

# ON AN UPPER BOUND FOR JENSEN'S INEQUALITY

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*Key words:* Jensen's discrete inequality, global bounds, generalized A-G inequality.

*Abstract:* In this paper we shall give another global upper bound for Jensen's discrete inequality which is better than existing ones. For instance, we determine a new converse for the generalized  $A - G$  inequality.



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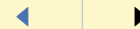
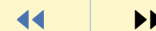
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## 1. Introduction

Throughout this paper,  $\tilde{x} = \{x_i\}$  is a finite sequence of real numbers belonging to a fixed closed interval  $I = [a, b]$ ,  $a < b$ , and  $\tilde{p} = \{p_i\}$ ,  $\sum p_i = 1$  is a sequence of positive weights associated with  $\tilde{x}$ . If  $f$  is a convex function on  $I$ , then the well-known Jensen's inequality [1, 4] asserts that:

$$(1.1) \quad 0 \leq \sum p_i f(x_i) - f\left(\sum p_i x_i\right).$$

One can see that the lower bound zero is of global nature since it does not depend on  $\tilde{p}$  and  $\tilde{x}$  but only on  $f$  and the interval  $I$ , whereupon  $f$  is convex.

An upper global bound (i.e. depending on  $f$  and  $I$  only) for Jensen's inequality is given by Dragomir [3].

**Theorem 1.1.** *If  $f$  is a differentiable convex mapping on  $I$ , then we have*

$$(1.2) \quad \sum p_i f(x_i) - f\left(\sum p_i x_i\right) \leq \frac{1}{4}(b-a)(f'(b) - f'(a)) := D_f(a, b).$$

In [5] we obtain an upper global bound without a differentiability restriction on  $f$ . Namely, we proved the following:

**Theorem 1.2.** *If  $\tilde{p}$ ,  $\tilde{x}$  are defined as above, we have*

$$(1.3) \quad \sum p_i f(x_i) - f\left(\sum p_i x_i\right) \leq f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) := S_f(a, b),$$

for any  $f$  that is convex over  $I := [a, b]$ .

In many cases the bound  $S_f(a, b)$  is better than  $D_f(a, b)$ .



For instance, for  $f(x) = -x^s$ ,  $0 < s < 1$ ;  $f(x) = x^s$ ,  $s > 1$ ;  $I \subset \mathbb{R}^+$ , we have that

$$(1.4) \quad S_f(a, b) \leq D_f(a, b),$$

for each  $s \in (0, 1) \cup (1, 2] \cup [3, +\infty)$ .

In this article we establish another global bound  $T_f(a, b)$  for Jensen's inequality, which is better than both of the aforementioned bounds  $D_f(a, b)$  and  $S_f(a, b)$ .

Finally, we determine  $T_f(a, b)$  in the case of the generalized  $A - G$  inequality.

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## 2. Results

Our main result is contained in the following

**Theorem 2.1.** Let  $f$ ,  $\tilde{p}$ ,  $\tilde{x}$  be defined as above and  $p, q > 0$ ,  $p + q = 1$ . Then

$$(2.1) \quad \sum p_i f(x_i) - f\left(\sum p_i x_i\right) \leq \max_p [pf(a) + qf(b) - f(pa + qb)] \\ := T_f(a, b).$$

*Remark 1.* It is easy to see that  $g(p) := pf(a) + (1 - p)f(b) - f(pa + (1 - p)b)$  is concave for  $0 \leq p \leq 1$  with  $g(0) = g(1) = 0$ . Hence, there exists a unique positive  $\max_p g(p) = T_f(a, b)$ .

The next theorem demonstrates that the inequality (2.1) is stronger than (1.2) or (1.3).

**Theorem 2.2.** Let  $\tilde{I}$  be the domain of a convex function  $f$  and  $I := [a, b] \subset \tilde{I}$ . Then

I.  $T_f(a, b) \leq D_f(a, b)$ ;

II.  $T_f(a, b) \leq S_f(a, b)$ ,

for each  $I \subset \tilde{I}$ .

The following well known  $A - G$  inequality [4] asserts that

$$(2.2) \quad A(\tilde{p}, \tilde{x}) \geq G(\tilde{p}, \tilde{x}),$$

where

$$(2.3) \quad A(\tilde{p}, \tilde{x}) := \sum p_i x_i; \quad G(\tilde{p}, \tilde{x}) := \prod x_i^{p_i},$$



are generalized arithmetic, i.e., geometric means, respectively.

Applying Theorems 2.1 (cf [2]) and 2.2 with  $f(x) = -\log x$ , one obtains the following converses of the  $A - G$  inequality:

$$(2.4) \quad 1 \leq \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq \exp \left( \frac{(b-a)^2}{4ab} \right)$$

and

$$(2.5) \quad 1 \leq \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq \frac{(a+b)^2}{4ab}.$$

Since  $1+x \leq e^x$ ,  $x \in \mathbb{R}$ , putting  $x = \frac{(b-a)^2}{4ab}$ , one can see that the inequality (2.5) is stronger than (2.4) for each  $a, b \in \mathbb{R}^+$ .

An even stronger converse of the  $A - G$  inequality can be obtained by applying Theorem 2.1.

**Theorem 2.3.** *Let  $\tilde{p}, \tilde{x}, A(\tilde{p}, \tilde{x}), G(\tilde{p}, \tilde{x})$  be defined as above and  $x_i \in [a, b]$ ,  $0 < a < b$ .*

*Denote  $u := \log(b/a)$ ;  $w := (e^u - 1)/u$ . Then*

$$(2.6) \quad 1 \leq \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq \frac{w}{e} \exp \frac{1}{w} := T(w).$$

Comparing the bounds  $D, S$  and  $T$ , i.e., (2.4), (2.5) and (2.6) for arbitrary  $\tilde{p}$  and  $x_i \in [a, 2a]$ ,  $a > 0$ , we obtain

$$(2.7) \quad 1 \leq \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq e^{1/8} \approx 1.1331,$$

$$(2.8) \quad 1 \leq \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq 9/8 = 1.125,$$

and

$$(2.9) \quad 1 \leq \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq 2(e \log 2)^{-1} \approx 1.0615$$

respectively.

*Remark 2.* One can see that  $T(w) = S(t)$ , where Specht's ratio  $S(t)$  is defined as

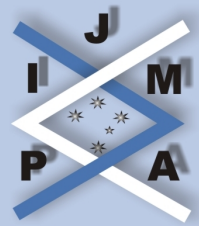
$$(2.10) \quad S(t) := \frac{t^{1/(t-1)}}{e \log t^{1/(t-1)}}$$

with  $t = b/a$ .

It is known [6, 7] that  $S(t)$  represents the best possible upper bound for the  $A - G$  inequality with uniform weights, i.e.

$$(2.11) \quad S(t)(x_1 x_2 \cdots x_n)^{\frac{1}{n}} \geq \frac{x_1 + x_2 + \cdots + x_n}{n} \left( \geq (x_1 x_2 \cdots x_n)^{\frac{1}{n}} \right).$$

Therefore Theorem 2.3 shows that Specht's ratio is the best upper bound for the generalized  $A - G$  inequality also.



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### 3. Proofs

*Proof of Theorem 2.1.* Since  $x_i \in [a, b]$ , there is a sequence  $\{\lambda_i\}$ ,  $\lambda_i \in [0, 1]$ , such that  $x_i = \lambda_i a + (1 - \lambda_i)b$ .

Hence

$$\begin{aligned} & \sum p_i f(x_i) - f\left(\sum p_i x_i\right) \\ &= \sum p_i f(\lambda_i a + (1 - \lambda_i)b) - f\left(\sum p_i (\lambda_i a + (1 - \lambda_i)b)\right) \\ &\leq \sum p_i (\lambda_i f(a) + (1 - \lambda_i)f(b)) - f\left(a \sum p_i \lambda_i + b \sum p_i (1 - \lambda_i)\right) \\ &= f(a) \left(\sum p_i \lambda_i\right) + f(b) \left(1 - \sum p_i \lambda_i\right) - f\left[a \left(\sum p_i \lambda_i\right) + b \left(1 - \sum p_i \lambda_i\right)\right]. \end{aligned}$$

Denoting  $\sum p_i \lambda_i := p$ ;  $1 - \sum p_i \lambda_i := q$ , we have that  $0 \leq p, q \leq 1$ ,  $p + q = 1$ . Consequently,

$$\begin{aligned} \sum p_i f(x_i) - f\left(\sum p_i x_i\right) &\leq pf(a) + qf(b) - f(pa + qb) \\ &\leq \max_p [pf(a) + qf(b) - f(pa + qb)] \\ &:= T_f(a, b), \end{aligned}$$

and the proof of Theorem 2.1 is complete.  $\square$

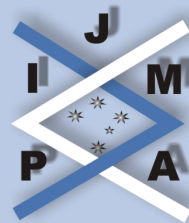
*Proof of Theorem 2.2.*

#### Part I.

Since  $f$  is convex (and differentiable, in this case), we have

$$(3.1) \quad \forall x, t \in I : f(x) \geq f(t) + (x - t)f'(t).$$





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In particular,

$$(3.2) \quad f(pa + qb) \geq f(a) + q(b - a)f'(a); \quad f(pa + qb) \geq f(b) + p(a - b)f'(b).$$

Therefore

$$\begin{aligned} pf(a) + qf(b) - f(pa + qb) &= p(f(a) - f(pa + qb)) + q(f(b) - f(pa + qb)) \\ &\leq p(q(a - b)f'(a)) + q(p(b - a)f'(b)) \\ &= pq(b - a)(f'(b) - f'(a)). \end{aligned}$$

Hence

$$\begin{aligned} T_f(a, b) &:= \max_p [pf(a) + qf(b) - f(pa + qb)] \\ &\leq \max_p [pq(b - a)(f'(b) - f'(a))] \\ &= \frac{1}{4}(b - a)(f'(b) - f'(a)) \\ &:= D_f(a, b). \end{aligned}$$

## Part II.

We shall prove that, for each  $0 \leq p, q, p + q = 1$ ,

$$(3.3) \quad pf(a) + qf(b) - f(pa + qb) \leq f(a) + f(b) - 2f\left(\frac{a+b}{2}\right).$$

Indeed,

$$\begin{aligned} pf(a) + qf(b) - f(pa + qb) &= f(a) + f(b) - (qf(a) + pf(b)) - f(pa + qb) \\ &\leq f(a) + f(b) - (f(qa + pb) + f(pa + qb)) \end{aligned}$$

$$\begin{aligned} &\leq f(a) + f(b) - 2f\left(\frac{1}{2}(qa + pb) + \frac{1}{2}(pa + qb)\right) \\ &= f(a) + f(b) - 2f\left(\frac{a+b}{2}\right). \end{aligned}$$

Since the right-hand side of the above inequality does not depend on  $p$ , we immediately get

$$(3.4) \quad T_f(a, b) \leq S_f(a, b).$$

□

*Proof of Theorem 2.3.* By Theorem 2.1, applied with  $f(x) = -\log x$ , we obtain

$$\begin{aligned} 0 &\leq \log \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \\ &\leq T_{-\log x}(a, b) \\ &= \max_p [\log(pa + qb) - p \log a - q \log b]. \end{aligned}$$

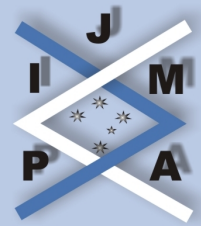
Using standard arguments it is easy to find that the unique maximum is attained at the point  $p$ :

$$(3.5) \quad p = \frac{b}{b-a} - \frac{1}{\log b - \log a}.$$

Since  $0 < a < b$ , we get  $0 < p < 1$  and after some calculations, it follows that

$$(3.6) \quad 0 \leq \log \frac{A(\tilde{p}, \tilde{x})}{G(\tilde{p}, \tilde{x})} \leq \log \left( \frac{b-a}{\log b - \log a} \right) + \frac{a \log b - b \log a}{b-a} - 1.$$

Putting  $\log(b/a) := u$ ,  $(e^u - 1)/u := w$  and taking the exponent, one obtains the result of Theorem 2.3. □



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