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APPLICATIONS OF NUNOKAWA'S THEOREM

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ABSTRACT. The object of the present paper is to give applications of the Nunokawa Theorem [Proc. Japan Acad. Ser. A Math. Sci. 69 (1993), 234-237]. Our results have some interesting examples as special cases .

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1. INTRODUCTION

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. It is known that the class

$$B(\mu) = \left\{ f(z) \in \mathcal{A} : \operatorname{Re} \left\{ f'(z) \left\{ \frac{f(z)}{z} \right\}^{\mu-1} \right\} > 0, \;\; \mu > 0, \, z \in U \right\} \tag{1.2}$$

is the class of univalent functions in U ([3]).

To derive our main theorem, we need the following lemma due to Nunokawa [2].

Lemma 1.1. *Let $p(z)$ be analytic in U, with $p(0) = 1$ and $p(z) \neq 0$ $(z \in U)$. If there exists a point $z_0 \in U$, such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \textit{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha \;\; (\alpha > 0),$$

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162-04

then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

where $k \geq 1$ *when* $\arg p(z_0) = \frac{\pi}{2}\alpha$ *and* $k \leq -1$ *when* $\arg p(z_0) = -\frac{\pi}{2}\alpha$

In [1] , Miller and Mocanu proved the following theorem.

Theorem A. *Let* $\beta_0 = 1.21872...$, *be the solution of*

$$\beta\pi = \frac{3}{2}\pi - \tan^{-1}\beta$$

and let

$$\alpha = \alpha(\beta) = \beta + \frac{2}{\pi}\tan^{-1}\beta$$

for $0 < \beta < \beta_0$.

If $p(z)$ *is analytic in* U, *with* $p(0) = 1$, *then*

$$p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \Rightarrow p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}$$

or

$$|\arg(p(z) + zp'(z))| < \frac{\pi}{2}\alpha \Rightarrow |\arg p(z)| < \frac{\pi}{2}\beta.$$

Corresponding to Theorem A, we will obtain a result which is useful in obtaining applications of analytic function theory.

2. MAIN RESULTS

Now we derive:

Theorem 2.1. *Let* $p(z)$ *be analytic in* U, *with* $p(0) = 1$ *and* $p(z) \neq 0$ $(z \in U)$ *and suppose that*

$$|\arg(p(z) + \beta zp'(z))| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\beta\alpha\right) \qquad (\alpha > 0, \beta > 0),$$

then we have

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \textit{for} \quad z \in U.$$

Proof. If there exists a point $z_0 \in U$, such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha \ \ (\alpha > 0),$$

then from Lemma 1.1, we have
(i) for the case $\arg p(z_0) = \frac{\pi}{2}\alpha$,

$$\begin{aligned}\arg(p(z) + \beta z_0 p'(z_0)) &= \arg\ p(z_0)\left\{1 + \beta\frac{z_0 p'(z_0)}{p(z_0)}\right\} \\ &= \frac{\pi}{2}\alpha + \arg(1 + i\beta\alpha k) \geq \frac{\pi}{2}\alpha + \tan^{-1}\beta\alpha.\end{aligned}$$

This contradicts our condition in the theorem.
(ii) for the case $\arg p(z_0) = -\frac{\pi}{2}\alpha$, the application of the same method as in (i) shows that

$$\arg(p(z) + \beta z_0 p'(z_0)) \leq -\left(\frac{\pi}{2}\alpha + \tan^{-1}\beta\alpha\right).$$

This also contradicts the assumption of the theorem, hence the theorem is proved. □

Making $p(z) = f'(z)$ for $f(z) \in \mathcal{A}$ in Theorem 2.1, we have

Example 2.1. If $f(z) \in \mathcal{A}$ satisfies

$$|\arg(f'(z) + \beta z f''(z))| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\beta\alpha\right)$$

then we have

$$|\arg\ f'(z)| < \frac{\pi}{2}\alpha\,,$$

where $\alpha > 0,\ \beta > 0$ and $z \in U$.

Further, taking $p(z) = \frac{f(z)}{z}$ for $f(z) \in \mathcal{A}$ in Theorem 2.1, we have

Example 2.2. If $f(z) \in \mathcal{A}$ satisfies

$$\left|\arg\{(1-\beta)\frac{f(z)}{z} + \beta f'(z)\}\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\beta\alpha\right),$$

then we have

$$\left|\arg\ \frac{f(z)}{z}\right| < \frac{\pi}{2}\alpha\,,$$

where $\alpha > 0,\ 0 < \beta \leq 1$ and $z \in U$.

Theorem 2.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\arg f'(z)\left\{\frac{f(z)}{z}\right\}^{\mu-1}\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\frac{\alpha}{\mu}\right),$$

then we have

$$\left|\arg\ \left\{\frac{f(z)}{z}\right\}^{\mu}\right| < \frac{\pi}{2}\alpha,$$

where $\alpha > 0,\ \mu > 0$ and $z \in U$.

Proof. Let $p(z) = \left\{\frac{f(z)}{z}\right\}^{\mu}$, $\mu > 0$, then we have

$$p(z) + \frac{1}{\mu}zp'(z) = f'(z)\left\{\frac{f(z)}{z}\right\}^{\mu-1}$$

and the statements of the theorem directly follow from Theorem 2.1. □

Theorem 2.3. *Let $\mu > 0\,, c + \mu > 0$ and $\alpha > 0$. If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\arg f'(z)\left\{\frac{f(z)}{z}\right\}^{\mu-1}\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\frac{\alpha}{\mu+c}\right), \quad (z \in U)$$

then $F(z) = [I_{\mu,c}(f)](z)$ defined by

$$I_{\mu,c}f(z) = \left[\frac{\mu+c}{z^c}\int_0^z f^{\mu}(t)t^{c-1}dt\right]^{\frac{1}{\mu}}, \qquad ([I_{\mu,c}(f)](z)/z \neq 0 \textit{ in } U)$$

satisfies

$$\left|\arg F'(z)\left\{\frac{F(z)}{z}\right\}^{\mu-1}\right| < \frac{\pi}{2}\alpha.$$

Proof. Consider the function p defined by

$$p(z) = F'(z)\left\{\frac{F(z)}{z}\right\}^{\mu-1} \quad (z \in U).$$

Then we easily see that

$$p(z) + \frac{1}{\mu + c} zp'(z) = f'(z)\left\{\frac{f(z)}{z}\right\}^{\mu-1},$$

and the statements of the theorem directly follow from Theorem 2.1. □

Theorem 2.4. *Let a function $f(z) \in \mathcal{A}$ satisfy the following inequalities*

$$\left|\arg f'(z)\left\{\frac{z}{f(z)}\right\}^{\mu+1}\right| < \frac{\pi}{2}\left(-\alpha + \frac{2}{\pi}\tan^{-1}\frac{\alpha}{\mu}\right), \quad (z \in U) \tag{2.1}$$

for some α $(0 < \alpha \le 1)$, $(0 < \mu < 1)$. *Then*

$$\left|\arg\left\{\frac{f(z)}{z}\right\}^{\mu}\right| < \frac{\pi}{2}\alpha\,.$$

Proof. Let us define the function $p(z)$ by $p(z) = \left(\frac{f(z)}{z}\right)^{\mu}$, $(0 < \mu < 1)$. Then $p(z)$ satisfies

$$f'(z)\left\{\frac{z}{f(z)}\right\}^{\mu+1} = \frac{1}{p(z)}\left(1 + \frac{1}{\mu}\frac{zp'(z)}{p(z)}\right).$$

If there exists a point $z_0 \in U$, such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha,$$

then from Lemma 1.1, we have:
(i) for the case $\arg p(z_0) = \frac{\pi}{2}\alpha$,

$$\begin{aligned}\arg f'(z_0)\left\{\frac{z}{f(z_0)}\right\}^{\mu+1} &= \arg\left\{\frac{1}{p(z_0)}\left(1 + \frac{1}{\mu}\frac{zp'(z_0)}{p(z_0)}\right)\right\} \\ &= -\frac{\pi}{2}\alpha + \arg\left(1 + \frac{i\alpha k}{\mu}\right) \\ &\ge -\frac{\pi}{2}\alpha + \tan^{-1}\frac{\alpha}{\mu}.\end{aligned}$$

This contradicts our condition in the theorem.
(ii) for the case $\arg\, p(z_0) = -\frac{\pi}{2}\alpha$, the application of the same method as in (i) shows that

$$\arg f'(z_0)\left\{\frac{z}{f(z_0)}\right\}^{\mu+1} \le -\left(-\frac{\pi}{2}\alpha + \tan^{-1}\frac{\alpha}{\mu}\right).$$

This also contradicts the assumption of the theorem, hence the theorem is proved. □

Theorem 2.5. *Let $f(z) \in \mathcal{A}$ satisfy the condition (2.1) and let*

$$F(z) = \left[\frac{c-\mu}{z^{c-\mu}}\int_0^z \left\{\frac{t}{f(t)}\right\}^{\mu} dt\right]^{-\frac{1}{\mu}}, \tag{2.2}$$

where $c - \mu > 0$. *Then*

$$\left| \arg F'(z) \left\{ \frac{z}{F(z)} \right\}^{\mu+1} \right| < \frac{\pi}{2}\alpha.$$

Proof. If we put

$$p(z) = F'(z) \left\{ \frac{z}{F(z)} \right\}^{\mu+1},$$

then from (2.2) we have

$$p(z) + \frac{1}{c-\mu} z p'(z) = f'(z) \left\{ \frac{z}{f(z)} \right\}^{\mu+1}.$$

The statements of the theorem then directly follow from Theorem 2.1. □

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