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ON QUADRATURE RULES, INEQUALITIES AND ERROR BOUNDS

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ABSTRACT. The order structure of the set of six operators connected with quadrature rules is established in the class of 5–convex functions. An error bound of the Lobatto quadrature rule with five knots is given for less regular functions as in the classical result.

Key words and phrases: Approximate integration, Quadrature rules, Convex functions of higher order.

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1. INTRODUCTION

For $f : [-1,1] \to \mathbb{R}$ we consider six operators approximating the integral mean value $\frac{1}{2}\int_{-1}^{1} f(x)dx$. They are

$$\mathcal{C}(f) := \frac{1}{3}\left(f\left(-\frac{\sqrt{2}}{2}\right) + f(0) + f\left(\frac{\sqrt{2}}{2}\right)\right),$$
$$\mathcal{G}_2(f) := \frac{1}{2}\left(f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)\right),$$
$$\mathcal{G}_3(f) := \frac{4}{9}f(0) + \frac{5}{18}\left(f\left(-\frac{\sqrt{15}}{5}\right) + f\left(\frac{\sqrt{15}}{5}\right)\right),$$
$$\mathcal{L}_4(f) := \frac{1}{12}\left(f(-1)+f(1)\right) + \frac{5}{12}\left(f\left(-\frac{\sqrt{5}}{5}\right) + f\left(\frac{\sqrt{5}}{5}\right)\right),$$
$$\mathcal{L}_5(f) := \frac{16}{45}f(0) + \frac{1}{20}\left(f(-1)+f(1)\right) + \frac{49}{180}\left(f\left(-\frac{\sqrt{21}}{7}\right) + f\left(\frac{\sqrt{21}}{7}\right)\right),$$
$$\mathcal{S}(f) := \frac{1}{6}\left(f(-1)+f(1)\right) + \frac{2}{3}f(0).$$

All of them are connected with the very well known rules of approximate integration: Chebyshev quadrature, Gauss–Legendre quadrature with two and three knots, Lobatto quadrature with four and five knots and Simpson's Rule, respectively (see e.g. [4, 8, 9, 10, 11]).

In the paper [6] the order structure of the set of above operators was investigated in the class of 3–convex functions. In this note we establish all possible inequalities between these operators in the class of 5–convex functions. As an application we give an error bound of the operator $\mathcal{L}_5$

190-07

for six times differentiable functions instead of eight times differentiable ones as in the classical result.

In this paper only 5–convex functions on $[-1,1]$ are considered. Recall that the function $f:[-1,1]\to\mathbb{R}$ is called 5–*convex* if

$$D(x_1,\dots,x_7;f):=\begin{vmatrix} 1 & \dots & 1\\ x_1 & \dots & x_7\\ x_1^2 & \dots & x_7^2\\ x_1^3 & \dots & x_7^3\\ x_1^4 & \dots & x_7^4\\ x_1^5 & \dots & x_7^5\\ f(x_1) & \dots & f(x_7)\end{vmatrix}\geq 0 \tag{1.1}$$

for any $x_1,\dots,x_7$ such that $-1\leq x_1<\dots<x_7\leq 1$. More detailed introductory notes concerning higher–order convexity were given in [6]. For a wide treatment of this topic we refer the reader to Popoviciu's thesis [3], the very well known books [2] and [5] and to Hopf's thesis [1], where it appeared (without the name) for the first time.

2. RESULTS

Let us start with four technical results.

Lemma 2.1. *If* $f:[-1,1]\to\mathbb{R}$ *is an even 5–convex function then*

$$\begin{aligned}(w^2-u^2)(v^2-u^2)(w^2-v^2)f(0)+u^2w^2(w^2-u^2)f(v)&\\ \leq w^2v^2(w^2-v^2)f(u)+v^2u^2(v^2-u^2)f(w)&\end{aligned}$$

for any $0<u<v<w\leq 1$.

Proof. Fix $0<u<v<w\leq 1$. By 5–convexity, $D(-w,-v,-u,0,u,v,w;f)\geq 0$. Expand this determinant by the last row and perform elementary computations on Vandermonde determinants. □

Lemma 2.2. *If* $f:[-1,1]\to\mathbb{R}$ *is 5–convex then so is the function* $[-1,1]\ni x\mapsto f(-x)$.

Proof. This result is well known from the theory of convex functions of higher order and it holds in fact for convex functions of any odd order (cf. e.g. [3]). However, the proof is easy if we use the condition (1.1) and elementary properties of determinants. □

By (1.1) it is obvious that a sum of two 5–convex functions is also 5–convex. Then we have the following.

Lemma 2.3. *If* $f:[-1,1]\to\mathbb{R}$ *is 5–convex then so is its even part, i.e. the function*

$$f_e(x)=\frac{f(x)+f(-x)}{2},\quad x\in[-1,1].$$

Record also the trivial

Lemma 2.4. *If* $\mathcal{T}\in\{\mathcal{C},\mathcal{G}_2,\mathcal{G}_3,\mathcal{L}_4,\mathcal{L}_5,\mathcal{S}\}$ *then* $\mathcal{T}(f)=\mathcal{T}(f_e)$ *for any* $f:[-1,1]\to\mathbb{R}$.

Now we establish all possible inequalities between the considered operators in the class of 5–convex functions.

Theorem 2.5. *If* $f:[-1,1]\to\mathbb{R}$ *is 5–convex then* $\mathcal{G}_3(f)\leq\mathcal{L}_5(f)\leq\mathcal{L}_4(f)$. *In the class of 5–convex functions the operators* $\mathcal{G}_2$, $\mathcal{C}$, $\mathcal{S}$ *are not comparable both with each other and with* $\mathcal{G}_3$, $\mathcal{L}_4$, $\mathcal{L}_5$.

Proof. By Lemmas 2.3 and 2.4, it is enough to prove the desired inequalities for even 5–convex functions. Using Lemma 2.1 for $u = \frac{\sqrt{21}}{7}$, $v = \frac{\sqrt{15}}{5}$, $w = 1$ we obtain $\mathcal{G}_3(f) \le \mathcal{L}_5(f)$. The inequality $\mathcal{L}_5(f) \le \mathcal{L}_4(f)$ we get for $u = \frac{\sqrt{5}}{5}$, $v = \frac{\sqrt{21}}{7}$, $w = 1$.

Now let $f = \exp$, $g = 1 - \cos$. Both functions are 5–convex on $[-1, 1]$ since their derivatives of the sixth order are nonnegative on this interval (cf. [2, 3, 5], for a quick reference cf. also [7]). See the table below.

Operator	$\mathcal{G}_2$	$\mathcal{C}$	$\mathcal{S}$	$\mathcal{G}_3$	$\mathcal{L}_5$	$\mathcal{L}_4$
f	1.17135	1.17373	1.18103	1.17517	1.17520	1.17524
g	0.16209	0.15984	0.15323	0.15850	0.15853	0.15857

Then

$$\mathcal{G}_2(f) < \mathcal{C}(f) < \mathcal{G}_3(f) < \mathcal{L}_5(f) < \mathcal{L}_4(f) < \mathcal{S}(f),$$
$$\mathcal{S}(g) < \mathcal{G}_3(g) < \mathcal{L}_5(g) < \mathcal{L}_4(g) < \mathcal{C}(g) < \mathcal{G}_2(g),$$

which proves the second part of the statement. □

Remark 1. By the example given in the above proof one could expect that the inequality

$$\min\{\mathcal{G}_2, \mathcal{C}, \mathcal{S}\} \le \mathcal{G}_3 \le \mathcal{L}_5 \le \mathcal{L}_4 \le \max\{\mathcal{G}_2, \mathcal{C}, \mathcal{S}\}$$

holds in the class of 5–convex functions. However this is not the case since for a 5–convex function $h(x) = x^6 - \frac{3}{2}x^4 + \frac{1}{6}$ we have

$$\mathcal{G}_2(h) = \frac{1}{27}, \quad \mathcal{C}(h) = \mathcal{S}(h) = 0, \quad \mathcal{G}_3(h) = -\frac{1}{75}, \quad \mathcal{L}_5(h) = \frac{1}{105}, \quad \mathcal{L}_4(h) = \frac{1}{25},$$

so

$$\mathcal{G}_3(h) < \mathcal{C}(h) = \mathcal{S}(h) < \mathcal{L}_5(h) < \mathcal{G}_2(h) < \mathcal{L}_4(h).$$

Let us comment on the results of Theorem 2.5. The set $\{\mathcal{C}, \mathcal{G}_2, \mathcal{G}_3, \mathcal{L}_4, \mathcal{L}_5, \mathcal{S}\}$ has 15 two–element subsets. That is why maximally 15 inequalities may be established between the operators considered. For 3–convex functions we have proved in [6] that 12 inequalities hold true and only 3 fail. We can see that for 5–convex functions the situation is quite different: only 3 inequalities are true, the rest are false. Moreover, the operators $\mathcal{G}_2$, $\mathcal{C}$, $\mathcal{S}$ comparable for 3–convex functions are not comparable for 5–convex ones, while the operators $\mathcal{G}_3$, $\mathcal{L}_4$, $\mathcal{L}_5$ comparable for 5–convex functions are not comparable for 3–convex ones.

The classical error bound of the quadrature $\mathcal{L}_5$ depends on the derivative of eighth order (cf. [4, 10]). Similarly to the results of the papers [6, 7] we give an error bound of this quadrature for less regular functions: in this paper for six–times differentiable functions. Let $\mathcal{I}(f) := \frac{1}{2}\int_{-1}^{1} f(x)dx$. For $f \in \mathcal{C}^6([-1, 1])$ denote

$$M(f) := \sup\left\{\left|f^{(6)}(x)\right| : x \in [-1, 1]\right\}.$$

Corollary 2.6. *If $f \in \mathcal{C}^6([-1, 1])$ then $|\mathcal{L}_5(f) - \mathcal{I}(f)| \le \frac{M(f)}{15750}$.*

Proof. It is well known (cf. [4, 9]) that if $f \in \mathcal{C}^6([-1, 1])$, then $\mathcal{I}(f) = \mathcal{G}_3(f) + \frac{f^{(6)}(\xi)}{31500}$ for some $\xi \in (-1, 1)$. Assume for a while that f is 5–convex. Hence by Theorem 2.5

$$\mathcal{I}(f) \le \mathcal{L}_5(f) + \frac{f^{(6)}(\xi)}{31500}.$$

Thus we arrive at

$$\mathcal{I}(f) - \mathcal{L}_5(f) \le \frac{M(f)}{31500}. \tag{2.1}$$

Now let $f \in \mathcal{C}^6([-1,1])$ be an arbitrary function and let $g(x) = \frac{M(f)x^6}{720}$. Then $\left|f^{(6)}\right| \le g^{(6)}$ on $[-1,1]$, whence $(g-f)^{(6)} \ge 0$ and $(g+f)^{(6)} \ge 0$ on $[-1,1]$. This implies that $g-f$ and $g+f$ are 5–convex on $[-1,1]$. It is easy to see that $M(g-f) \le 2M(f)$ and $M(g+f) \le 2M(f)$. Then we infer by 5–convexity and (2.1),

$$\mathcal{I}(g-f) - \mathcal{L}_5(g-f) \le \frac{M(g-f)}{31500} \le \frac{M(f)}{15750} \quad \text{and}$$

$$\mathcal{I}(g+f) - \mathcal{L}_5(g+f) \le \frac{M(g+f)}{31500} \le \frac{M(f)}{15750}.$$

It is easy to see that $\mathcal{I}(g) = \mathcal{L}_5(g)$. Since the operators $\mathcal{I}$, $\mathcal{L}_5$ are linear, then

$$-\mathcal{I}(f) + \mathcal{L}_5(f) \le \frac{M(f)}{15750} \quad \text{and} \quad \mathcal{I}(f) - \mathcal{L}_5(f) \le \frac{M(f)}{15750},$$

which concludes the proof. □

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