ON AN UPPER BOUND FOR JENSEN'S INEQUALITY

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Abstract: In this paper we shall give another global upper bound for Jensen's discrete inequality which is better than existing ones. For instance, we determine a new converse for the generalized $A - G$ inequality.

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1. Introduction

Throughout this paper, $\tilde{x} = \{x_i\}$ is a finite sequence of real numbers belonging to a fixed closed interval $I = [a, b]$, $a < b$, and $\tilde{p} = \{p_i\}$, $\sum p_i = 1$ is a sequence of positive weights associated with $\tilde{x}$. If f is a convex function on I, then the well-known Jensen's inequality [1, 4] asserts that:

$$0 \le \sum p_i f(x_i) - f\left(\sum p_i x_i\right). \tag{1.1}$$

One can see that the lower bound zero is of global nature since it does not depend on $\tilde{p}$ and $\tilde{x}$ but only on f and the interval I, whereupon f is convex.

An upper global bound (i.e. depending on f and I only) for Jensen's inequality is given by Dragomir [3].

Theorem 1.1. *If f is a differentiable convex mapping on I, then we have*

$$\sum p_i f(x_i) - f\left(\sum p_i x_i\right) \le \frac{1}{4}(b-a)(f'(b) - f'(a)) := D_f(a, b). \tag{1.2}$$

In [5] we obtain an upper global bound without a differentiability restriction on f. Namely, we proved the following:

Theorem 1.2. *If $\tilde{p}$, $\tilde{x}$ are defined as above, we have*

$$\sum p_i f(x_i) - f\left(\sum p_i x_i\right) \le f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) := S_f(a, b), \tag{1.3}$$

for any f that is convex over $I := [a, b]$.

In many cases the bound $S_f(a, b)$ is better than $D_f(a, b)$.

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For instance, for $f(x) = -x^s$, $0 < s < 1$; $f(x) = x^s$, $s > 1$; $I \subset \mathbb{R}^+$, we have that

$$S_f(a,b) \le D_f(a,b), \tag{1.4}$$

for each $s \in (0,1) \bigcup (1,2] \bigcup [3,+\infty)$.

In this article we establish another global bound $T_f(a,b)$ for Jensen's inequality, which is better than both of the aforementioned bounds $D_f(a,b)$ and $S_f(a,b)$.

Finally, we determine $T_f(a,b)$ in the case of the generalized $A - G$ inequality.

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2. Results

Our main result is contained in the following

Theorem 2.1. *Let* $f,\ \tilde{p}, \tilde{x}$ *be defined as above and* $p, q > 0,\ p + q = 1$. *Then*

$$\sum p_i f(x_i) - f\left(\sum p_i x_i\right) \leq \max_p [pf(a) + qf(b) - f(pa + qb)] \tag{2.1}$$
$$:= T_f(a, b).$$

Remark 1. It is easy to see that $g(p) := pf(a) + (1-p)f(b) - f(pa + (1-p)b)$ is concave for $0 \leq p \leq 1$ with $g(0) = g(1) = 0$. Hence, there exists a unique positive $\max_p g(p) = T_f(a, b)$.

The next theorem demonstrates that the inequality (2.1) is stronger than (1.2) or (1.3).

Theorem 2.2. *Let* $\tilde{I}$ *be the domain of a convex function* f *and* $I := [a, b] \subset \tilde{I}$. *Then*

I. $T_f(a, b) \leq D_f(a, b)$;

II. $T_f(a, b) \leq S_f(a, b)$,

for each $I \subset \tilde{I}$.

The following well known $A - G$ inequality [4] asserts that

$$A(\tilde{p}, \tilde{x}) \geq G(\tilde{p}, \tilde{x}), \tag{2.2}$$

where

$$A(\tilde{p}, \tilde{x}) := \sum p_i x_i; \quad G(\tilde{p}, \tilde{x}) := \prod x_i^{p_i}, \tag{2.3}$$

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are generalized arithmetic, i.e., geometric means, respectively.

Applying Theorems 2.1 (cf [2]) and 2.2 with $f(x) = -\log x$, one obtains the following converses of the $A - G$ inequality:

$$1 \leq \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq \exp\left(\frac{(b-a)^2}{4ab}\right) \tag{2.4}$$

and

$$1 \leq \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq \frac{(a+b)^2}{4ab}. \tag{2.5}$$

Since $1 + x \leq e^x$, $x \in \mathbb{R}$, putting $x = \frac{(b-a)^2}{4ab}$, one can see that the inequality (2.5) is stronger than (2.4) for each $a, b \in \mathbb{R}^+$.

An even stronger converse of the $A - G$ inequality can be obtained by applying Theorem 2.1.

Theorem 2.3. *Let $\tilde{p}, \tilde{x}, A(\tilde{p}, \tilde{x}), G(\tilde{p}, \tilde{x})$ be defined as above and $x_i \in [a, b]$, $0 < a < b$.*

Denote $u := \log(b/a)$; $w := (e^u - 1)/u$. Then

$$1 \leq \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq \frac{w}{e} \exp\frac{1}{w} := T(w). \tag{2.6}$$

Comparing the bounds D, S and T, i.e., (2.4), (2.5) and (2.6) for arbitrary $\tilde{p}$ and $x_i \in [a, 2a]$, $a > 0$, we obtain

$$1 \leq \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq e^{1/8} \approx 1.1331, \tag{2.7}$$

$$1 \leq \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq 9/8 = 1.125, \tag{2.8}$$

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and

$$1 \le \frac{A(\tilde{p},\tilde{x})}{G(\tilde{p},\tilde{x})} \le 2(e\log 2)^{-1} \approx 1.0615 \tag{2.9}$$

respectively.

Remark 2. One can see that $T(w) = S(t)$, where Specht's ratio $S(t)$ is defined as

$$S(t) := \frac{t^{1/(t-1)}}{e\log t^{1/(t-1)}} \tag{2.10}$$

with $t = b/a$.

It is known [6, 7] that $S(t)$ represents the best possible upper bound for the $A-G$ inequality with uniform weights, i.e.

$$S(t)(x_1x_2\cdots x_n)^{\frac{1}{n}} \ge \frac{x_1+x_2+\cdots+x_n}{n} \left(\ge (x_1x_2\cdots x_n)^{\frac{1}{n}}\right). \tag{2.11}$$

Therefore Theorem 2.3 shows that Specht's ratio is the best upper bound for the generalized $A-G$ inequality also.

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3. Proofs

Proof of Theorem 2.1. Since $x_i \in [a,b]$, there is a sequence $\{\lambda_i\}$, $\lambda_i \in [0,1]$, such that $x_i = \lambda_i a + (1-\lambda_i)b$.

Hence

$$\begin{aligned}
&\sum p_i f(x_i) - f\left(\sum p_i x_i\right) \\
&= \sum p_i f(\lambda_i a + (1-\lambda_i)b) - f\left(\sum p_i(\lambda_i a + (1-\lambda_i)b)\right) \\
&\le \sum p_i(\lambda_i f(a) + (1-\lambda_i)f(b)) - f(a\sum p_i\lambda_i + b\sum p_i(1-\lambda_i) \\
&= f(a)\left(\sum p_i\lambda_i\right) + f(b)\left(1-\sum p_i\lambda_i\right) - f\left[a\left(\sum p_i\lambda_i\right) + b\left(1-\sum p_i\lambda_i\right)\right].
\end{aligned}$$

Denoting $\sum p_i\lambda_i := p$; $1-\sum p_i\lambda_i := q$, we have that $0 \le p,q \le 1$, $p+q=1$. Consequently,

$$\begin{aligned}
\sum p_i f(x_i) - f\left(\sum p_i x_i\right) &\le pf(a) + qf(b) - f(pa+qb) \\
&\le \max_p [pf(a) + qf(b) - f(pa+qb)] \\
&:= T_f(a,b),
\end{aligned}$$

and the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2.

Part I.

Since f is convex (and differentiable, in this case), we have

(3.1) $$\forall x,t \in I: \ f(x) \ge f(t) + (x-t)f'(t).$$

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In particular,

$$f(pa+qb) \geq f(a) + q(b-a)f'(a); \quad f(pa+qb) \geq f(b) + p(a-b)f'(b). \tag{3.2}$$

Therefore

$$\begin{aligned} pf(a) + qf(b) - f(pa+qb) &= p(f(a) - f(pa+qb)) + q(f(b) - f(pa+qb)) \\ &\leq p(q(a-b)f'(a)) + q(p(b-a)f'(b)) \\ &= pq(b-a)(f'(b) - f'(a)). \end{aligned}$$

Hence

$$\begin{aligned} T_f(a,b) &:= \max_p [pf(a) + qf(b) - f(pa+qb)] \\ &\leq \max_p [pq(b-a)(f'(b) - f'(a))] \\ &= \frac{1}{4}(b-a)(f'(b) - f'(a)) \\ &:= D_f(a,b). \end{aligned}$$

Part II.

We shall prove that, for each $0 \leq p, q, p+q = 1$,

$$pf(a) + qf(b) - f(pa+qb) \leq f(a) + f(b) - 2f\left(\frac{a+b}{2}\right). \tag{3.3}$$

Indeed,

$$\begin{aligned} pf(a) + qf(b) - f(pa+qb) &= f(a) + f(b) - (qf(a) + pf(b)) - f(pa+qb) \\ &\leq f(a) + f(b) - (f(qa+pb) + f(pa+qb)) \end{aligned}$$

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$$\leq f(a)+f(b)-2f\left(\frac{1}{2}(qa+pb)+\frac{1}{2}(pa+qb)\right)$$
$$= f(a)+f(b)-2f\left(\frac{a+b}{2}\right).$$

Since the right-hand side of the above inequality does not depend on p, we immediately get

$$T_f(a,b) \leq S_f(a,b). \tag{3.4}$$

□

Proof of Theorem 2.3. By Theorem 2.1, applied with $f(x) = -\log x$, we obtain

$$\begin{aligned} 0 &\leq \log \frac{A(\tilde{p},\tilde{x})}{G(\tilde{p},\tilde{x})} \\ &\leq T_{-\log x}(a,b) \\ &= \max_p[\log(pa+qb) - p\log a - q\log b]. \end{aligned}$$

Using standard arguments it is easy to find that the unique maximum is attained at the point p:

$$p = \frac{b}{b-a} - \frac{1}{\log b - \log a}. \tag{3.5}$$

Since $0 < a < b$, we get $0 < p < 1$ and after some calculations, it follows that

$$0 \leq \log \frac{A(\tilde{p},\tilde{x})}{G(\tilde{p},\tilde{x})} \leq \log\left(\frac{b-a}{\log b - \log a}\right) + \frac{a\log b - b\log a}{b-a} - 1. \tag{3.6}$$

Putting $\log(b/a) := u$, $(e^u - 1)/u := w$ and taking the exponent, one obtains the result of Theorem 2.3. □

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