S. Dhompongsa.



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## Contents

# 1. Introduction

It is well known that normal structure and uniform normal structure play an important role in fixed point theory. So it is worthwhile studying the relationship between uniform normal structure and other geometrical constants of Banach spaces. Recently J. Gao [5] proved that $\delta(1+\epsilon) > \epsilon/2$ implies that a Banach space $X$ has uniform normal structure. Kato et al. [6] obtained

$$N(X) \geq \left(C_{\mathrm{NJ}}(X) - \frac{1}{4}\right)^{-\frac{1}{2}}, \tag{1.1}$$

which implies that $X$ has uniform normal structure if $C_{\mathrm{NJ}}(X) < 5/4$. S. Dhompongsa et al. [3, 4] proved that $C_{\mathrm{NJ}}(X) < (3+\sqrt{5})/4$ or $C_{\mathrm{NJ}}(a, X) < (1+a)^2/(1+a^2)$ for some $a \in [0,1]$ implies that $X$ has uniform normal structure. However $C_{\mathrm{NJ}}(a, X) < (1+a)^2/(1+a^2)$ is not a sharp condition for $X$ to have uniform normal structure especially when $a$ is close to 0. Our aim is to improve the result of S. Dhompongsa.

We shall assume throughout this paper that $X$ is a Banach space and $X^*$ its dual space. We will use $S_X$ to denote the unit sphere of $X$. A Banach space $X$ is called non-trivial if $\dim X \geq 2$. A Banach space $X$ is called *uniformly nonsquare* if for any $x, y \in S_X$ there exists $\delta > 0$, such that either $\|x-y\|/2 \leq 1-\delta$, or $\|x+y\|/2 \leq 1-\delta$. Uniformly nonsquare spaces are superreflexive. Let $C$ be a nonempty bounded convex subset of $X$. The number $\operatorname{diam} C = \sup\{\|x-y\| : x, y \in C\}$ is called the *diameter* of $C$. The number $r(C) = \inf\{\sup\{\|x-y\| : x \in C\} : y \in C\}$ is called the *Chebyshev radius* of $C$. By $Z(C)$ we will denote the set of all $x \in C$ at which this infimum is attained. It is called the *Chebyshev center* of $C$. Bynum [2] introduced the following normal structure coefficient

$$N(X) = \inf\{\mathrm{diam} C\}, \tag{1.2}$$

where the infimum is taken over all closed convex subsets $C$ of $X$ with $r(C) = 1$. Obviously $1 \leq N(X) \leq 2$ and $X$ is said to have *uniform normal structure* provided $N(X) > 1$. Moreover if $X$ is reflexive, then the infimum in the definition of $N(X)$ may as well be taken over all convex hulls of finite subsets of $X$ [1]. In connection with a famous work of Jordan-von Neumann concerning inner products, the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of $X$ was introduced by Clarkson as the smallest constant $C$ for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

holds for all $x, y$ with $(x, y) \neq (0, 0)$. If $C$ is the best possible in the right hand side of the above inequality then so is $1/C$ on the left. Hence

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ not both zero}\right\}. \tag{1.3}$$

The statements without explicit reference have been taken from Kato et al. [6]. In [3] S. Dhompongsa generalized this definition in the following sense.

$$\begin{aligned} &C_{\mathrm{NJ}}(a, X) \\ &= \sup\left\{\frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x, y, z \in X \text{ not all zero and } \|y-z\| \leq a\|x\|\right\} \end{aligned} \tag{1.4}$$

where $a$ is a nonnegative parameter. Obviously, $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(0, X)$.

# 2. Main Results

Our proofs are based on an idea due to S. Prus [7]. Let $C$ be a convex hull of a finite subset of $X$. Since $C$ is compact, there exists an element $y \in C$ such that

$$\sup_{x \in C} \|x - y\| = r(C). \tag{2.1}$$

Translating the set $C$ we can assume that $y = 0$. The following result is [7, Theorem 2.1].

**Proposition 2.1.** *Let $C$ be a nonempty compact convex subset of a finite dimensional Banach space $X$ and $x_0 \in C$. If $x_0 \in Z(C)$, then there exist elements $x_1 \dots, x_n \in C$, functionals $x_1^*, \dots, x_n^* \in S_{X^*}$, and nonnegative scalars $\lambda_1, \dots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i = 1$,*

$$x_i^*(x_0 - x_i) = \|x_0 - x_i\| = r(C)$$

*for $i = 1, \dots, n$ and*

$$\sum_{i=1}^n \lambda_i x_i^*(x - x_0) \geq 0$$

*for every $x \in C$.*

**Theorem 2.2.** *Let $X$ be a non-trivial Banach space with the normal structure constant $N(X)$. Then for each $a$, $0 \leq a \leq 1$,*

$$N(X) \geq \sqrt{\frac{\max_{r \in [a,1]} f(r)}{C_{\mathrm{NJ}}(a, X)}}, \tag{2.2}$$

*where*

$$f(r) = \frac{(1+r)^2 + (1+a)^2}{2(1+r^2)}, \qquad r \in [a,1].$$

*Proof.*
**Case 1:** If $C_{\mathrm{NJ}}(a,X) = 2$, it suffices to note that

$$\max_{a \le r \le 1} f(r) = \max_{a \le r \le 1} \frac{(1+r)^2 + (1+a)^2}{2(1+r^2)} \le \max_{a \le r \le 1} \frac{(1+r)^2 + (1+r)^2}{2(1+r^2)} \le 2.$$

In this case our estimate is a trivial one.
**Case 2:** If $C_{\mathrm{NJ}}(a,X) < 2$, then $X$ is uniformly nonsquare and therefore reflexive [3]. Now let $C$ be a convex hull of a finite subset of $X$ such that $r(C) = 1$ and $\operatorname{diam} C = d$. We can assume that $\sup\{\|x\| : x \in C\} = 1$ and by Proposition 2.1 we get elements $x_1 \dots, x_n$, norm-one functionals $x_1^*, \dots, x_n^*$ and non-negative numbers $\lambda_1, \dots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i = 1$, $x_i^*(-x_i) = \|x_i\| = 1$ for $i = 1, \dots, n$ and $\sum_{i=1}^n \lambda_i x_i^*(x_j) \ge 0$ for $j = 1, \dots, n$. For any $r \in [a,1]$, let us set

$$x_{i,j} = \frac{x_i - x_j}{d}, \quad y_{i,j} = \frac{r}{d} x_i, \quad z_{i,j} = \frac{(r-a)x_i + a x_j}{d} \quad \text{for} \quad i,j = 1, \dots, n.$$

Obviously $\|x_{i,j}\| \le 1,\ \|y_{i,j}\| \le r,\ \|z_{i,j}\| \le r$, and $\|y_{i,j} - z_{i,j}\| = a\|x_{i,j}\|$. It follows that

$$\sum_{i,j=1}^n \lambda_i \lambda_j \left[\|x_{i,j} + y_{i,j}\|^2 + \|x_{i,j} - z_{i,j}\|^2\right]$$
$$\ge \sum_{j=1}^n \lambda_j \sum_{i=1}^n \lambda_i [x_i^*(x_{i,j} + y_{i,j})]^2 + \sum_{i=1}^n \lambda_i \sum_{j=1}^n \lambda_j [x_j^*(x_{i,j} - z_{i,j})]^2$$

$$= \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i \left[\frac{1+r}{d} + \frac{1}{d} x_i^*(x_j)\right]^2 + \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i \left[\frac{1+a}{d} + \frac{1+a-r}{d} x_j^*(x_i)\right]^2$$

$$= \frac{(1+r)^2}{d^2} + \frac{2(1+r)}{d^2} \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i x_i^*(x_j) + (1/d^2) \sum_{j=1}^{n} \lambda_j \sum_{i=1}^{n} \lambda_i [x_i^*(x_j)]^2$$

$$+ \frac{(1+a)^2}{d^2} + \frac{2(1+a)(1+a-r)}{d^2} \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \lambda_j x_j^*(x_i)$$

$$+ \frac{(1+a-r)^2}{d^2} \sum_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \lambda_j [x_j^*(x_i)]^2$$

$$\geq \frac{(1+r)^2}{d^2} + \frac{(1+a)^2}{d^2} \qquad \text{for any } r \in [a,1].$$

Therefore there exist $i,\ j$ such that

$$\|x_{i,j} + y_{i,j}\|^2 + \|x_{i,j} - z_{i,j}\|^2 \geq \frac{(1+r)^2}{d^2} + \frac{(1+a)^2}{d^2}.$$

From the definition of the generalized Jordan-von Neumann constant we obtain that

$$C_{\mathrm{NJ}}(a,X) \geq \frac{\|x_{i,j} + y_{i,j}\|^2 + \|x_{i,j} - z_{i,j}\|^2}{2\|x_{i,j}\|^2 + \|y_{i,j}\|^2 + \|z_{i,j}\|^2} \geq \frac{(1+r)^2 + (1+a)^2}{2(1+r^2)d^2},$$

which implies

$$d \geq \sqrt{\frac{\max_{r\in[a,1]} f(r)}{C_{\mathrm{NJ}}(a,X)}}.$$

Since $C$ is arbitrary, we obtain the desired estimate (2.2). $\square$

**Lemma 2.3.** *Let* $0 \leq a \leq 1$ *and* $r_0 = \left(\sqrt{4+(1+a)^4} - (1+a)^2\right) \Big/ 2$. *Then* $a \leq r_0$ *if* $a \in \left[0, \sqrt{2}-1\right]$ *and* $a \geq r_0$ *if* $a \in \left[\sqrt{2}-1, 1\right]$.

*Proof.* If $a \in [0, \sqrt{2}-1]$ then

$$\begin{aligned} 4+(1+a)^4 - [(1+a)^2+2a]^2 &= 4(1-a-3a^2-a^3) \\ &= -4(a+1)\left(a+1+\sqrt{2}\right)\left(a+1-\sqrt{2}\right) \\ &\geq 0, \end{aligned}$$

which implies $\sqrt{4+(1+a)^4} \geq (1+a)^2 + 2a$. Therefore

$$r_0 - a = \frac{\sqrt{4+(1+a)^4} - (1+a)^2}{2} - a \geq 0.$$

Thus we obtain that $r_0 \geq a$ if $a \in [0, \sqrt{2}-1]$. Similarly we get $r_0 \leq a$ if $a \in [\sqrt{2}-1, 1]$. ☐

**Theorem 2.4.** *Let* $X$ *be a non-trivial Banach space with the generalized Jordan-von Neumann constant* $C_{\mathrm{NJ}}(a,X)$. *If*

$$C_{\mathrm{NJ}}(a,X) < \frac{2+(1+a)^2+\sqrt{4+(1+a)^4}}{4} \quad \text{for some } a \in \left[0, \sqrt{2}-1\right], \tag{2.3}$$

*or*

$$C_{\mathrm{NJ}}(a,X) < \frac{(1+a)^2}{1+a^2} \quad \text{for some } a \in \left[\sqrt{2}-1, 1\right], \tag{2.4}$$

*then $X$ has uniform normal structure.*

*Proof.* Let

$$f(r) := \frac{(1+r)^2+(1+a)^2}{2(1+r^2)}, \qquad r_0 = \frac{\sqrt{4+(1+a)^4}-(1+a)^2}{2}.$$

First we note that $f(r)$ is increasing on $[0, r_0]$, and decreasing on $[r_0, 1]$.
**Case 1:** If $a \in \left[0, \sqrt{2}-1\right]$, then $r_0 \in [a, 1]$ by Lemma 2.3, which implies

$$\max_{r\in[a,1]} f(r) = f(r_0) = \frac{2+(1+a)^2+\sqrt{4+(1+a)^4}}{4}.$$

By (2.2) and (2.3) we obtain that

$$N(X) \geq \sqrt{\frac{\max_{r\in[a,1]} f(r)}{C_{\mathrm{NJ}}(a,X)}} > 1$$

and hence $X$ has uniform normal structure.
**Case 2:** If $a \in \left[\sqrt{2}-1, 1\right]$, then $r_0 \leq a$ by Lemma 2.3 and thus $f(r)$ is decreasing on $[a, 1]$, which implies

$$\max_{r\in[a,1]} f(r) = f(a) = \frac{(1+a)^2}{1+a^2}.$$

By (2.2) and (2.4) we obtain that

$$N(X) \geq \sqrt{\frac{\max_{r\in[a,1]} f(r)}{C_{\mathrm{NJ}}(a,X)}} > 1$$

and hence $X$ has uniform normal structure. □

Note that

$$\frac{2+(1+a)^2+\sqrt{4+(1+a)^4}}{4} > \frac{(1+a)^2}{1+a^2} \qquad \text{for all } a \in \left[0, \sqrt{2}-1\right).$$

Thus this gives a strong improvement of [3, Theorem 3.6] and [4, Corollary 3.8].

**Corollary 2.5 ([3, Theorem 3.6]).** *$X$ has uniform normal structure if* $C_{\mathrm{NJ}}(X) < (3+\sqrt{5})/4$.

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