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MULTIPLICATION OF SUBHARMONIC FUNCTIONS

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ABSTRACT. We study subharmonic functions in the unit ball of $\mathbb{R}^N$, with either a Bloch–type growth or a growth described through integral conditions involving some involutions of the ball. Considering mappings $u \mapsto gu$ between sets of functions with a prescribed growth, we study how the choice of these sets is related to the growth of the function g.

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1. INTRODUCTION

This paper is devoted to functions u which are defined in the unit ball B_N of $\mathbb{R}^N$ (relative to the Euclidean norm $|\cdot|$), whose growth is described by the above boundedness on B_N of $x \mapsto (1-|x|^2)^\alpha\, v(x)$ for some parameter α. The function v may denote merely u or some integral involving u and involutions Φ_x (precise definitions and notations will be detailed in Section 2). In the first (resp. second) case, u is said to belong to the set $\mathcal{X}$ (resp. $\mathcal{Y}$). Given a function g defined on B_N, we try to obtain links between the growth of g and information on such mappings as

$$\mathcal{Y} \to \mathcal{X},$$
$$u \mapsto gu.$$

This work is motivated by the situation known in the case of holomorphic functions f in the unit disk D of $\mathbb{C}$. Such a function is said to belong to the Bloch space $\mathcal{B}_\lambda$ if

$$||f||_{\mathcal{B}_\lambda} := |f(0)| + \sup_{z\in D}(1-|z|^2)^\lambda\, |f'(z)| < +\infty.$$

It is said to belong to the space $BMOA_\mu$ if

$$||f||^2_{BMOA_\mu} := |f(0)|^2 + \sup_{a\in D}\int_D (1-|z|^2)^{2\mu-2}\, |f'(z)|^2 (1-|\varphi_a(z)|^2)\, dA(z) < +\infty$$

with $dA(z)$ the normalized area measure element on D and $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$.

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Given h a holomorphic function on D, the operator $I_h : f \mapsto I_h(f)$ defined by:

$$(I_h(f))(z) = \int_0^z h(\zeta)\, f'(\zeta)\, d\zeta \qquad \forall z \in D$$

was studied for instance in [7] where it was proved that $I_h : BMOA_\mu \to \mathcal{B}_\lambda$ is bounded (with respect to the above norms) if and only if $h \in \mathcal{B}_{\lambda-\mu+1}$ (assuming $1 < \mu < \lambda$).

Since $|f'|^2$ is subharmonic in the unit ball of $\mathbb{R}^2$, the question naturally arose whether some similar phenomena occur for subharmonic functions in B_N for $N \geq 2$.

2. NOTATIONS AND MAIN RESULTS

Let $B_N = \{x \in \mathbb{R}^N \;:\; |x| < 1\}$ with $N \in \mathbb{N}$, $N \geq 2$ and $|\cdot|$ the Euclidean norm in $\mathbb{R}^N$. Given $a \in B_N$, let $\Phi_a : B_N \to B_N$ denote the involution defined by:

$$\Phi_a(x) = \frac{a - P_a(x) - \sqrt{1-|a|^2}\,Q_a(x)}{1 - \langle x, a\rangle} \qquad \forall x \in B_N,$$

where

$$\langle x, a\rangle = \sum_{j=1}^N x_j\, a_j\,, \qquad P_a(x) = \frac{\langle x, a\rangle}{|a|^2}\, a\,, \qquad Q_a(x) = x - P_a(x)$$

for all $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ and $a = (a_1, a_2, \dots, a_N) \in \mathbb{R}^N$, with $P_a(x) = 0$ if $a = 0$. We refer to [4, pp. 25–26] and [1, p. 115] for the main properties of the map Φ_a (initially defined in the unit ball of $\mathbb{C}^N$). For instance, we will make use of the relation:

$$1 - |\Phi_a(x)|^2 = \frac{(1-|a|^2)\,(1-|x|^2)}{(1-\langle x, a\rangle)^2}.$$

In the following, α, β, γ and λ are given real numbers, with $\gamma \geq 0$.

Definition 2.1. Let $\mathcal{X}_\lambda$ denote the set of all functions $u : B_N \to [-\infty, +\infty[$ satisfying:

$$M_{\mathcal{X}_\lambda}(u) := \sup_{x \in B_N} (1-|x|^2)^\lambda\, u(x) < +\infty.$$

Let $\mathcal{Y}_{\alpha,\beta,\gamma}$ denote the set of all measurable functions $u : B_N \to [-\infty, +\infty[$ satisfying:

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) := \sup_{a \in B_N} (1-|a|^2)^\alpha \int_{B_N} (1-|x|^2)^\beta\, u(x)\, (1-|\Phi_a(x)|^2)^\gamma\, dx < +\infty.$$

The subset $\mathcal{SX}_\lambda$ (resp. $\mathcal{SY}_{\alpha,\beta,\gamma}$) gathers all $u \in \mathcal{X}_\lambda$ (resp. $u \in \mathcal{Y}_{\alpha,\beta,\gamma}$) which moreover are subharmonic and non–negative. The subset $\mathcal{RSY}_{\alpha,\beta,\gamma}$ gathers all $u \in \mathcal{SY}_{\alpha,\beta,\gamma}$ which moreover are radial.

Remark 1. When $\lambda < 0$ (resp. $\alpha + \beta < -N$ or $\alpha < -\gamma$), the set $\mathcal{SX}_\lambda$ (resp. $\mathcal{SY}_{\alpha,\beta,\gamma}$) merely reduces to the single function $u \equiv 0$ (see Propositions 6.2, 6.3 and 6.4).

In Proposition 3.1 and Corollary 3.2, we will establish that $\mathcal{SY}_{\alpha,\beta,\gamma} \subset \mathcal{SX}_{\alpha+\beta+N}$ and that there exists a constant $C > 0$ such that

$$M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(gu) \leq C\, M_{\mathcal{X}_\lambda}(g)\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u)$$

for all $u \in \mathcal{SY}_{\alpha,\beta,\gamma}$ and all $g \in \mathcal{X}_\lambda$ with $M_{\mathcal{X}_\lambda}(g) \geq 0$. We will next study whether some kind of a “converse” holds and obtain the following:

Theorem 2.1. *Given $\lambda \in \mathbb{R}$ and $g : B_N \to [0, +\infty[$ a subharmonic function satisfying:*

$$\exists C' > 0 \qquad M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(gu) \leq C'\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \qquad \forall u \in \mathcal{SY}_{\alpha,\beta,\gamma},$$

then $g \in \mathcal{X}_{\lambda+\frac{N-1}{2}}$ in each of the six cases gathered in the following Table 2.1.

case	α	β	γ
(i)	$\alpha = \frac{N+1}{2} + \beta$	$\beta > -\frac{N+1}{2}$	$\gamma > \max(\alpha, -1-\beta)$
(ii)	$\alpha = \beta + 1$	$\beta > -\frac{N+3}{4}$	$\gamma > \lvert 1+\beta\rvert$
(iii)	$\alpha = \frac{N+1}{2} - \gamma$	$\beta \geq -\gamma$	$\frac{N+1}{4} < \gamma < \frac{N+1}{2}$
(iv)	$\alpha = 1$	$\beta \geq 0$	$\gamma > 1$
(v)	$\alpha = 1 + \beta - \gamma$	$\beta > -1$	$\frac{1+\beta}{2} < \gamma < \beta + \frac{N+3}{4}$
(vi)	$\alpha = \frac{\beta+1}{2}$	$\beta \geq -\frac{1}{2}$	$\gamma > \left\lvert \frac{1+\beta}{2} \right\rvert$

Table 2.1: Six situations where Theorem 2.1 shows that g belongs to the set $\mathcal{X}_{\lambda+\frac{N-1}{2}}$.

Theorem 2.2. *Given $\lambda \in \mathbb{R}$ and g a subharmonic function defined on B_N, satisfying:*

$$\exists C'' > 0 \qquad M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(gu) \leq C''\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \qquad \forall u \in \mathcal{RSY}_{\alpha,\beta,\gamma},$$

then $g \in \mathcal{SX}_{\lambda+\alpha+\frac{N-1}{2}}$ provided that $\alpha \geq 0$, $\beta \geq -\frac{N+1}{2}$, $\gamma > \frac{N-1}{2}$.

3. SOME PRELIMINARIES

Notation 3.1. Given $a \in B_N$ and $R \in]0,1[$, let $B(a, R_a) = \{x \in B_N \,:\, |x-a| < R_a\}$ with

$$R_a = R\,\frac{1-|a|^2}{1+R|a|}.$$

Proposition 3.1. *There exists a $C > 0$ depending only on N, β, γ, such that:*

$$M_{\mathcal{X}_{\alpha+\beta+N}}(u) \leq C\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \qquad \forall u \in \mathcal{SY}_{\alpha,\beta,\gamma}.$$

Proof. Let some $R \in]0,1[$ be fixed in the following. Since $u \geq 0$, we obtain for any $a \in B_N$:

$$\begin{aligned} M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) &\geq (1-|a|^2)^\alpha \int_{B_N} (1-|x|^2)^\beta\, u(x)\, (1-|\Phi_a(x)|^2)^\gamma\, dx \\ &\geq (1-|a|^2)^\alpha \int_{B(a,R_a)} (1-|x|^2)^\beta\, u(x)\, (1-|\Phi_a(x)|^2)^\gamma\, dx. \end{aligned}$$

It follows from Lemma 1 of [6] that

$$B(a,R_a) \subset E(a,R) = \{x \in B_N \,:\, |\Phi_a(x)| < R\},$$

hence:

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \geq (1-R^2)^\gamma\, (1-|a|^2)^\alpha \int_{B(a,R_a)} (1-|x|^2)^\beta\, u(x)\, dx \tag{3.1}$$

as $\gamma \geq 0$. From Lemmas 1 and 5 of [5], it is known that

$$\frac{1-R}{1+R} \leq \frac{1-|x|^2}{1-|a|^2} \leq 2 \qquad \forall x \in B(a,R_a).$$

Let $C_\beta = \left(\frac{1-R}{1+R}\right)^\beta$ if $\beta \geq 0$ and $C_\beta = 2^\beta$ if $\beta < 0$. Hence

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \geq C_\beta (1-R^2)^\gamma \, (1-|a|^2)^{\alpha+\beta} \int_{B(a,R_a)} u(x)\, dx.$$

The volume of $B(a,R_a)$ is $\sigma_N \frac{(R_a)^N}{N}$ with $\sigma_N = \frac{2\,\pi^{N/2}}{\Gamma(N/2)}$ the area of the unit sphere S_N in $\mathbb{R}^N$ (see [2, p. 29]) and $R_a \geq \frac{R}{1+R}\,(1-|a|^2)$. The subharmonicity of u now provides:

$$\begin{aligned} M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) &\geq C_\beta (1-R^2)^\gamma \, (1-|a|^2)^{\alpha+\beta}\, u(a)\, \sigma_N \, \frac{(R_a)^N}{N} \\ &\geq C_\beta \, \frac{\sigma_N}{N} \, \frac{R^N (1-R)^\gamma}{(1+R)^{N-\gamma}} \, (1-|a|^2)^{\alpha+\beta+N}\, u(a). \end{aligned}$$

□

Corollary 3.2. *Let $g \in \mathcal{X}_\lambda$ with $M_{\mathcal{X}_\lambda}(g) \geq 0$. Then:*

$$M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(gu) \leq C\, M_{\mathcal{X}_\lambda}(g)\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \qquad \forall u \in \mathcal{SY}_{\alpha,\beta,\gamma}$$

where the constant C stems from Proposition 3.1.

Proof. Since $u \geq 0$, we have for any $x \in B_N$:

$$\begin{aligned} (1-|x|^2)^{\lambda+\alpha+\beta+N} g(x)\, u(x) &\leq M_{\mathcal{X}_\lambda}(g)\, (1-|x|^2)^{\alpha+\beta+N} u(x) \\ &\leq M_{\mathcal{X}_\lambda}(g)\, M_{\mathcal{X}_{\alpha+\beta+N}}(u) \end{aligned}$$

because of $M_{\mathcal{X}_\lambda}(g) \geq 0$. □

Lemma 3.3. *Given $a \in B_N$ and $R \in]0,1[$, the following holds for any $x \in B(a,R_a)$:*

$$\frac{1}{2} < \frac{1}{1+R\,|a|} \leq \frac{1-\langle x,a\rangle}{1-|a|^2} \leq \frac{1+2R\,|a|}{1+R\,|a|} < 2 \quad \textit{and} \quad \frac{1}{4} < \frac{1-\langle x,a\rangle}{1-|x|^2} < 2\,\frac{1+R}{1-R}\,.$$

Proof. Clearly $\langle x,a\rangle = \langle a+y,a\rangle = |a|^2 + \langle y,a\rangle$ with $|y| < R_a$. From the Cauchy-Schwarz inequality, it follows that $-R_a\,|a| \leq \langle y,a\rangle \leq R_a\,|a|$. Hence:

$$1-|a|^2 - R\,|a|\,\frac{1-|a|^2}{1+R|a|} \leq 1-\langle x,a\rangle \leq 1-|a|^2 + R\,|a|\,\frac{1-|a|^2}{1+R|a|}.$$

The term on the left equals

$$(1-|a|^2)\left(1-\frac{R\,|a|}{1+R|a|}\right) = (1-|a|^2)\frac{1}{1+R|a|}$$

and $1+R|a| < 2$. The term on the right equals

$$(1-|a|^2)\left(1+\frac{R\,|a|}{1+R|a|}\right),$$

with $\frac{R|a|}{1+R|a|} < 1$. Now

$$\frac{1-\langle x,a\rangle}{1-|x|^2} = \frac{1-\langle x,a\rangle}{1-|a|^2}\,\frac{1-|a|^2}{1-|x|^2}$$

and the last inequalities follow from Lemmas 1 and 5 of [5]. □

Lemma 3.4. *Let* $H = \{(s,t) \in \mathbb{R}^2 : t \geq 0,\ s^2+t^2 < 1\}$ *and* $P > -1$, $Q > -1$, $T > -1$. *Then*

$$\iint_H s^P\, t^Q\, (1-s^2-t^2)^T\, ds\, dt = \begin{cases} 0 & \text{if } P \text{ is odd;} \\ \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{Q+1}{2}\right)\Gamma(T+1)}{2\,\Gamma\left(\frac{P+Q}{2}+T+2\right)} & \text{if } P \text{ is even.} \end{cases}$$

Proof. With polar coordinates $s = r\,\cos\theta$, $t = r\,\sin\theta$, this integral turns into $I_1\, I_2$ with

$$I_1 = \int_0^1 r^{P+Q}\,(1-r^2)^T\, r\, dr \qquad \text{and} \qquad I_2 = \int_0^\pi (\cos\theta)^P\,(\sin\theta)^Q\, d\theta.$$

Keeping in mind the various expressions for the Beta function (see [3, pp. 67–68]):

$$\begin{aligned} B(x,y) &= \int_0^1 \xi^{x-1}\,(1-\xi)^{y-1}\, d\xi \\ &= 2\int_0^{\pi/2} (\cos\theta)^{2x-1}\,(\sin\theta)^{2y-1}\, d\theta = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \end{aligned}$$

(with $x > 0$ and $y > 0$), the change of variable $\omega = r^2$ leads to:

$$\begin{aligned} I_1 &= \frac{1}{2}\int_0^1 \omega^{\frac{P+Q}{2}}\,(1-\omega)^T\, d\omega \\ &= \frac{1}{2}\, B\left(\frac{P+Q}{2}+1, T+1\right) = \frac{\Gamma\left(\frac{P+Q}{2}+1\right)\,\Gamma(T+1)}{2\,\Gamma\left(\frac{P+Q}{2}+T+2\right)}. \end{aligned}$$

When P is odd, $I_2 = 0$ because $\cos(\pi-\theta) = -\cos(\theta)$. However, when P is even:

$$\begin{aligned} I_2 &= 2\int_0^{\pi/2} (\cos\theta)^P\,(\sin\theta)^Q\, d\theta \\ &= B\left(\frac{P+1}{2}, \frac{Q+1}{2}\right) = \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{Q+1}{2}\right)}{\Gamma\left(\frac{P+Q}{2}+1\right)}. \end{aligned}$$

□

Lemma 3.5. *Given* $A \geq 0$ *and* $a \in B_N$, *let* u *and* f_a *denote the functions defined on* B_N *by* $u(x) = \frac{1}{(1-|x|^2)^A}$ *and* $f_a(x) = \frac{1}{(1-\langle x,a\rangle)^A}$ $\forall x \in B_N$. *They are both subharmonic in* B_N.

Remark 2. u is radial, but not f_a.

Proof. For u, the result of Lemma 3.5 has already been proved in Proposition 1 of [5]. For any $j \in \{1, 2, \ldots, N\}$, we now compute:

$$\frac{\partial f_a}{\partial x_j}(x) = a_j\, A\,(1-\langle x,a\rangle)^{-A-1} \quad \text{and} \quad \frac{\partial^2 f_a}{\partial x_j^2}(x) = (a_j)^2\, A\,(A+1)(1-\langle x,a\rangle)^{-A-2},$$

so that:

$$(\Delta f_a)(x) = \frac{|a|^2\, A\,(A+1)}{(1-\langle x,a\rangle)^{A+2}} \geq 0 \qquad \forall x \in B_N.$$

□

Remark 3. Given $A \geq 0$, $A' \geq 0$, the function f_a defined on B_N by

$$f_a(x) = \frac{1}{(1-\langle x,a\rangle)^A(1-|x|^2)^{A'}}$$

is subharmonic too. The computation

$$(\Delta f_a)(x) \geq f_a(x) \left(\frac{A\,|a|}{1 - \langle x, a \rangle} - \frac{2A'|x|}{1 - |x|^2} \right)^2 \geq 0$$

is left to the reader.

Proposition 3.6. *Given $N \in \mathbb{N}$, $N > 3$, $(s, t, b_1, b_2) \in \mathbb{R}^4$ such that $|s\,b_1| + |t\,b_2| < 1$ and $P > 0$, let*

$$I_P(s, t, b_1, b_2) = \int_0^\pi \frac{(\sin\theta)^{N-3}\,d\theta}{(1 - s\,b_1 - t\,b_2\,\cos\theta)^P}.$$

Then

$$I_P(s, t, b_1, b_2) = \sqrt{\pi}\,\frac{\Gamma\left(\frac{N}{2} - 1\right)}{\Gamma(P)} \sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} \frac{\Gamma(j + 2k + P)}{k!\,j!\,\Gamma\left(\frac{N-1}{2} + k\right)} (b_1\,s)^j \left(\frac{t\,b_2}{2}\right)^{2k}.$$

Proof. As

$$\left|\frac{t\,b_2\,\cos\theta}{1 - s\,b_1}\right| \leq \left|\frac{t\,b_2}{1 - s\,b_1}\right| < 1,$$

the following development is valid:

$$\begin{aligned} I_P(s, t, b_1, b_2) &= \int_0^\pi \frac{(\sin\theta)^{N-3}\,d\theta}{(1 - s\,b_1)^P \left(1 - \frac{t\,b_2\,\cos\theta}{1 - s\,b_1}\right)^P} \\ &= \frac{1}{(1 - s\,b_1)^P} \sum_{n\in\mathbb{N}} \frac{\Gamma(n + P)}{n!\,\Gamma(P)} \left(\frac{t\,b_2}{1 - s\,b_1}\right)^n \int_0^\pi (\sin\theta)^{N-3} (\cos\theta)^n\,d\theta. \end{aligned}$$

The last integral vanishes when n is odd. When n is even ($n = 2k$), then

$$\begin{aligned} 2\int_0^{\pi/2} (\sin\theta)^{N-3} (\cos\theta)^{2k}\,d\theta &= B\left(\frac{N-2}{2}, k + \frac{1}{2}\right) \\ &= \frac{\Gamma\left(\frac{N-2}{2}\right)\,\Gamma\left(k + \frac{1}{2}\right)}{\Gamma\left(\frac{N-1}{2} + k\right)} \\ &= \frac{\Gamma\left(\frac{N-2}{2}\right)\,(2k)!\,\sqrt{\pi}}{\Gamma\left(\frac{N-1}{2} + k\right)\,2^{2k}\,k!} \end{aligned}$$

by [3, p. 40]. Hence:

$$I_P(s, t, b_1, b_2) = \frac{\Gamma\left(\frac{N-2}{2}\right)\,\sqrt{\pi}}{\Gamma(P)} \sum_{k\in\mathbb{N}} \frac{\Gamma(2k + P)}{\Gamma\left(\frac{N-1}{2} + k\right)\,2^{2k}\,k!} \frac{(t\,b_2)^{2k}}{(1 - s\,b_1)^{2k+P}}.$$

The result follows from the expansion

$$\frac{\Gamma(2k + P)}{(1 - s\,b_1)^{2k+P}} = \sum_{j\in\mathbb{N}} \frac{\Gamma(j + 2k + P)}{j!} (b_1\,s)^j.$$

□

4. PROOF OF THEOREM 2.1

The cases *(i), (ii), (iii), (iv), (v)* and *(vi)* of Theorem 2.1 will be proved separately at the end of this section.

Theorem 4.1. *Given $A > 0$, $P > 0$, $T > -1$ and $N \in \mathbb{N}$ ($N \geq 2$) such that $1 \leq A + P \leq N + 1 + 2T$, let*

$$I_{A,P,T}(a,b) = \int_{B_N} \frac{(1-|x|^2)^T}{(1-\langle x,a\rangle)^A\,(1-\langle x,b\rangle)^P}\,dx \qquad \forall a \in B_N, \forall b \in B_N$$

and τ a number satisfying both $\frac{P-A}{2} < \tau < P$ and $0 \leq \tau \leq \frac{A+P}{2}$. Then

$$I_{A,P,T}(a,b) \leq \frac{K}{(1-|a|^2)^{\frac{A+P}{2}-\tau}(1-|b|^2)^{\tau}} \qquad \forall a \in B_N, \forall b \in B_N$$

where the constant K is independent of a and b.

Example 4.1. If $P > A$ and $\tau = \frac{A+P}{2}$, then

$$I_{A,P,T}(a,b) \leq \frac{K}{(1-|b|^2)^{\frac{A+P}{2}}} \qquad \forall a \in B_N, \forall b \in B_N,$$

with

$$K = 2^{A+P-1}\,\pi^{\frac{N-1}{2}}\,\frac{\Gamma(T+1)}{\Gamma(P)}\,\Gamma\left(\frac{P-A}{2}\right).$$

Example 4.2. If $P < A$ and $\tau = 0$, then

$$I_{A,P,T}(a,b) \leq \frac{K}{(1-|a|^2)^{\frac{A+P}{2}}} \qquad \forall a \in B_N, \forall b \in B_N,$$

with

$$K = 2^{A+P-1}\,\pi^{\frac{N-1}{2}}\,\frac{\Gamma(T+1)}{\Gamma(A)}\,\Gamma\left(\frac{A-P}{2}\right).$$

Proof. Up to a unitary transform, we assume $a = (|a|, 0, 0, \ldots, 0)$ and $b = (b_1, b_2, 0, \ldots, 0)$.

Proof of Theorem 4.1 in the case $N > 3$. Polar coordinates in $\mathbb{R}^N$ provide the formulas: $x_1 = r\,\cos\theta_1$ with $r = |x|$, $x_2 = r\,\sin\theta_1\,\cos\theta_2$ (the formulas for $x_3, \ldots, x_N$ are available in [9, p. 15]) where θ_1, θ_2,…, $\theta_{N-2} \in]0,\pi[$ and $\theta_{N-1} \in]0, 2\pi[$. The volume element dx becomes $r^{N-1}\,dr\,d\sigma^{(N)}$ where $d\sigma^{(N)}$ denotes the area element on S_N, with

$$d\sigma^{(N)} = (\sin\theta_1)^{N-2}(\sin\theta_2)^{N-3}d\theta_1\,d\theta_2\,d\sigma^{(N-2)}$$

(see [9, p. 15] for full details). Here $\theta_2 \in]0,\pi[$ since $N > 3$. In the following, we will write $s = r\,\cos\theta_1$ and $t = r\,\sin\theta_1$, thus $\langle x, b\rangle = s\,b_1 + t\,b_2\,\cos\theta_2$ and

$$I_{A,P,T}(a,b) = \sigma_{N-2}\int_0^{\pi}\int_0^1 \frac{(1-r^2)^T\,r^{N-1}\,(\sin\theta_1)^{N-2}\,I_P(s,t,b_1,b_2)}{(1-|a|\,s)^A}\,dr\,d\theta_1 \tag{4.1}$$

with $I_P(s,t,b_1,b_2)$ defined in the previous proposition. From [2, p. 29] we notice that

$$\sigma_{N-2}\,\Gamma\left(\frac{N-2}{2}\right)\sqrt{\pi} = 2\,\pi^{\frac{N-1}{2}}.$$

The expansion

$$\frac{1}{(1-|a|\,s)^A} = \sum_{\ell\in\mathbb{N}} \frac{\Gamma(\ell+A)}{\ell!\,\Gamma(A)}\,(|a|\,s)^{\ell}$$

leads to:

$$I_{A,P,T}(a,b) = \frac{2\,\pi^{\frac{N-1}{2}}}{\Gamma(P)\,\Gamma(A)} \sum_{(k,j,\ell)\in\mathbb{N}^3} \frac{\Gamma(j+2k+P)\,\Gamma(\ell+A)}{k!\,j!\,\ell!\,\Gamma(\frac{N-1}{2}+k)}\,(b_1)^j \left(\frac{b_2}{2}\right)^{2k} |a|^\ell\, J_{k,j,\ell}$$

where

$$\begin{aligned} J_{k,j,\ell} &= \int_0^\pi \int_0^1 s^{j+\ell}\, t^{2k}\,(1-r^2)^T\, r^{N-1} (\sin\theta_1)^{N-2}\, dr\, d\theta_1 \\ &= \iint_H s^{j+\ell}\, t^{2k+N-2} (1-s^2-t^2)^T ds\, dt \end{aligned}$$

with H as in Lemma 3.4. Now $J_{k,j,\ell} = 0$ unless $j+\ell = 2h$ ($h \in \mathbb{N}$). Thus:

$$\begin{aligned} &I_{A,P,T}(a,b) \\ &= \frac{\pi^{\frac{N-1}{2}}}{\Gamma(P)\,\Gamma(A)} \sum_{(k,h)\in\mathbb{N}^2} \sum_{j=0}^{2h} \frac{\Gamma(j+2k+P)\,\Gamma(2h-j+A)\,\Gamma\left(h+\frac{1}{2}\right)\,\Gamma(T+1)}{k!\,j!\,(2h-j)!\,\Gamma\left(k+h+\frac{N}{2}+T+1\right)}\,(b_1)^j \left(\frac{b_2}{2}\right)^{2k} |a|^{2h-j} \end{aligned}$$

Taking [3, p. 40] into account:

$$\begin{aligned} I_{A,P,T}(a,b) = \frac{\pi^{\frac{N}{2}}\,\Gamma(T+1)}{\Gamma(P)\,\Gamma(A)} &\sum_{(k,h)\in\mathbb{N}^2} \sum_{j=0}^{2h} \frac{(2h)!\,B(j+2k+P, 2h-j+A)}{2^{2h+2k}\,h!\,k!\,j!\,(2h-j)!} \\ &\times \frac{\Gamma(2k+P+2h+A)}{\Gamma(k+h+\frac{N}{2}+T+1)}\, b_1^j\, b_2^{2k}\, |a|^{2h-j}. \end{aligned} \tag{4.2}$$

Let

$$L = \frac{2^{P+A-1}\,\Gamma(T+1)}{\Gamma(P)\,\Gamma(A)}\,\pi^{\frac{N-1}{2}}.$$

The duplication formula

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right)$$

(see [3, p. 45]) is applied with $2z = 2k+P+2h+A$. Now

$$\Gamma(k+h+\frac{A+P+1}{2}) \le \Gamma\left(k+h+\frac{N}{2}+T+1\right)$$

since Γ increases on $[1,+\infty[$ and

$$1 \le k+h+\frac{A+P+1}{2} \le k+h+\frac{N}{2}+T+1.$$

This leads to:

$$\begin{aligned} &I_{A,P,T}(a,b) \\ &\le L \sum_{(k,h)\in\mathbb{N}^2} \sum_{j=0}^{2h} \frac{(2h)!\,B(j+2k+P, 2h-j+A)\,\Gamma\left(k+h+\frac{A+P}{2}\right)}{h!\,k!\,j!\,(2h-j)!}\, b_1^j b_2^{2k} |a|^{2h-j} \\ &= L \sum_{(k,h)\in\mathbb{N}^2} \frac{\Gamma\left(k+h+\frac{A+P}{2}\right)}{h!\,k!}\, b_2^{2k} \sum_{j=0}^{2h} \frac{(2h)!}{j!\,(2h-j)!}\, b_1^j\, |a|^{2h-j}\, B(j+2k+P, 2h-j+A). \end{aligned}$$

The last sum turns into:

$$\begin{aligned}
&\sum_{j=0}^{2h} \frac{(2h)!\, b_1^j\, |a|^{2h-j}}{j!\,(2h-j)!} \int_0^1 \xi^{j+2k+P-1}\,(1-\xi)^{2h-j+A-1}\,d\xi \\
&= \int_0^1 \left(\sum_{j=0}^{2h} \frac{(2h)!\,(b_1\,\xi)^j\,[(1-\xi)\,|a|]^{2h-j}}{j!\,(2h-j)!} \right) \xi^{2k+P-1}\,(1-\xi)^{A-1}\,d\xi \\
&= \int_0^1 [b_1\,\xi + |a|\,(1-\xi)]^{2h}\,\xi^{2k+P-1}\,(1-\xi)^{A-1}\,d\xi.
\end{aligned}$$

Hence the majorant of $I_{A,P,T}(a,b)$ becomes:

$$\begin{aligned}
&L \int_0^1 \sum_{k\in\mathbb{N}} \frac{(b_2\,\xi)^{2k}}{k!} \left(\sum_{h\in\mathbb{N}} \frac{\Gamma(h+k+\frac{A+P}{2})}{h!} [b_1\,\xi + |a|\,(1-\xi)]^{2h} \right) \xi^{P-1}\,(1-\xi)^{A-1}\,d\xi \\
&= L \int_0^1 \sum_{k\in\mathbb{N}} \frac{\Gamma(k+\frac{A+P}{2})\,(b_2\,\xi)^{2k}}{k!} \left(\frac{1}{1-[b_1\,\xi + |a|\,(1-\xi)]^2} \right)^{k+\frac{A+P}{2}} \xi^{P-1}(1-\xi)^{A-1}\,d\xi
\end{aligned}$$

according to the expansion

$$\frac{\Gamma(C)}{(1-X)^C} = \sum_{h\in\mathbb{N}} \frac{\Gamma(h+C)}{h!}\,X^h$$

with $|X| < 1$ when $C > 0$ (see [8, p. 53]). Here $X = [b_1\,\xi + |a|\,(1-\xi)]^2$ belongs to $]-1,1[$ since b_1 and $|a|$ do and $\xi \in [0,1]$. The same expansion now applies with

$$C = \frac{A+P}{2} \qquad \text{and} \qquad X = \frac{(b_2\,\xi)^2}{1-[b_1\,\xi + |a|\,(1-\xi)]^2}$$

since $|X| < 1$, as

$$\begin{aligned}
\delta(\xi) &:= (b_2\,\xi)^2 + [b_1\,\xi + |a|\,(1-\xi)]^2 \\
&= |b|^2\,\xi^2 + |a|^2\,(1-\xi)^2 + 2\xi(1-\xi)\,b_1\,|a| \\
&\le |b|^2\,\xi^2 + |a|^2\,(1-\xi)^2 + 2\xi(1-\xi)|b|\,|a| \\
&= [\xi\,|b| + |a|\,(1-\xi)]^2 < 1.
\end{aligned}$$

Hence

$$\begin{aligned}
&I_{A,P,T}(a,b) \\
&\le L \int_0^1 \frac{\Gamma\left(\frac{A+P}{2}\right)}{\left(1-\frac{(b_2\,\xi)^2}{1-[b_1\,\xi+|a|\,(1-\xi)]^2}\right)^{\frac{A+P}{2}}} \frac{\xi^{P-1}\,(1-\xi)^{A-1}\,d\xi}{(1-[b_1\,\xi + |a|\,(1-\xi)]^2)^{\frac{A+P}{2}}} \\
&= L\cdot\Gamma\left(\frac{A+P}{2}\right) \int_0^1 \frac{\xi^{P-1}\,(1-\xi)^{A-1}\,d\xi}{(1-[b_1\,\xi + |a|\,(1-\xi)]^2 - (b_2\,\xi)^2)^{\frac{A+P}{2}}}.
\end{aligned}$$

Now

$$\begin{aligned}
1-\delta(\xi) &\ge 1 - [\xi\,|b| + |a|\,(1-\xi)]^2 \\
&\ge 1 - [\xi\,|b| + (1-\xi)]^2 \\
&= \xi(1-|b|)[2-\xi(1-|b|)] \\
&\ge \xi(1-|b|^2)
\end{aligned}$$

since

$$[2-\xi(1-|b|)]-(1+|b|)=(1-\xi)(1-|b|)\geq 0.$$

Similarly,

$$1-\delta(\xi)\geq(1-\xi)(1-|a|^2).$$

Thus

$$\frac{1}{[1-\delta(\xi)]^{\frac{A+P}{2}}}\leq\frac{1}{[(1-\xi)(1-|a|^2)]^{\frac{A+P}{2}-\tau}[\xi(1-|b|^2)]^{\tau}}$$

since $\tau\geq 0$ and $\frac{A+P}{2}-\tau\geq 0$. Finally:

$$I_{A,P,T}(a,b)\leq\frac{L\cdot\Gamma\left(\frac{A+P}{2}\right)}{(1-|a|^2)^{\frac{A+P}{2}-\tau}(1-|b|^2)^{\tau}}\int_0^1\xi^{P-\tau-1}\,(1-\xi)^{A+\tau-\frac{A+P}{2}-1}\,d\xi.$$

This integral converges since $P-\tau>0$ and

$$A+\tau-\frac{A+P}{2}=\frac{A-P}{2}+\tau>0.$$

Now the result follows with

$$K=L\cdot\Gamma\left(\frac{A+P}{2}\right)B\left(P-\tau,\frac{A-P}{2}+\tau\right)=L\,\Gamma(P-\tau)\,\Gamma\left(\frac{A-P}{2}+\tau\right).$$

Proof of Theorem 4.1 in the case $N=3$. Here

$$I_{A,P,T}(a,b)=\int_0^{\pi}\int_0^1\frac{(1-r^2)^T\,r^2\,(\sin\theta_1)\,J_P(s,t,b_1,b_2)}{(1-|a|\,s)^A}dr\,d\theta_1,$$

where

$$J_P(s,t,b_1,b_2)=\int_0^{2\pi}\frac{d\theta_2}{(1-s\,b_1-t\,b_2\,\cos\theta_2)^P}=2\,I_P(s,t,b_1,b_2)$$

with $I_P(s,t,b_1,b_2)$ as in Proposition 3.6, with $N=3$. Hence $I_{A,P,T}(a,b)$ has the same expression as in Formula (4.1), with $N=3$, since $\sigma_1=2$. Thus the proof ends in the same manner as that above in the case $N>3$.

Proof of Theorem 4.1 in the case $N=2$. Now $x_1=s=r\,\cos\theta$ and $x_2=t=r\,\sin\theta$:

$$\begin{aligned}
&I_{A,P,T}(a,b)\\
&=\int_0^{2\pi}\int_0^1\frac{(1-r^2)^T\,r\,dr\,d\theta}{(1-|a|\,s)^A\,(1-s\,b_1-t\,b_2)^P}\\
&=\int_{B_2}\sum_{\ell\in\mathbb{N}}\frac{\Gamma(\ell+A)}{\ell!\,\Gamma(A)}\,(|a|\,s)^{\ell}\sum_{n\in\mathbb{N}}\frac{(t\,b_2)^n}{n!\,\Gamma(P)}\frac{\Gamma(n+P)}{(1-s\,b_1)^{n+P}}\,(1-s^2-t^2)^T\,ds\,dt\\
&=\sum_{(\ell,n,j)\in\mathbb{N}^3}\frac{\Gamma(\ell+A)\,|a|^{\ell}\,(b_2)^n\,\Gamma(j+n+P)\,(b_1)^j}{\ell!\,\Gamma(A)\,n!\,\Gamma(P)\,j!}\int_{B_2}s^{\ell+j}\,t^n\,(1-s^2-t^2)^T\,ds\,dt.
\end{aligned}$$

The last integral vanishes when n is odd or $\ell+j$ odd. Otherwise ($n=2k$ and $\ell+j=2h$), it equals

$$2\int_H s^{\ell+j}\,t^n\,(1-s^2-t^2)^T\,ds\,dt=\frac{\Gamma\left(h+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)\,\Gamma(T+1)}{\Gamma(k+h+T+2)}$$

by Lemma 3.4 and turns into

$$\frac{n!\,(2h)!\,\pi\,\Gamma(T+1)}{2^{2h+2k}\,h!\,k!\,\Gamma(k+h+T+2)}$$

according to [3, p. 40]. Thus $I_{A,P,T}(a,b)$ is again recognized as Formula (4.2) now with $N=2$ and the proof ends as for the case $N>3$. $\square$

We now present an example of a family of functions $\{f_a\}_a$ which is uniformly bounded above in $\mathcal{Y}_{\alpha,\beta,\gamma}$:

Corollary 4.2. *Given* $\beta > -\frac{N+1}{2}$ *(*$N\geq 2$*) let* $\alpha = \frac{N+1}{2}+\beta$ *and* $\gamma > \max(\alpha, -1-\beta)$. *For any* $a\in B_N$ *let* f_a *denote the function defined by:* $f_a(x) = \frac{1}{(1-\langle x,a\rangle)^{2\alpha}}$, $\forall x \in B_N$. *Then* $f_a \in \mathcal{Y}_{\alpha,\beta,\gamma}$, $\forall a \in B_N$. *Moreover, there exists* $K>0$ *such that* $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(f_a) \leq K$, $\forall a \in B_N$.

Remark 4. This constant K is the same as that in the previous theorem, with $A=2\alpha$, $P=2\gamma$ and $T=\beta+\gamma$.

Proof. With the above choices for parameters A, P, T, we actually have: $P>A>0$, $T>-1$ and

$$A+P = 2\alpha+2\gamma = N+1+2\beta+2\gamma = N+1+2T > 1.$$

The conditions $0\leq\tau\leq\alpha+\gamma$ together with $\gamma-\alpha<\tau<2\gamma$ reduce to: $\gamma-\alpha<\tau\leq\alpha+\gamma$. Let

$$J_b(f_a) = (1-|b|^2)^\alpha \int_{B_N} (1-|x|^2)^\beta\, f_a(x)\,(1-|\Phi_b(x)|^2)^\gamma\, dx. \tag{4.3}$$

Now

$$\begin{aligned} J_b(f_a) &= (1-|b|^2)^{\alpha+\gamma} \int_{B_N} \frac{(1-|x|^2)^{\beta+\gamma}}{(1-\langle x,a\rangle)^{N+1+2\beta}\,(1-\langle x,b\rangle)^{2\gamma}}\, dx \\ &\leq K \qquad \forall a \in B_N, \forall b\in B_N \end{aligned}$$

according to Theorem 4.1 applied with $\tau = \alpha+\gamma = \frac{A+P}{2}$. $\square$

4.1. **Proof of Theorem 2.1 in the case *(i)*.** Given $R\in]0,1[$, the subharmonicity of g provides for any $a\in B_N$ the majoration:

$$g(a) \leq \frac{1}{V_a}\int_{B(a,R_a)} g(x)\,dx$$

with V_a the volume of $B(a,R_a)$. From Lemma 3.3, it is clear that:

$$1 \leq \left(2\,\frac{1+R}{1-R}\,\frac{1-|x|^2}{1-\langle x,a\rangle}\right)^A \qquad \forall x\in B(a,R_a)$$

with $A=2\alpha>0$. Now $g(x)\geq 0$, $\forall x\in B_N$. With f_a as in Corollary 4.2, this leads to:

$$V_a\, g(a) \leq \left(2\,\frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} (1-|x|^2)^A\, f_a(x)\, g(x)\, dx.$$

Now

$$A = \alpha+\beta+\frac{N+1}{2} = \alpha+\beta+N-\frac{N-1}{2},$$

thus

$$\begin{aligned} V_a\, g(a) &\leq \left(2\,\frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{(1-|x|^2)^{\lambda+\alpha+\beta+N}\, f_a(x)\, g(x)}{(1-|x|^2)^{\lambda+\frac{N-1}{2}}}\, dx \\ &\leq C'K\left(2\,\frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{dx}{(1-|x|^2)^{\lambda+\frac{N-1}{2}}} \end{aligned}$$

from Corollary 4.2. Lemmas 1 and 5 of [5] provide

$$\left(\frac{1-|x|^2}{1-|a|^2}\right)^{\lambda+\frac{N-1}{2}} \geq C_{\lambda+\frac{N-1}{2}} \qquad \forall x \in B(a, R_a),$$

with $C_{\lambda+\frac{N-1}{2}}$ defined in the same pattern as C_β in the proof of Proposition 3.1. Finally:

$$V_a\, g(a) \leq \frac{C'K}{C_{\lambda+\frac{N-1}{2}}} \left(2\,\frac{1+R}{1-R}\right)^A \frac{V_a}{(1-|a|^2)^{\lambda+\frac{N-1}{2}}},$$

thus

$$M_{\mathcal{X}_{\lambda+\frac{N-1}{2}}}(g) \leq \frac{C'K}{C_{\lambda+\frac{N-1}{2}}} \left(2\,\frac{1+R}{1-R}\right)^{2\alpha} \qquad \forall R \in]0,1[.$$

The majorant is an increasing function with respect to R. Letting R tend toward 0^+, we get:

$$M_{\mathcal{X}_{\lambda+\frac{N-1}{2}}}(g) \leq \frac{C'K}{C_{\lambda+\frac{N-1}{2}}}\, 2^{2\alpha}.$$

4.2. **Proof of Theorem 2.1 in the case *(ii)*.** Here we work with f_a defined by:

$$f_a(x) = \frac{1}{(1-\langle x, a\rangle)^A} \qquad \text{where} \qquad A = \alpha+\beta+N.$$

Theorem 4.1 applies once again, with $A = N+1+2\beta > \frac{N-1}{2} > 0$, $P = 2\gamma > 0$ and $T = \beta+\gamma > -1$ (because $\gamma > -1-\beta$). Condition $A+P = N+1+2T$ is fulfilled too. Moreover $\tau := \alpha+\gamma = \beta+\gamma+1$ satisfies both $0 \leq \tau \leq \beta+\gamma+\frac{N+1}{2}$ (obviously $0 < \beta+\gamma+1$ and $1 < \frac{N+1}{2}$) and $\gamma-\beta-\frac{N+1}{2} < \tau < 2\gamma$:

$$\tau-\gamma+\beta+\frac{N+1}{2} = 2\beta+\frac{N+3}{2} > 0 \quad \text{and} \quad 2\gamma-\tau = \gamma-1-\beta > 0.$$

With such a choice for τ we have

$$\frac{A+P}{2} - \tau = \frac{N+1}{2} - 1 = \frac{N-1}{2},$$

thus

$$I_{A,P,T}(a,b) \leq \frac{K}{(1-|a|^2)^{\frac{N+1}{2}-1}(1-|b|^2)^{\alpha+\gamma}} \qquad \forall a \in B_N, \forall b \in B_N. \tag{4.4}$$

Hence, $J_b(f_a)$ defined in Formula (4.3) now satisfies

$$J_b(f_a) \leq \frac{K}{(1-|a|^2)^{\frac{N-1}{2}}} \qquad \forall a \in B_N, \forall b \in B_N. \tag{4.5}$$

In other words,

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(f_a) \leq \frac{K}{(1-|a|^2)^{\frac{N-1}{2}}} \qquad \forall a \in B_N. \tag{4.6}$$

This implies:

$$M_{\mathcal{X}_{\lambda+\alpha+\beta+N}}(g\, f_a) \leq \frac{C'K}{(1-|a|^2)^{\frac{N-1}{2}}} \qquad \forall a \in B_N. \tag{4.7}$$

With R and V_a as in the previous proof, we obtain here:

$$V_a\, g(a) \le \left(2\,\frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{(1-|x|^2)^{\lambda+\alpha+\beta+N}\, f_a(x)\, g(x)}{(1-|x|^2)^{\lambda}}\, dx$$
$$\le \frac{C'K}{(1-|a|^2)^{\frac{N-1}{2}}} \left(2\,\frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{dx}{(1-|x|^2)^{\lambda}}$$

and the last integral is majorized by $\frac{V_a}{C_\lambda\,(1-|a|^2)^\lambda}$ with C_λ defined similarly to C_β in the proof of Proposition 3.1. Finally:

$$M_{\mathcal{X}_{\lambda+\frac{N-1}{2}}}(g) \le \frac{C'K}{C_\lambda}\, 2^{N+1+2\beta}.$$

4.3. **Proof of Theorem 2.1 in the case *(iii)* .** Here f_a is defined by:

$$f_a(x) = \frac{1}{(1-\langle x,a\rangle)^A (1-|x|^2)^{\beta+\gamma}} \qquad \forall x \in B_N,$$

where $A = N+1-2\gamma > 0$. Theorem 4.1 is applied with $P = 2\gamma > 0$ and $T = 0 > -1$. Thus

$$A+P = N+1 = N+1+2T.$$

We have to choose τ satisfying both

$$0 \le \tau \le \frac{N+1}{2} \quad \text{and} \quad 2\gamma - \frac{N+1}{2} < \tau < 2\gamma.$$

Now

$$\tau := \frac{N+1}{2} = \frac{A+P}{2} = \alpha+\gamma$$

fulfills the last condition since:

$$2\gamma - \tau = 2\left(\gamma - \frac{N+1}{4}\right) > 0 \quad \text{and} \quad \tau - 2\gamma + \frac{N+1}{2} = 2\left(\frac{N+1}{2} - \gamma\right) > 0.$$

Formula (4.3) implies $J_b(f_a) \le K$ for all $a \in B_N$ and all $b \in B_N$. Thus $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(f_a) \le K$, $\forall a \in B_N$. As before,

$$V_a\, g(a) \le \left(2\,\frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{(1-|x|^2)^{A+\beta+\gamma}\, g(x)}{(1-\langle x,a\rangle)^A\,(1-|x|^2)^{\beta+\gamma}}\, dx.$$

Now

$$A+\beta+\gamma = N+1-\gamma+\beta$$
$$= N+1+\alpha-\frac{N+1}{2}+\beta$$
$$= \alpha+\beta+N-\frac{N-1}{2},$$

whence

$$V_a\, g(a) \le \left(2\,\frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{(1-|x|^2)^{\alpha+\beta+N}\, f_a(x)\, g(x)}{(1-|x|^2)^{\frac{N-1}{2}}}\, dx$$
$$\le C'K\left(2\,\frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{dx}{(1-|x|^2)^{\lambda+\frac{N-1}{2}}}$$

and the proof ends as in the case *(i)*. Here

$$M_{\mathcal{X}_{\lambda+\frac{N-1}{2}}}(g) \le \frac{C'K}{C_{\lambda+\frac{N-1}{2}}}\, 2^{N+1-2\gamma}.$$

4.4. **Proof of Theorem 2.1 in the case *(iv)*.** Here f_a is defined by:

$$f_a(x) = \frac{1}{(1-\langle x,a\rangle)^A(1-|x|^2)^\beta} \qquad \forall x \in B_N,$$

where $A = N+1$, $T=\gamma$ and $P=2\gamma$ thus $A+P = N+1+2T$, allowing us to use Theorem 4.1, with $\tau = \alpha+\gamma = 1+\gamma$ (since $0 \le \tau \le \frac{N+1}{2}+\gamma$ and $\gamma - \frac{N+1}{2} < \tau < 2\gamma$). Hence Inequalities (4.4), (4.5), (4.6) and (4.7) follow. Now

$$V_a\, g(a) \le \left(2\,\frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} (1-|x|^2)^{A+\beta} f_a(x)\, g(x)\, dx. \tag{4.8}$$

Since $A+\beta = \alpha+\beta+N$, this turns into:

$$V_a\, g(a) \le \left(2\,\frac{1+R}{1-R}\right)^A \int_{B(a,R_a)} \frac{M\mathcal{X}_{\lambda+\alpha+\beta+N}(g\, f_a)}{(1-|x|^2)^\lambda}\, dx$$

and the proof ends as in the case *(ii)*, here with:

$$M\mathcal{X}_{\lambda+\frac{N-1}{2}}(g) \le \frac{C'K}{C_\lambda}\, 2^{N+1}.$$

4.5. **Proof of Theorem 2.1 in the case *(v)*.** Here f_a is defined by:

$$f_a(x) = \frac{1}{(1-\langle x,a\rangle)^A(1-|x|^2)^\gamma} \qquad \forall x \in B_N,$$

where

$$A = N+1+2(\beta-\gamma) > N+1-\frac{N+3}{2} = \frac{N-1}{2} > 0.$$

With $P = 2\gamma > 0$ and $T=\beta$, the condition $A+P = N+1+2T$ of Theorem 4.1 is fulfilled. Moreover $\tau := \alpha+\gamma = 1+\beta$ satisfies

$$0 \le \tau \le \frac{N+1}{2}+\beta \quad \text{and} \quad 2\gamma - \frac{N+1}{2} - \beta < \tau < 2\gamma$$

since:

$$2\gamma-\tau = 2\gamma-(1+\beta) > 0 \quad \text{and} \quad \tau - 2\gamma + \frac{N+1}{2} + \beta = -2\gamma + \frac{N+3}{2} + 2\beta > 0.$$

Again

$$\frac{A+P}{2} - \tau = \frac{N+1}{2} - 1 = \frac{N-1}{2}$$

and inequalities (4.4) to (4.7) follow. Formula (4.8) still holds with $(1-|x|^2)^{A+\gamma}$ instead of $(1-|x|^2)^{A+\beta}$. Here

$$A+\gamma = N+1+2\beta-\gamma = N+\alpha+\beta$$

and the conclusion follows as in the previous case. Finally:

$$M\mathcal{X}_{\lambda+\frac{N-1}{2}}(g) \le \frac{C'K}{C_\lambda}\, 2^{N+1+2(\beta-\gamma)}.$$

4.6. **Proof of Theorem 2.1 in the case *(vi)*.** Here f_a is defined by:

$$f_a(x) = \frac{1}{(1-\langle x,a\rangle)^A(1-|x|^2)^\alpha} \qquad \forall x \in B_N$$

with $A = N+\beta > \frac{N-1}{2} > 0$, $P = 2\gamma > 0$, $T = \frac{\beta-1}{2}+\gamma > -1$ (actually $T+1 = \frac{\beta+1}{2}+\gamma > 0$). The use of Theorem 4.1 is allowed since

$$A+P = N+1+\beta-1+2\gamma = N+1+2T.$$

Now $\tau := \alpha+\gamma = \frac{\beta+1}{2}+\gamma$ satisfies $0 \le \tau \le \frac{N+\beta}{2}+\gamma$ (because of $\gamma > -\frac{\beta+1}{2}$). Moreover $\gamma - \frac{N+\beta}{2} < \tau < 2\gamma$ is fulfilled too since

$$\frac{\beta+1}{2} < \gamma \quad \text{and} \quad \beta+1+(N+\beta) = 1+N+2\beta > 0.$$

In addition,

$$\frac{A+P}{2} - \tau = \frac{N+\beta}{2} - \frac{\beta+1}{2} = \frac{N-1}{2}.$$

Again it induces Formula (4.6). With $(1-|x|^2)^{A+\beta}$ replaced by $(1-|x|^2)^{A+\alpha}$, inequality (4.8) remains valid. Since $A+\alpha = N+\alpha+\beta$, the conclusion is once again obtained in a similar way as in the cases *(iv)* and *(v)*, here with

$$M_{\mathcal{X}_{\lambda+\frac{N-1}{2}}}(g) \le \frac{C'K}{C_\lambda}\, 2^{N+\beta}.$$

5. THE SITUATION WITH RADIAL SUBHARMONIC FUNCTIONS

5.1. The example of $u : x \mapsto (1-|x|^2)^{-A}$ with $A \ge 0$.

Proposition 5.1. *Given $P \ge 1$, $T > -1$ and $N \in \mathbb{N}$ $(N \ge 2)$ such that $P \le N+1+2T$, let*

$$I_{P,T}(b) = \int_{B_N} \frac{(1-|x|^2)^T}{(1-\langle x,b\rangle)^P}\, dx \qquad \forall b \in B_N.$$

Then

$$I_{P,T}(b) \le \frac{K'}{(1-|b|^2)^{P/2}} \qquad \forall b \in B_N\,,$$

(equality holds when $P = N+1+2T$) with

$$K' = \frac{\Gamma(T+1)}{\Gamma\left(\frac{P+1}{2}\right)}\, \pi^{\frac{N}{2}}.$$

Proof. Letting $A \to 0^+$ in Theorem 4.1, the majorization of $I_{P,T}(b)$ is an immediate result, since K (as a function of A) tends towards K': see Example 4.1. Nonetheless, we still have to show that equality holds in the case $P = N+1+2T$.

Proof in the case $N \ge 3$. Up to a unitary transform, we may assume $b = (|b|, 0, 0, \dots, 0)$, so that $\langle x,b\rangle = |b|\, x_1 = |b|\, r\, \cos\theta_1$ with $\theta_1 \in]0,\pi[$ (we will have $\theta_1 \in]0,2\pi[$ in the case $N=2$). Now

$$dx = r^{N-1}\,(\sin\theta_1)^{N-2}\, dr\, d\theta_1\, d\sigma^{(N-1)},$$

with the same notations as in the proof of Theorem 4.1. Here:

$$I_{P,T}(b) = \sigma_{N-1} \int_0^\pi \int_0^1 \frac{(1-r^2)^T\, r^{N-1}\,(\sin\theta_1)^{N-2}}{(1-|b|\, r\, \cos\theta_1)^P}\, dr\, d\theta_1.$$

Then

$$I_{P,T}(b) = \sigma_{N-1} \sum_{n\in\mathbb{N}} \frac{\Gamma(n+P)}{n!\,\Gamma(P)}\,|b|^n \iint_H s^n\, t^{N-2}\,(1-s^2-t^2)^T\,ds\,dt \tag{5.1}$$

with $s = r\cos\theta_1$ and $t = r\sin\theta_1$. This integral vanishes for odd n. If $n = 2k$, its value is given by Lemma 3.4. Thus

$$I_{P,T}(b) = \frac{\sigma_{N-1}\,\Gamma\left(\frac{N-1}{2}\right)\Gamma(T+1)}{2\,\Gamma(P)} \sum_{k\in\mathbb{N}} \frac{|b|^{2k}\,\Gamma\left(k+\frac{1}{2}\right)\,\Gamma(2k+P)}{(2k)!\,\Gamma\left(k+\frac{N}{2}+T+1\right)}.$$

Now [2, p. 29] and [3, p. 40] lead to:

$$I_{P,T}(b) = \frac{\Gamma(T+1)}{\Gamma(P)}\,\pi^{\frac{N-1}{2}} \sum_{k\in\mathbb{N}} \frac{|b|^{2k}\sqrt{\pi}\,\Gamma(2k+P)}{2^{2k}\,k!\,\Gamma\left(k+\frac{N}{2}+T+1\right)}.$$

Through the duplication formula ([3, p. 45]), it follows that:

$$\begin{aligned} I_{P,T}(b) &= \frac{\Gamma(T+1)}{\Gamma(P)}\,\pi^{\frac{N-1}{2}} \sum_{k\in\mathbb{N}} \frac{|b|^{2k}\,2^{2k+P-1}\,\Gamma\left(k+\frac{P}{2}\right)\,\Gamma\left(k+\frac{P+1}{2}\right)}{2^{2k}\,k!\,\Gamma\left(k+\frac{N}{2}+T+1\right)} \\ &= K' \sum_{k\in\mathbb{N}} \frac{\Gamma\left(k+\frac{P}{2}\right)}{k!\,\Gamma\left(\frac{P}{2}\right)}\,|b|^{2k} \end{aligned}$$

with

$$K' = \frac{\Gamma(T+1)}{\Gamma(P)}\,\pi^{\frac{N-1}{2}}\,2^{P-1}\Gamma\left(\frac{P}{2}\right).$$

Another application of the duplication formula provides the final expression of K'.

Proof in the case $N = 2$. Now

$$I_{P,T}(b) = \int_0^{2\pi}\int_0^1 \frac{(1-r^2)^T\,r}{(1-|b|\,r\,\cos\theta)^P}\,dr\,d\theta.$$

Then

$$I_{P,T}(b) = \sum_{n\in\mathbb{N}} \frac{\Gamma(n+P)}{n!\,\Gamma(P)}\,|b|^n \left(\int_0^1 r^{n+1}\,(1-r^2)^T\,dr\right)\left(\int_0^{2\pi}(\cos\theta)^n\,d\theta\right).$$

The last integral equals $2\int_0^{\pi}(\cos\theta)^n\,d\theta$ for any n. As $\sigma_1 = 2$, here we recognize the same expression as in formula (5.1), replacing N by 2. Hence the same conclusion. ☐

Corollary 5.2. *Given* $\alpha \geq 0$, $\beta \geq -\frac{N+1}{2}$ *and* $\gamma > \frac{N-1}{2}$, *let* $A = \frac{N+1}{2}+\beta$ *and* u *defined on* B_N *by:*

$$u(x) = \frac{1}{(1-|x|^2)^A} \qquad \forall x \in B_N.$$

Then $u \in \mathcal{RSY}_{\alpha,\beta,\gamma}$ *and* $M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \leq K'$ *where* K' *stems from Proposition 5.1 (with* $P = 2\gamma > 1$ *and* $T = \beta+\gamma-A = \gamma - \frac{N+1}{2} > -1$*).*

Proof. The subharmonicity of u follows from Lemma 3.5 since $A \geq 0$. Let $J_b(u)$ be defined similarly as in formula (4.3). Then

$$J_b(u) = (1-|b|^2)^{\alpha+\gamma}\int_{B_N} \frac{(1-|x|^2)^{\beta+\gamma-A}}{(1-\langle x,b\rangle)^P}\,dx.$$

As

$$N+1+2T = N+1+2\gamma-(N+1) = P,$$

Proposition 5.1 provides:

$$J_b(u) \leq (1-|b|^2)^{\alpha+\gamma}\,\frac{K'}{(1-|b|^2)^{P/2}} \leq K'$$

since $\alpha \geq 0$. The conclusion proceeds from

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) = \sup_{b\in B_N} J_b(u).$$

□

5.2. **Proof of Theorem 2.2.** Let A and u be defined as in Corollary 5.2. With R and V_a as in the proof of Theorem 2.1:

$$\begin{aligned} V_a\, g(a) &\leq \int_{B(a,R_a)} (1-|x|^2)^A\, u(x)\, g(x)\, dx \\ &= \int_{B(a,R_a)} \frac{(1-|x|^2)^{\lambda+\alpha+\beta+N}\, u(x)\, g(x)\, dx}{(1-|x|^2)^{\lambda+\alpha+\frac{N-1}{2}}} \end{aligned}$$

since:

$$A = \frac{N+1}{2} + \beta = \beta + N - \frac{N-1}{2}.$$

This leads to:

$$\begin{aligned} V_a\, g(a) &\leq C''\, K' \int_{B(a,R_a)} \frac{dx}{(1-|x|^2)^{\lambda+\alpha+\frac{N-1}{2}}} \\ &\leq \frac{C''\, K'\, V_a}{C_{\lambda+\alpha+\frac{N-1}{2}}}\, \frac{1}{(1-|a|^2)^{\lambda+\alpha+\frac{N-1}{2}}}, \end{aligned}$$

with $C_{\lambda+\alpha+\frac{N-1}{2}}$ defined in the same way as C_β in the proof of Proposition 3.1. We obtain finally:

$$M_{\mathcal{X}_{\lambda+\alpha+\frac{N-1}{2}}}(g) \leq \frac{C''\, K'}{C_{\lambda+\alpha+\frac{N-1}{2}}}.$$

6. ANNEX: THE SETS $\mathcal{SX}_\lambda$ AND $\mathcal{SY}_{\alpha,\beta,\gamma}$ FOR SOME SPECIAL VALUES OF λ, α, β, γ

Throughout the paper, it was assumed that $\gamma \geq 0$. When $\gamma \leq 0$, the set $\mathcal{SY}_{\alpha,\beta,\gamma}$ is related to other sets of the same kind by:

Proposition 6.1. *Given $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\gamma \leq 0$, then*

$$\mathcal{Y}^+_{\alpha+\gamma,\beta+\gamma,0} \subset \mathcal{Y}^+_{\alpha,\beta,\gamma} \subset \mathcal{Y}^+_{\alpha+s\gamma,\beta-s\gamma,0} \qquad \forall s \in [-1,1],$$

where $\mathcal{Y}^+_{\alpha,\beta,\gamma}$ denotes the subset of $\mathcal{Y}_{\alpha,\beta,\gamma}$ consisting of all non-negative $u \in \mathcal{Y}_{\alpha,\beta,\gamma}$ (not necessarily subharmonic).

Proof. For any $a \in B_N$ and $x \in B_N$, the following holds:

$$(1-|a|^2)^\alpha(1-|x|^2)^\beta(1-|\Phi_a(x)|^2)^\gamma = (1-|a|^2)^{\alpha+\gamma}(1-|x|^2)^{\beta+\gamma}(1-\langle a,x\rangle)^{-2\gamma}. \tag{6.1}$$

Since $\langle a,x\rangle \in]-1,1[$ through the Cauchy-Schwarz inequality, we have $(1-\langle a,x\rangle)^{-2\gamma} \leq 2^{-2\gamma}$ as $-2\gamma \geq 0$. If $u \in \mathcal{Y}_{\alpha+\gamma,\beta+\gamma,0}$ and $u(x) \geq 0$, $\forall x \in B_N$, then $u \in \mathcal{Y}_{\alpha,\beta,\gamma}$ with

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \leq 2^{-2\gamma}\, M_{\mathcal{Y}_{\alpha+\gamma,\beta+\gamma,0}}(u).$$

Also, $\langle a,x\rangle < |a|$ and $\langle a,x\rangle < |x|$, thus

$$(1-\langle a,x\rangle)^{(s-1)\gamma} \geq (1-|a|)^{(s-1)\gamma} \quad \text{and} \quad (1-\langle a,x\rangle)^{(-s-1)\gamma} \geq (1-|x|)^{(-s-1)\gamma}$$

since $(s-1)\gamma \geq 0$ and $(-s-1)\gamma \geq 0$. Moreover

$$1-|a| = \frac{1-|a|^2}{1+|a|} \geq \frac{1-|a|^2}{2} \quad \text{and} \quad 1-|x| \geq \frac{1-|x|^2}{2},$$

thus

$$(1-\langle a,x\rangle)^{-2\gamma} \geq (1-|a|^2)^{(s-1)\gamma}(1-|x|^2)^{(-s-1)\gamma}\left(\frac{1}{2}\right)^{-2\gamma}.$$

Finally

$$(1-|a|^2)^{\alpha}(1-|x|^2)^{\beta}(1-|\Phi_a(x)|^2)^{\gamma} \geq 2^{2\gamma}(1-|a|^2)^{\alpha+s\gamma}(1-|x|^2)^{\beta-s\gamma}.$$

Any non-negative $u \in \mathcal{Y}_{\alpha,\beta,\gamma}$ then belongs to $\mathcal{Y}_{\alpha+s\gamma,\beta-s\gamma,0}$ with

$$M_{\mathcal{Y}_{\alpha+s\gamma,\beta-s\gamma,0}}(u) \leq 2^{-2\gamma}\, M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u).$$

□

Remark 5. Even with $\gamma \leq 0$, Proposition 3.1 still holds, since

$$\begin{aligned}(1-|\Phi_a(x)|^2)^{\gamma} &= \left(\frac{1-\langle a,x\rangle}{1-|x|^2}\right)^{-\gamma}\left(\frac{1-\langle a,x\rangle}{1-|a|^2}\right)^{-\gamma} \\ &\geq \left(\frac{1}{2}\right)^{-\gamma}\left(\frac{1}{4}\right)^{-\gamma} = 2^{3\gamma} \quad \forall x \in B(a,R_a)\end{aligned}$$

according to Lemma 3.3. For the proof of Proposition 3.1 in the case $\gamma \leq 0$, it is enough to replace $(1-R^2)^{\gamma}$ in formula (3.1) by $2^{3\gamma}$.

Proposition 6.2. *If $\lambda < 0$, then the set $\mathcal{SX}_{\lambda}$ contains only the function $u \equiv 0$ on B_N.*

Proof. Given $u \in \mathcal{SX}_{\lambda}$ and $\xi \in B_N$, let $r \in]|\xi|,1[$. Then

$$u(\xi) \leq \max_{|x|\leq r} u(x) = \max_{|x|=r} u(x)$$

according to the maximum principle (see [2, pp. 48–49]). Thus

$$0 \leq u(\xi) \leq M_{\mathcal{X}_{\lambda}}(u)\,(1-r^2)^{-\lambda}$$

which tends towards 0 as $r \to 1^-$ (since $-\lambda > 0$). Finally $u(\xi) = 0$. □

Remark 6. When $\alpha < 0$, it is not compulsory that $\mathcal{SY}_{\alpha,\beta,\gamma} = \{0\}$. For instance, with α, β, γ as in case *(ii)* of Theorem 2.1, we have $\alpha = \beta + 1 > \frac{1-N}{4}$. It is thus possible to choose β in such a way that $\alpha < 0$. In Subsection 4.2 we have an example of function $f_a \in \mathcal{SY}_{\alpha,\beta,\gamma}$ (with a fixed in B_N) and this function is not vanishing. Similarly $\beta < 0$ does not imply $\mathcal{SY}_{\alpha,\beta,\gamma} = \{0\}$. In Table 2.1 we have several examples of such situations: see Subsections 4.1 to 4.6 for examples of non-vanishing subharmonic functions belonging to such sets $\mathcal{SY}_{\alpha,\beta,\gamma}$.

Proposition 6.3. *Let $\gamma \in \mathbb{R}$ and $(\alpha,\beta) \in \mathbb{R}^2$ such that $\alpha+\beta < -N$, then $\mathcal{SY}_{\alpha,\beta,\gamma} = \{0\}$.*

Proof. Given $R \in]0,1[$, let $K_{R,\gamma} = (1-R^2)^{\gamma}$ if $\gamma \geq 0$, or $K_{R,\gamma} = 2^{3\gamma}$ if $\gamma \leq 0$. Then: $(1-|\Phi_a(x)|^2)^{\gamma} \geq K_{R,\gamma}$, $\forall a \in B_N$, $\forall x \in B(a,R_a)$ according to Remark 5 (also remember that $|\Phi_a| < R$ on $B(a,R_a)$, see [6]). With C_β as in the proof of Proposition 3.1, the following inequalities hold for any $u \in \mathcal{SY}_{\alpha,\beta,\gamma}$ and any $a \in B_N$. The second inequality is based upon $u \geq 0$ and the last one makes use of the suharmonicity of u.

$$\begin{aligned}(1-|a|^2)^{-\alpha} M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) &\geq \int_{B_N} (1-|x|^2)^\beta u(x) (1-|\Phi_a(x)|^2)^\gamma dx \\ &\geq K_{R,\gamma} \int_{B(a,R_a)} (1-|x|^2)^\beta u(x)\, dx \\ &\geq K_{R,\gamma} C_\beta (1-|a|^2)^\beta \int_{B(a,R_a)} u(x)\, dx \\ &\geq K_{R,\gamma} C_\beta (1-|a|^2)^\beta V_a u(a)\end{aligned}$$

where the volume V_a of $B(a, R_a)$ satisfies:

$$V_a \geq \frac{\sigma_N}{N} \left(\frac{R}{1+R}\right)^N (1-|a|^2)^N$$

(see the end of the proof of Proposition 3.1). Thus

$$u(a) \leq \kappa (1-|a|^2)^{-\alpha-\beta-N} \qquad \forall a \in B_N,$$

the constant $\kappa > 0$ being independent of a.

Given $\xi \in B_N$, the maximum principle now provides for any $r \in]|\xi|, 1[$:

$$0 \leq u(\xi) \leq \max_{|x|\leq r} u(x) = \max_{|x|=r} u(x) \leq \kappa (1-r^2)^{-\alpha-\beta-N}$$

which tends towards 0 as $r \to 1^-$, since $-\alpha-\beta-N > 0$. Hence $u(\xi) = 0$. □

Proposition 6.4. *Given $\gamma \geq 0$, $\alpha < -\gamma$ and $\beta \in \mathbb{R}$, then $\mathcal{SY}_{\alpha,\beta,\gamma} = \{0\}$.*

Proof. Since $1 - \langle x, a\rangle \in]0, 2[$, we have $(1-\langle a, x\rangle)^{-2\gamma} \geq 2^{-2\gamma}$, $\forall x \in B_N$, $\forall a \in B_N$. Given $u \in \mathcal{SY}_{\alpha,\beta,\gamma}$, $\xi \in B_N$ and $r \in]0, 1-|\xi|[$, the formula (6.1) leads to:

$$M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \geq (1-|a|^2)^{\alpha+\gamma} 2^{-2\gamma} \int_{B(\xi,r)} (1-|x|^2)^{\beta+\gamma} u(x)\, dx \qquad \forall a \in B_N$$

since $u \geq 0$ on $B_N \supset B(\xi, r)$. Now $|x| \leq |\xi| + r$, $\forall x \in B(\xi, r)$. Let $L_\xi = [1-(|\xi|+r)^2]^{\beta+\gamma}$ if $\beta+\gamma \geq 0$, or $L_\xi = 1$ if $\beta+\gamma \leq 0$. Then

$$(1-|a|^2)^{-\alpha-\gamma} 2^{2\gamma} M_{\mathcal{Y}_{\alpha,\beta,\gamma}}(u) \geq L_\xi \int_{B(\xi,r)} u(x)\, dx \geq L_\xi \frac{\sigma_N}{N} r^N u(\xi) \qquad \forall a \in B_N$$

since u is subharmonic and the volume of $B(\xi, r)$ is $\frac{\sigma_N}{N} r^N$. Finally, with ξ fixed, we have:

$$0 \leq u(\xi) \leq \kappa_\xi (1-|a|^2)^{-\alpha-\gamma} \qquad \forall a \in B_N,$$

the constant $\kappa_\xi > 0$ being independent of a. Hence $u(\xi) = 0$, letting $|a| \to 1^-$. □

REFERENCES

[1] A.B. ALEKSANDROV, Function theory in the ball, in: *Several Complex Variables, Part II*, Encyclopedia of Mathematical Sciences, Volume 8, Springer Verlag, 1994.

[2] W.K. HAYMAN AND P.B. KENNEDY, *Subharmonic Functions, Vol. I*, London Mathematical Society Monographs, No. 9. Academic Press, London–New York, 1976.

[3] R. REMMERT, *Classical Topics in Complex Function Theory*, Graduate Texts in Mathematics, 172. Springer-Verlag, New York, 1998.

[4] W. RUDIN, Function theory in the unit ball of $\mathbb{C}^n$, *Fundamental Principles of Mathematical Science*, 241, Springer-Verlag, New York-Berlin, 1980.

[5] R. SUPPER, Bloch and gap subharmonic functions, *Real Analysis Exchange*, **28**(2) (2003), 395–414.

[6] R. SUPPER, Subharmonic functions with a Bloch type growth, *Integral Transforms and Special Functions*, **16**(7) (2005), 587–596.

[7] R. YONEDA, Pointwise multipliers from BMOA^{α} to BMOA^{β}, *Complex Variables, Theory and Applications*, **49**(14) (2004), 1045–1061.

[8] K.H. ZHU, *Operator Theory in Function Spaces*, Monographs and Textbooks in Pure and Applied Mathematics, 139. Marcel Dekker, Inc., New York, 1990.

[9] C. ZUILY, *Distributions et équations aux dérivées partielles*, Collection Méthodes, Hermann, Paris, 1986.