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ON CHAOTIC ORDER OF INDEFINITE TYPE

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ABSTRACT. Let A, B be J-selfadjoint matrices with positive eigenvalues and $I \geqq^J A, I \geqq^J B$. Then it is proved as an application of Furuta inequality of indefinite type that

$$\mathrm{Log}\, A \geqq^J \mathrm{Log}\, B$$

if and only if

$$A^r \geqq^J (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$$

for all $p > 0$ and $r > 0$.

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In [2], T. Ando gave inequalities for matrices on an (indefinite) inner product space; for instance,

Proposition 1 ([2, Theorem 4]). *Let A, B be J-selfadjoint matrices with $\sigma(A), \sigma(B) \subsetneqq (\alpha, \beta)$. Then*

$$A \geqq^J B \Rightarrow f(A) \geqq^J f(B)$$

for any operator monotone function $f(t)$ on (α, β).

Since the principal branch $\mathrm{Log}\, x$ of the logarithm is operator monotone, as a corollary, we have

Corollary 2. *For J-selfadjoint matrices A, B with positive eigenvalues and $A \geqq^J B$, we have*

$$\mathrm{Log}\, A \geqq^J \mathrm{Log}\, B.$$

In this note, we give a characterization of this inequality relation, called a chaotic order, for J-selfadjoint matrices A, B with positive eigenvalues and $I \geqq^J A, I \geqq^J B$.

Before giving our theorem, we recall basic facts about matrices on an (indefinite) inner product space. We refer the reader to [3].

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Let $M_n(\mathbb{C})$ be the set of all complex n-square matrices acting on $\mathbb{C}^n$ and let $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{C}^n$; $\langle x,y\rangle := \sum_{i=1}^n x_i\overline{y_i}$ for $x=(x_i)$, $y=(y_i)\in\mathbb{C}^n$. For a selfadjoint involution $J\in M_n(\mathbb{C})$; $J=J^*$ and $J^2=I$, we consider the (indefinite) inner product $[\cdot,\cdot]$ on $\mathbb{C}^n$ given by

$$[x,y] := \langle Jx,y\rangle \quad (x,y\in\mathbb{C}^n).$$

The J-adjoint matrix $A^\sharp$ of $A\in M_n(\mathbb{C})$ is defined as

$$[Ax,y] = [x,A^\sharp y] \quad (x,y\in\mathbb{C}^n).$$

In other words, $A^\sharp = JA^*J$. A matrix $A\in M_n(\mathbb{C})$ is said to be J-selfadjoint if $A^\sharp = A$ or $JA^*J = A$. And for J-selfadjoint matrices A and B, the J-order, denoted as $A \geqq^J B$, is defined by

$$[Ax,x] \geqq [Bx,x] \quad (x\in\mathbb{C}^n).$$

A matrix $A\in M_n(\mathbb{C})$ is called J-positive if $A\geqq^J O$, or

$$[Ax,x]\geqq 0 \quad (x\in\mathbb{C}^n).$$

A matrix $A\in M_n(\mathbb{C})$ is said to be a J-contraction if $I\geqq^J A^\sharp A$ or $[x,x]\geqq[Ax,Ax]$ $(x\in\mathbb{C}^n)$. We remark that $I\geqq^J A$ implies that all eigenvalues of A are real. Hence, for a J-contraction A all eigenvalues of $A^\sharp A$ are real. In fact, by a result of Potapov-Ginzburg (see [3, Chapter 2, Section 4]), all eigenvalues of $A^\sharp A$ are non-negative.

We also recall facts in [6]:

Proposition 3 ([6, Theorem 2.6])**.** *Let A,B be J-selfadjoint matrices with non-negative eigenvalues and $0<\alpha<1$. If*

$$I\geqq^J A\geqq^J B,$$

then J-selfadjoint powers $A^\alpha,\ B^\alpha$ are well defined and

$$I\geqq^J A^\alpha\geqq^J B^\alpha.$$

Proposition 4 ([6, Lemma 3.1])**.** *Let A,B be J-selfadjoint matrices with non-negative eigenvalues and $I\geqq^J A$, $I\geqq^J B$. Then the eigenvalues of ABA are non-negative and*

$$I\geqq^J A^\lambda$$

for all $\lambda>0$.

We also have a generalization; Furuta inequality of indefinite type:

Proposition 5 ([6, Theorem 3.4])**.** *Let A,B be J-selfadjoint matrices with non-negative eigenvalues and $I\geqq^J A\geqq^J B$. For each $r\geqq 0$,*

$$\left(A^{\frac{r}{2}}A^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}\geqq^J\left(A^{\frac{r}{2}}B^pA^{\frac{r}{2}}\right)^{\frac{1}{q}}$$

holds for all $p\geqq 0, q\geqq 1$ with $(1+r)q\geqq p+r$.

Remark 6. Let $0<\alpha<1$. For J-selfadjoint matrices A,B with positive eigenvalues and $A\geqq^J B$, we have

$$A^\alpha\geqq^J B^\alpha,$$

by applying Proposition 1 to the operator monotone function x^α whose principal branch is considered. Hence,

$$\frac{A^\alpha-I}{\alpha}\geqq^J\frac{B^\alpha-I}{\alpha}.$$

We remark that A^α is given by the Dunford integral and that

$$\frac{A^\alpha-I}{\alpha}=\frac{1}{2\pi i}\int_C\frac{\zeta^\alpha-1}{\alpha}(\zeta I-A)^{-1}d\zeta,$$

where C is a closed rectifiable contour in the domain of ζ^α with positive direction surrounding all eigenvalues of A in its interior. Since

$$\frac{\zeta^\alpha - 1}{\alpha} \to Log\, \zeta \quad (\alpha \to 0)$$

uniformly for ζ, we also have Corollary 2.

Our theorem is as follows:

Theorem 7. *Let A, B be J-selfadjoint matrices with positive eigenvalues and $I \geqq^J A, I \geqq^J B$. Then the following statements are equivalent*:

(i) $\mathrm{Log}\, A \geqq^J \mathrm{Log}\, B$.
(ii) $A^r \geqq^J (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all* $p > 0$ *and* $r > 0$.

Here, principal branches of the functions are considered.

This theorem, as well as the corresponding result on a Hilbert space ([1, 4, 5, 7]), can be obtained and the similar approach in [7] also works. But careful arguments are necessary, and this is the reason for the present note.

Proof. (ii) $\Longrightarrow$ (i): Assume that

$$A^r \geqq^J \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right)^{\frac{r}{p+r}}$$

for all $p > 0$ and $r > 0$. Then by Corollary 2, we have

$$r(p+r)\mathrm{Log}\, A \geqq^J r\mathrm{Log}\, \left(A^{\frac{r}{2}} B^p A^{\frac{r}{2}}\right).$$

Dividing this inequality by $r > 0$ and taking p, r as $p = 1, r \to 0$, we have (i).
(i) $\Longrightarrow$ (ii): Since

$$I \geqq^J A, B,$$

by assumption, it follows from Corollary 2 that

$$O = \mathrm{Log}\, I \geqq^J \mathrm{Log}\, A, \mathrm{Log}\, B.$$

Hence, for $n \in \mathbb{N}$

$$I \geqq^J I + \frac{1}{n}\, \mathrm{Log}\, A =: A_1, \quad I + \frac{1}{n}\, \mathrm{Log}\, B =: B_1.$$

For a sufficiently large n, all eigenvalues of A_1, B_1 are positive. Applying Proposition 5 to A_1, B_1 and $np, nr, \frac{nr+np}{nr}$(resp.) as p, r, q(resp.), we get

$$A_1^{nr} \geqq^J \left(A_1^{\frac{nr}{2}} B_1^{np} A_1^{\frac{nr}{2}}\right)^{\frac{nr}{np+nr}} \tag{$\sharp$}$$

for all $p > 0, q > 0$. Recall that

$$\lim_{n\to\infty} \left(I + \frac{A}{n}\right)^n = e^A$$

for any matrix A and that $e^{\mathrm{Log}\, X} = X$ for any matrix X with all eigenvalues positive. Therefore, taking n as $n \to \infty$ in the inequality ($\sharp$), we obtain the conclusion. □

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