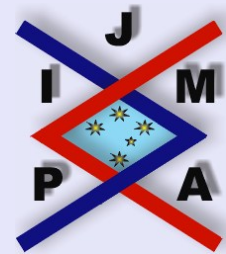


SUBORDINATION RESULTS FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY A LINEAR OPERATOR

B.A. FRASIN

Department of Mathematics
Al al-Bayt University
P.O. Box: 130095
Mafraq, Jordan

*E*Mail: bafrasin@yahoo.com



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Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

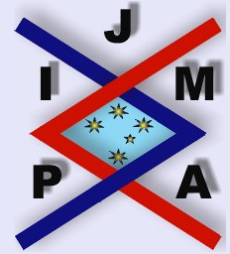
In this paper, we derive several interesting subordination results for certain class of analytic functions defined by the linear operator $\mathcal{L}(a, c)f(z)$ which introduced and studied by Carlson and Shaffer [2].

2000 Mathematics Subject Classification: Primary 30C45; Secondary 30A10, 30C80.

Key words: Analytic functions, Hadamard product, Subordinating factor sequence.

Contents

1	Introduction and Definitions	3
2	Main Theorem	8
	References	



Subordination Results for a Class of Analytic Functions Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 2 of 16

1. Introduction and Definitions

Let \mathcal{A} denote the class of functions of the form:

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\Delta = \{z : |z| < 1\}$. For two functions $f(z)$ and $g(z)$ given by

$$(1.2) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} c_n z^n$$

their Hadamard product (or convolution) is defined by

$$(1.3) \quad (f * g)(z) := z + \sum_{n=2}^{\infty} a_n c_n z^n.$$

Define the function $\phi(a, c; z)$ by

$$(1.4) \quad \phi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (c \notin \mathbb{Z}_0^- := \{0, -1, -2, \dots\}, z \in \Delta),$$

where $(\lambda)_n$ is the Pochhammer symbol given, in terms of Gamma functions,

$$(1.5) \quad (\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & n = 0, \\ \lambda(\lambda + 1)(\lambda + 2) \dots (\lambda + n - 1), & n \in \mathbb{N} : \{1, 2, \dots\}. \end{cases}$$



Subordination Results for a Class of Analytic Functions Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 3 of 16

Corresponding to the function $\phi(a, c; z)$, Carlson and Shaffer [2] introduced a linear operator $\mathcal{L}(a, c) : \mathcal{A} \rightarrow \mathcal{A}$ by

$$(1.6) \quad \mathcal{L}(a, c)f(z) := \phi(a, c; z) * f(z),$$

or, equivalently, by

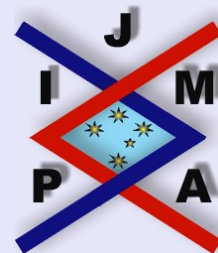
$$\mathcal{L}(a, c)f(z) := z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \quad (z \in \Delta).$$

Note that $\mathcal{L}(1, 1)f(z) = f(z)$, $\mathcal{L}(2, 1)f(z) = zf'(z)$ and $\mathcal{L}(3, 1)f(z) = zf'(z) + \frac{1}{2}z^2f''(z)$.

For $-1 \leq \alpha < 1$, $\beta \geq 0$, we let $\mathcal{L}(a, c; \alpha, \beta)$ consist of functions f in \mathcal{A} satisfying the condition

$$(1.7) \quad \operatorname{Re} \left\{ \frac{a\mathcal{L}(a+1, c)f(z)}{\mathcal{L}(a, c)f(z)} - (a-1) \right\} > \beta \left| \frac{a\mathcal{L}(a+1, c)f(z)}{\mathcal{L}(a, c)f(z)} - a \right| + \alpha, \quad (z \in \Delta)$$

The family $\mathcal{L}(a, c; \alpha, \beta)$ is of special interest for it contains many well-known as well as many new classes of analytic univalent functions. For $\mathcal{L}(1, 1; \alpha, 0)$, we obtain the family of starlike functions of order α ($0 \leq \alpha < 1$) and $\mathcal{L}(2, 1; \alpha, 0)$ is the family of convex functions of order α ($0 \leq \alpha < 1$). For $\mathcal{L}(1, 1; 0, \beta)$ and $\mathcal{L}(2, 1; 0, \beta)$, we obtain the class of uniformly β -starlike functions and uniformly β -convex functions, respectively, introduced by Kanas and Winsiowska



Subordination Results for a Class of Analytic Functions Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 4 of 16

([3],[4]) (see also the work of Kanas and Srivastava [5], Goodman ([7],[8]), Rønning ([10],[11]), Ma and Minda [9] and Gangadharan et al. [6]).

Before we state and prove our main result we need the following definitions and lemmas.

Definition 1.1 (Subordination Principle). Let $g(z)$ be analytic and univalent in Δ . If $f(z)$ is analytic in Δ , $f(0) = g(0)$, and $f(\Delta) \subset g(\Delta)$, then we see that the function $f(z)$ is subordinate to $g(z)$ in Δ , and we write $f(z) \prec g(z)$.

Definition 1.2 (Subordinating Factor Sequence). A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f(z)$ is analytic, univalent and convex in Δ , we have the subordination given by

$$(1.8) \quad \sum_{n=2}^{\infty} b_n a_n z^n \prec f(z) \quad (z \in \Delta, a_1 = 1).$$

Lemma 1.1 ([14]). The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if

$$(1.9) \quad \operatorname{Re} \left\{ 1 + 2 \sum_{n=1}^{\infty} b_n z^n \right\} > 0 \quad (z \in \Delta).$$

Lemma 1.2. If

$$(1.10) \quad \sum_{n=2}^{\infty} \sigma_n(a, c; \alpha, \beta) |a_n| \leq 1 - \alpha$$



Subordination Results for a
Class of Analytic Functions
Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 5 of 16

where, for convenience,

$$(1.11) \quad \sigma_n(a, c; \alpha, \beta) := \frac{(1 + \beta)(a)_n + [1 - \alpha - a(1 + \beta)](a)_{n-1}}{(c)_{n-1}}$$

$$(-1 \leq \alpha < 1; \beta \geq 0, n \geq 2),$$

then $f(z) \in \mathcal{L}(a, c; \alpha, \beta)$.

Proof. It suffices to show that

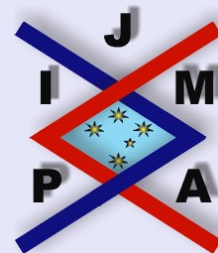
$$\beta \left| \frac{a\mathcal{L}(a+1, c)f(z)}{\mathcal{L}(a, c)f(z)} - a \right| - \operatorname{Re} \left\{ \frac{a\mathcal{L}(a+1, c)f(z)}{\mathcal{L}(a, c)f(z)} - a \right\} \leq 1 - \alpha.$$

We have

$$\begin{aligned} & \beta \left| \frac{a\mathcal{L}(a+1, c)f(z)}{\mathcal{L}(a, c)f(z)} - a \right| - \operatorname{Re} \left\{ \frac{a\mathcal{L}(a+1, c)f(z)}{\mathcal{L}(a, c)f(z)} - a \right\} \\ & \leq (1 + \beta) \left| \frac{a\mathcal{L}(a+1, c)f(z)}{\mathcal{L}(a, c)f(z)} - a \right| \\ & \leq \frac{(1 + \beta) \sum_{n=2}^{\infty} \left(\frac{a(a+1)_{n-1} - a(a)_{n-1}}{(c)_{n-1}} \right) |a_n| |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} |a_n| |z|^{n-1}} \\ & \leq \frac{(1 + \beta) \sum_{n=2}^{\infty} \left(\frac{(a)_n - a(a)_{n-1}}{(c)_{n-1}} \right) |a_n|}{1 - \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} |a_n|}. \end{aligned}$$

The last expression is bounded above by $1 - \alpha$ if

$$\sum_{n=2}^{\infty} \frac{(1 + \beta)(a)_n + [1 - \alpha - a(1 + \beta)](a)_{n-1}}{(c)_{n-1}} |a_n| \leq 1 - \alpha$$



Subordination Results for a Class of Analytic Functions Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 6 of 16

and the proof is complete. \square

Let $\mathcal{L}^*(a, c; \alpha, \beta)$ denote the class of functions $f(z) \in \mathcal{A}$ whose coefficients satisfy the condition (1.10). We note that $\mathcal{L}^*(a, c; \alpha, \beta) \subseteq \mathcal{L}(a, c; \alpha, \beta)$.



**Subordination Results for a
Class of Analytic Functions
Defined by a Linear Operator**

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 7 of 16

2. Main Theorem

Employing the techniques used earlier by Srivastava and Attiya [13], Attiya [1] and Singh [12], we state and prove the following theorem.

Theorem 2.1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^*(a, c; \alpha, \beta)$ where $-1 \leq \alpha < 1$; $\beta \geq 0$; $a > 0$; $c > 0$. Also let \mathcal{K} denote the familiar class of functions $f(z) \in \mathcal{A}$ which are also univalent and convex in Δ . Then*

$$(2.1) \quad \frac{\sigma_2(a, c; \alpha, \beta)}{2[1 - \alpha + \sigma_2(a, c; \alpha, \beta)]} (f * g)(z) \prec g(z) \quad (z \in \Delta; g \in \mathcal{K}),$$

and

$$(2.2) \quad \operatorname{Re}(f(z)) > -\frac{1 - \alpha + \sigma_2(a, c; \alpha, \beta)}{\sigma_2(a, c; \alpha, \beta)}, \quad (z \in \Delta).$$

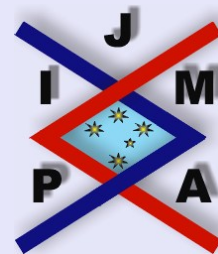
The constant $\frac{\sigma_2(a, c; \alpha, \beta)}{2[1 - \alpha + \sigma_2(a, c; \alpha, \beta)]}$ is the best estimate.

Proof. Let $f(z) \in \mathcal{L}^*(a, c; \alpha, \beta)$ and let $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. Then

$$(2.3) \quad \frac{\sigma_2(a, c; \alpha, \beta)}{2[1 - \alpha + \sigma_2(a, c; \alpha, \beta)]} (f * g)(z) \\ = \frac{\sigma_2(a, c; \alpha, \beta)}{2[1 - \alpha + \sigma_2(a, c; \alpha, \beta)]} \left(z + \sum_{n=2}^{\infty} a_n c_n z^n \right).$$

Thus, by Definition 1.2, the assertion of our theorem will hold if the sequence

$$(2.4) \quad \left\{ \frac{\sigma_2(a, c; \alpha, \beta)}{2[1 - \alpha + \sigma_2(a, c; \alpha, \beta)]} a_n \right\}_{n=1}^{\infty}$$



Subordination Results for a
Class of Analytic Functions
Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 8 of 16

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.1, this will be the case if and only if

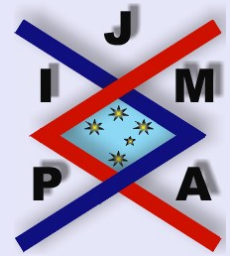
$$(2.5) \quad \operatorname{Re} \left\{ 1 + 2 \sum_{n=1}^{\infty} \frac{\sigma_2(a, c; \alpha, \beta)}{2[1 - \alpha + \sigma_2(a, c; \alpha, \beta)]} a_n z^n \right\} > 0 \quad (z \in \Delta).$$

Now

$$\begin{aligned} & \operatorname{Re} \left\{ 1 + \frac{\sigma_2(a, c; \alpha, \beta)}{1 - \alpha + \sigma_2(a, c; \alpha, \beta)} \sum_{n=1}^{\infty} a_n z^n \right\} \\ &= \operatorname{Re} \left\{ 1 + \frac{\sigma_2(a, c; \alpha, \beta)}{1 - \alpha + \sigma_2(a, c; \alpha, \beta)} z \right. \\ &\quad \left. + \frac{1}{1 - \alpha + \sigma_2(a, c; \alpha, \beta)} \sum_{n=1}^{\infty} \sigma_2(a, c; \alpha, \beta) a_n z^n \right\} \\ &\geq 1 - \left\{ \frac{\sigma_2(a, c; \alpha, \beta)}{1 - \alpha + \sigma_2(a, c; \alpha, \beta)} r \right. \\ &\quad \left. - \frac{1}{1 - \alpha + \sigma_2(a, c; \alpha, \beta)} \sum_{n=1}^{\infty} \sigma_n(a, c; \alpha, \beta) a_n r^n \right\}. \end{aligned}$$

Since $\sigma_n(a, c; \alpha, \beta)$ is an increasing function of n ($n \geq 2$)

$$1 - \left\{ \frac{\sigma_2(a, c; \alpha, \beta)}{1 - \alpha + \sigma_2(a, c; \alpha, \beta)} r \right. \\ \left. - \frac{1}{1 - \alpha + \sigma_2(a, c; \alpha, \beta)} \sum_{n=1}^{\infty} \sigma_n(a, c; \alpha, \beta) a_n r^n \right\}$$



Subordination Results for a Class of Analytic Functions Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 9 of 16

$$\begin{aligned}
 &> 1 - \frac{\sigma_2(a, c; \alpha, \beta)}{1 - \alpha + \sigma_2(a, c; \alpha, \beta)} r - \frac{1 - \alpha}{1 - \alpha + \sigma_2(a, c; \alpha, \beta)} r \quad (|z| = r) \\
 &> 0.
 \end{aligned}$$

Thus (2.5) holds true in Δ . This proves the inequality (2.1). The inequality (2.2) follows by taking the convex function $g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n$ in (2.1). To prove the sharpness of the constant $\frac{\sigma_2(a, c; \alpha, \beta)}{2[1 - \alpha + \sigma_2(a, c; \alpha, \beta)]}$, we consider the function $f_0(z) \in \mathcal{L}^*(a, c; \alpha, \beta)$ given by

$$(2.6) \quad f_0(z) = z - \frac{1 - \alpha}{\sigma_2(a, c; \alpha, \beta)} z^2 \quad (-1 \leq \alpha < 1; \beta \geq 0).$$

Thus from (2.1), we have

$$(2.7) \quad \frac{\sigma_2(a, c; \alpha, \beta)}{2[1 - \alpha + \sigma_2(a, c; \alpha, \beta)]} f_0(z) \prec \frac{z}{1-z}.$$

It can easily verified that

$$(2.8) \quad \min \left\{ \operatorname{Re} \left(\frac{\sigma_2(a, c; \alpha, \beta)}{2[1 - \alpha + \sigma_2(a, c; \alpha, \beta)]} f_0(z) \right) \right\} = -\frac{1}{2} \quad (z \in \Delta),$$

This shows that the constant $\frac{\sigma_2(a, c; \alpha, \beta)}{2[1 - \alpha + \sigma_2(a, c; \alpha, \beta)]}$ is best possible. \square

Corollary 2.2. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^*(1, 1; \alpha, \beta)$ and satisfy the condition

$$(2.9) \quad \sum_{n=2}^{\infty} [n(1 + \beta) - (\alpha + \beta)] |a_n| \leq 1 - \alpha$$



Subordination Results for a
Class of Analytic Functions
Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 10 of 16

then

$$(2.10) \quad \frac{\beta + 2 - \alpha}{2(\beta + 3 - 2\alpha)}(f * g)(z) \prec g(z) \\ (-1 \leq \alpha < 1; \beta \geq 0; z \in \Delta; g \in \mathcal{K})$$

and

$$(2.11) \quad \operatorname{Re}(f(z)) > -\frac{\beta + 3 - 2\alpha}{\beta + 2 - \alpha}, \quad (z \in \Delta).$$

The constant $\frac{\beta+2-\alpha}{2(\beta+3-2\alpha)}$ is the best estimate.

Corollary 2.3. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^*(1, 1; \alpha, 0)$ and satisfy the condition

$$(2.12) \quad \sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha,$$

then

$$(2.13) \quad \frac{2 - \alpha}{6 - 4\alpha}(f * g)(z) \prec g(z) \quad (z \in \Delta; g \in \mathcal{K})$$

and

$$(2.14) \quad \operatorname{Re}(f(z)) > -\frac{3 - 2\alpha}{2 - \alpha}, \quad (z \in \Delta).$$

The constant $\frac{2-\alpha}{6-4\alpha}$ is the best estimate.



Subordination Results for a
Class of Analytic Functions
Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 11 of 16

Putting $\alpha = 0$ in Corollary 2.3, we obtain

Corollary 2.4 ([12]). *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^*(1, 1; 0, 0)$ and satisfy the condition*

$$(2.15) \quad \sum_{n=2}^{\infty} n |a_n| \leq 1$$

then

$$(2.16) \quad \frac{1}{3}(f * g)(z) \prec g(z) \quad (z \in \Delta; g \in \mathcal{K})$$

and

$$(2.17) \quad \operatorname{Re}(f(z)) > -\frac{3}{2}, \quad (z \in \Delta).$$

The constant $1/3$ is the best estimate.

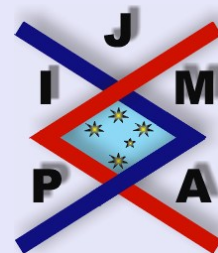
Corollary 2.5. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^*(2, 1; \alpha, \beta)$ and satisfy the condition*

$$(2.18) \quad \sum_{n=2}^{\infty} n[n(1 + \beta) - (\alpha + \beta)] |a_n| \leq 1 - \alpha,$$

then

$$(2.19) \quad \frac{\beta + 2 - \alpha}{2\beta + 5 - 3\alpha}(f * g)(z) \prec g(z)$$

$$(-1 \leq \alpha < 1; \beta \geq 0; z \in \Delta; g \in \mathcal{K})$$



Subordination Results for a Class of Analytic Functions Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 12 of 16

and

$$(2.20) \quad \operatorname{Re}(f(z)) > -\frac{2\beta + 5 - 3\alpha}{2(\beta + 2 - \alpha)}, \quad (z \in \Delta).$$

The constant $\frac{\beta+2-\alpha}{2\beta+5-3\alpha}$ is the best estimate.

Corollary 2.6. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^*(2, 1; \alpha, 0)$ and satisfy the condition

$$(2.21) \quad \sum_{n=2}^{\infty} n(n - \alpha) |a_n| \leq 1 - \alpha,$$

then

$$(2.22) \quad \frac{2 - \alpha}{5 - 3\alpha} (f * g)(z) \prec g(z) \quad (z \in \Delta; g \in \mathcal{K})$$

and

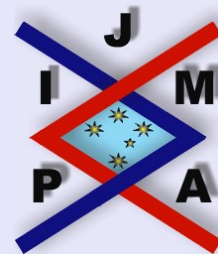
$$(2.23) \quad \operatorname{Re}(f(z)) > -\frac{5 - 3\alpha}{2(2 - \alpha)}, \quad (z \in \Delta).$$

The constant $\frac{2-\alpha}{5-3\alpha}$ is the best estimate.

Putting $\alpha = 0$ in Corollary 2.6, we obtain

Corollary 2.7. Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{L}^*(2, 1; 0, 0)$ and satisfy the condition

$$(2.24) \quad \sum_{n=2}^{\infty} n^2 |a_n| \leq 1$$



Subordination Results for a
Class of Analytic Functions
Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 13 of 16

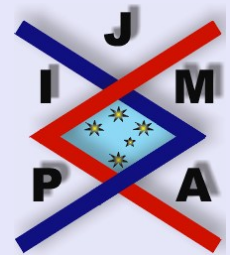
then

$$(2.25) \quad \frac{2}{5}(f * g)(z) \prec g(z) \quad (z \in \Delta; g \in \mathcal{K})$$

and

$$(2.26) \quad \operatorname{Re}(f(z)) > \frac{-5}{4}, \quad (z \in \Delta).$$

The constant $2/5$ is the best estimate.



**Subordination Results for a
Class of Analytic Functions
Defined by a Linear Operator**

B.A. Frasin

Title Page

Contents



Go Back

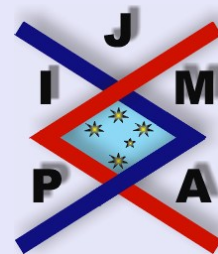
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Quit

Page 14 of 16

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Subordination Results for a
Class of Analytic Functions
Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



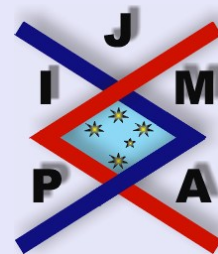
Go Back

Close

Quit

Page 15 of 16

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Subordination Results for a Class of Analytic Functions Defined by a Linear Operator

B.A. Frasin

Title Page

Contents



Go Back

Close

Quit

Page 16 of 16