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THE INERTIA OF HERMITIAN TRIDIAGONAL BLOCK MATRICES

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ABSTRACT. Let H be a partitioned tridiagonal Hermitian matrix. We characterized the possible inertias of H by a system of linear inequalities involving the orders of the blocks, the inertia of the diagonal blocks and the ranks the lower and upper subdiagonal blocks. From the main result can be derived some propositions on inertia sets of some symmetric sign pattern matrices.

Key words and phrases: Tridiagonal Block Matrices, Hermitian Matrices, Inertia, Patterned Matrix.

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1. PRELIMINARIES

Define the *inertia* of an $n \times n$ Hermitian matrix H as the triple $\mathrm{In}(H) = (\pi, \nu, \delta)$, where π, ν and $\delta = n - \pi - \nu$ are respectively the number of positive, negative, and zero eigenvalues. When n is given, we can specify $\mathrm{In}(H)$, by giving just π and ν, as $(\pi, \nu, *)$.

In the last decades the characterization of the inertias of Hermitian matrices with prescribed 2×2 and 3×3 block decompositions has been extensively investigated. In the first case, after the papers [18] and [2] in 1981, Cain and Marques de Sá established the following result.

Theorem 1.1 ([3])**.** *Let us consider nonnegative integers n_i, π_i, ν_i such that $\pi_i + \nu_i \leq n_i$, for $i = 1, 2$, and let $0 \leq r \leq R \leq \min\{n_1, n_2\}$. Then the following conditions are equivalent:*

(I) *For $i = 1, 2$, there exist $n_i \times n_i$ Hermitian matrices H_i and an $n_1 \times n_2$ matrix X such that $\mathrm{In}(H_i) = (\pi_i, \nu_i, *)$, $r \leq \mathrm{rank} X \leq R$ and*

$$H = \begin{bmatrix} H_1 & X \\ X^* & H_2 \end{bmatrix} \tag{1.1}$$

has inertia $(\pi, \nu, *)$.

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(II) *Let $k \in \{1,2\}$. Let W_k be any fixed Hermitian matrix of order n_k and inertia $(\pi_k, \nu_k, *)$. (I) holds with $H_k = W_{kk}$.*
(III) *Let W be any fixed $n_1 \times n_2$ matrix with $r \leq \mathrm{rank} W \leq R$. (I) holds with $X = W$.*
(IV) *For $k = 1, 2$, let W_{kk} be any fixed $n_k \times n_k$ Hermitian matrix with inertia $(\pi_k, \nu_k, *)$. (I) holds with $H_1 = W_{11}$ and $H_2 = W_{22}$.*
(V) *The following inequalities hold:*

$$\pi \geq \max\{\pi_1, \pi_2, r - \nu_1, r - \nu_2, \pi_1 + \pi_2 - R\},$$
$$\nu \geq \max\{\nu_1, \nu_2, r - \pi_1, r - \pi_2, \nu_1 + \nu_2 - R\},$$
$$\pi \leq \min\{n_1 + \pi_2, \pi_1 + n_2, \pi_1 + \pi_2 + R\},$$
$$\nu \leq \min\{n_1 + \nu_2, \nu_1 + n_2, \nu_1 + \nu_2 + R\},$$
$$\pi - \nu \leq \pi_1 + \pi_2,$$
$$\nu - \pi \leq \nu_1 + \nu_2,$$
$$\pi + \nu \geq \pi_1 + \nu_1 + \pi_2 + \nu_2 - R,$$
$$\pi + \nu \leq \min\{n_1 + n_2, \pi_1 + \nu_1 + n_2 + R, n_1 + \pi_2 + \nu_2 + R\}.$$

In this important theorem we can see how much influence the pair H_1, H_2 of complementary submatrices and the off-diagonal block X have on the inertia of H. In particular, if $H_1 = H_2 = 0$ in (1.1), then the inertias of H are characterized by the set $\{(k, k, n - 2k) \mid k = \mathrm{rank} X\}$.

Haynsworth, [15], established several links connecting the inertia triple of H with the inertia triples of certain principal submatrices of H. In 1992, Cain and Marques de Sá ([3]) extended the methods given by Haynsworth and Ostrowski in [16], for estimating and computing the inertia of certain skew-triangular block matrices. Later this result was improved in [11], which can have the following block tridiagonal version.

Theorem 1.2. *Let us consider nonnegative integers n_i, π_i, ν_i such that $\pi_i + \nu_i \leq n_i$, for $i = 1, 2, 3$, and let $0 \leq r_{i,i+1} \leq R_{i,i+1} \leq \min\{n_i, n_{i+1}\}$, for $i = 1, 2$. Then the following conditions are equivalent:*

(I) *For $i = 1, 2, 3$, and $j = 1, 2$, there exist $n_i \times n_i$ Hermitian matrices H_i and $n_j \times n_{j+1}$ matrices $X_{j,j+1}$ such that $\mathrm{In}(H_i) = (\pi_i, \nu_i, *)$, $r_{j,j+1} \leq \mathrm{rank} X_{j,j+1} \leq R_{j,j+1}$ and*

$$H = \begin{bmatrix} H_1 & X_{12} & 0 \\ X_{12}^* & H_2 & X_{23} \\ 0 & X_{23}^* & H_3 \end{bmatrix}$$

has inertia $(\pi, \nu, *)$.
(II) *Let $k \in \{1, 2, 3\}$. Let W_{kk} be any fixed $n_k \times n_k$ Hermitian matrix with inertia $(\pi_k, \nu_k, *)$. (I) holds with $H_k = W_{kk}$.*
(III) *Let $k \in \{1, 2\}$. Let $W_{k,k+1}$ be any fixed $n_k \times n_{k+1}$ matrix with $r_{k,k+1} \leq \mathrm{rank} W_{k,k+1} \leq R_{k,k+1}$. (I) holds with $X_{k,k+1} = W_{k,k+1}$.*
(IV) *For $k = 1, 2, 3$ let W_{kk} be any fixed $n_k \times n_k$ Hermitian matrix with inertia $(\pi_k, \nu_k, *)$. (I) holds with $H_1 = W_{11}$, $H_2 = W_{22}$ and $H_3 = W_{33}$.*
(V) *Let $(i, j, k) = (1, 2, 3)$ or $(2, 3, 1)$. Let W_{kk} be any fixed $n_k \times n_k$ Hermitian matrix with inertia $(\pi_k, \nu_k, *)$ and let W_{ij} be any fixed $n_i \times n_j$ matrix with $r_{ij} \leq \mathrm{rank}\, W_{ij} \leq R_{ij}$. (I) holds with $H_k = W_{kk}$ and $X_{ij} = W_{ij}$.*
(VI) *The following inequalities hold:*

$$\begin{aligned} \pi \geq \max\{\, & \pi_2, r_{12} - \nu_2, r_{23} - \nu_2, \\ & \pi_1 + r_{23} - \nu_2 - R_{12}, \pi_1 + r_{23} - \nu_3, \pi_3 + r_{12} - \nu_1, \\ & \pi_3 + r_{12} - \nu_2 - R_{23}, \pi_1 + \pi_2 - R_{12}, \pi_1 + \pi_3, \\ & \pi_2 + \pi_3 - R_{23}, \pi_1 + \pi_2 + \pi_3 - R_{12} - R_{23} \,\} \end{aligned}$$

$$\begin{aligned}
\nu \geq \max\{\, & \nu_2, r_{12}-\pi_2, r_{23}-\pi_2, \\
& \nu_1 + r_{23} - \pi_2 - R_{12}, \nu_1 + r_{23} - \pi_3, \\
& \nu_3 + r_{12} - \pi_1, \nu_3 + r_{12} - \pi_2 - R_{23}, \\
& \nu_1 + \nu_2 - R_{12}, \nu_1 + \nu_3, \nu_2 + \nu_3 - R_{23}, \\
& \nu_1 + \nu_2 + \nu_3 - R_{12} - R_{23} \,\} \ ,
\end{aligned}$$

$$\begin{aligned}
\pi \leq \min\{\, & n_1 + \pi_2 + n_3, \pi_1 + \pi_2 + \pi_3 + R_{12} + R_{23}, \\
& \pi_1 + \pi_2 + n_3 + R_{12}, \pi_1 + n_2 + \pi_3, n_1 + \pi_2 + \pi_3 + R_{23} \,\} \ , \\
\nu \leq \min\{\, & n_1 + \nu_2 + n_3, \nu_1 + \nu_2 + \nu_3 + R_{12} + R_{23}, \\
& \nu_1 + \nu_2 + n_3 + R_{12}, \nu_1 + n_2 + \nu_3, n_1 + \nu_2 + \nu_3 + R_{23} \,\} \ ,
\end{aligned}$$

$$\begin{aligned}
\pi + \nu \geq \max\{\, & \pi_1 + \nu_1 + \pi_2 + \nu_2 - R_{12}, \pi_2 + \nu_2 + \pi_3 + \nu_3 - R_{23}, \\
& \pi_1 + \nu_1 + \pi_2 + \nu_2 + \pi_3 + \nu_3 - R_{12} - R_{23}, \\
& \pi_1 + \nu_1 + 2r_{23} - \pi_2 - \nu_2 - R_{12}, \\
& \pi_3 + \nu_3 + 2r_{12} - \pi_2 - \nu_2 - R_{23} \,\} \ , \\
\pi + \nu \leq \min\{\, & n_1 + n_2 + n_3, \pi_1 + \nu_1 + n_2 + n_3 + R_{12}, \\
& n_1 + \pi_2 + \nu_2 + n_3 + R_{12} + R_{23}, n_1 + n_2 + \pi_3 + \nu_3 + R_{23}, \\
& \pi_1 + \nu_1 + \pi_2 + \nu_2 + n_3 + 2R_{12} + R_{23}, \\
& \pi_1 + \nu_1 + n_2 + \pi_3 + \nu_3 + R_{12} + R_{23}, \\
& n_1 + \pi_2 + \nu_2 + \pi_3 + \nu_3 + R_{12} + 2R_{23} \,\}
\end{aligned}$$

$$\begin{aligned}
\pi - \nu \leq \min\{\, & \pi_1 + \pi_2 + \pi_3, \\
& \pi_1 + \pi_2 + \pi_3 - \nu_1 + R_{12}, \pi_1 + \pi_2 + \pi_3 - \nu_3 + R_{23} \,\} \ , \\
\nu - \pi \leq \min\{\, & \nu_1 + \nu_2 + \nu_3, \\
& \nu_1 + \nu_2 + \nu_3 - \pi_1 + R_{12}, \nu_1 + \nu_2 + \nu_3 - \pi_3 + R_{23} \,\} \ .
\end{aligned}$$

Recently, Cohen and Dancis [5, 6, 7, 8] studied the classification of the ranks and inertias of Hermitian completion for some partially specified block band Hermitian matrix, also known as a *bordered matrix*, in terms of some linear inequalities involving inertias and ranks of specified submatrices. Several consequences have been also considered.

2. INERTIA OF A HERMITIAN TRIDIAGONAL BLOCK MATRIX

With a routine induction argument, based on the partitions developed in the proofs of the Theorem 2.1 of [4] or Theorem 3.1 of [11], after an analogous elimination process of redundant inequalities is possible to generalize the Theorem 1.2 to any tridiagonal block decomposition. Clearly Theorem 1.2 gives $n = 3$. (The case $n = 2$ is given by the Theorem 1.1.)

Let us consider the set $\pi_* = \{\pi_i,\, r_{i,i+1} - \nu_i,\, r_{i-1,i} - \nu_i \mid i = 1, \ldots, p\}$ and, by $\pi\nu-$duality, $\nu_* = \{\nu_i,\, r_{i,i+1} - \pi_i,\, r_{i-1,i} - \pi_i \mid i = 1, \ldots, p\}$. Denote by I^C the complementary of I and by I_{nc} (or J_{nc}) a subset of $\{1, \ldots, p\}$ of non-consecutive elements.

Theorem 2.1. *Let us assume that*

$$n_i \geq 0\ ,\ \pi_i \geq 0\ ,\ \nu_i \geq 0\ ,\ \pi_i + \nu_i \leq n_i\ , \quad \textit{for}\ \ i = 1, \ldots, p\ ,$$

and

$$0 \leq r_{i,i+1} \leq R_{i,i+1} \leq \min\{n_i, n_{i+1}\}\,, \quad \textit{for}\ \ i = 1, \ldots, p-1\ .$$

Then the following conditions are equivalent:

(I) *For* $i \in \{1, \ldots, p\}$, *and* $j \in \{1, \ldots, p-1\}$, *there exist* $n_i \times n_i$ *Hermitian matrices* H_i *and* $n_j \times n_{j+1}$ *matrices* $X_{j,j+1}$ *such that* $\operatorname{In}(H_i) = (\pi_i, \nu_i, *)$, $r_{j,j+1} \leq \operatorname{rank} X_{j,j+1} \leq R_{j,j+1}$ *and*

$$T_p = \begin{bmatrix} H_1 & X_{12} & & & \\ X_{12}^* & H_2 & X_{23} & & \\ & X_{23}^* & \ddots & \ddots & \\ & & \ddots & \ddots & X_{p-1,p} \\ & & & X_{p-1,p}^* & H_p \end{bmatrix} . \tag{2.1}$$

has inertia $(\pi, \nu, *)$.

(II) *Let* I *be any subset of* $\{1, \ldots, p\}$ *and* J *be any subset of non-consecutive elements of* $\{1, \ldots, p-1\}$, *such that* $j, j+1 \notin I$, *for any* $j \in J$. *Let* W_{kk} *be any fixed* $n_k \times n_k$ *Hermitian matrix with inertia* $(\pi_k, \nu_k, *)$, *for* $k \in I$, *and let* $W_{j,j+1}$ *be any fixed* $n_j \times n_{j+1}$ *matrix with* $r_{j,j+1} \leq \operatorname{rank} W_{j,j+1} \leq R_{j,j+1}$, *for* $j \in J$. *(I) holds with* $H_k = W_{kk}$ *and* $X_{j,j+1} = W_{j,j+1}$.

(III) *The following inequalities hold:*

$$\pi \geq \max \left\{ \sum_I \pi_* - \sum_{I \times I} R_{ij} \mid I \subset \{1, \ldots, p\} \right\} , \tag{2.2}$$

$$\nu \geq \max \left\{ \sum_I \nu_* - \sum_{I \times I} R_{ij} \mid I \subset \{1, \ldots, p\} \right\} , \tag{2.3}$$

$$\pi \leq \min \left\{ \sum_{I_{nc}} n_i + \sum_{I_{nc}^C} \pi_i + \sum_{I_{nc}^C \times I_{nc}^C} R_{ij} \mid I_{nc} \subset \{1, \ldots, p\} \right\} , \tag{2.4}$$

$$\nu \leq \min \left\{ \sum_{I_{nc}} n_i + \sum_{I_{nc}^C} \nu_i + \sum_{I_{nc}^C \times I_{nc}^C} R_{ij} \mid I_{nc} \subset \{1, \ldots, p\} \right\} , \tag{2.5}$$

$$\pi + \nu \geq \max \left\{ \sum_{i=1}^{p-1} r_{i,i+1} , \left\{ \sum_I (\pi + \nu)_* - \sum_{I \times I} R_{ij} \mid I \subset \{1, \ldots, p\} \right\} \right\} , \tag{2.6}$$

$$\pi + \nu \leq \min \left\{ \sum_I n_i + \sum_{I^C} (\pi_i + \nu_i + R_{i,i+1} + R_{i-1,i}) \mid I \subset \{1, \ldots, p\} \right\} , \tag{2.7}$$

$$\pi - \nu \leq \min \left\{ \sum_{i=1}^{p} \pi_i + \sum_{I_{nc}^C \times I_{nc}^C} R_{ij} + \sum_{I_{nc}} \nu_i - \sum_{J_{nc}} \nu_i \mid I_{nc} \cap J_{nc} \neq \emptyset \right\} , \tag{2.8}$$

$$\nu - \pi \leq \min \left\{ \sum_{i=1}^{p} \nu_i + \sum_{I_{nc}^C \times I_{nc}^C} R_{ij} + \sum_{I_{nc}} \pi_i - \sum_{J_{nc}} \pi_i \mid I_{nc} \cap J_{nc} \neq \emptyset \right\} . \tag{2.9}$$

In fact, suppose the result is true for T_p defined in (2.1). For T_{p+1} we may set

$$H_{p+1} = \begin{bmatrix} \tilde{H}_{p+1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{where} \quad \tilde{H}_{p+1} = \begin{bmatrix} I_{\pi_{p+1}} & 0 \\ 0 & I_{\nu_{p+1}} \end{bmatrix} .$$

This allows us to partition T_{p+1} as

$$T_{p+1} = \left[\begin{array}{ccccc|cc} H_1 & X_{12} & & & & & \\ X_{12}^* & H_2 & X_{23} & & & & \\ & X_{23}^* & \ddots & \ddots & & & \\ & & \ddots & \ddots & X_{p-1,p} & & \\ & & & X_{p-1,p}^* & H_p & Y & Z \\ \hline & & & & Y^* & \tilde{H}_{p+1} & 0 \\ & & & & Z^* & 0 & 0 \end{array}\right],$$

where $X_{p,p+1} = \left[\begin{array}{cc} Y & Z \end{array}\right]$. Consider now the nonsingular matrices U and V such that

$$UZV = \left[\begin{array}{cc} 0 & I_s \\ 0 & 0 \end{array}\right].$$

Then T_{p+1} is conjunctive to

$$\left[\begin{array}{cccccc|ccc} H_1 & X_{12} & & & & & & & \\ X_{12}^* & H_2 & X_{23} & & & & & & \\ & X_{23}^* & \ddots & \ddots & & & & & \\ & & \ddots & \ddots & 0 & \tilde{X}_{p-1,p} & & & \\ & & & 0 & 0 & 0 & 0 & 0 & I_s \\ & & & \tilde{X}_{p-1,p}^* & 0 & \tilde{H}_p & \tilde{X}_{p,p+1} & 0 & 0 \\ \hline & & & & 0 & \tilde{X}_{p,p+1}^* & \tilde{H}_{p+1} & & 0 \\ & & & & 0 & 0 & 0 & & 0 \\ & & & & I_s & 0 & & & \end{array}\right],$$

and, therefore, is conjunctive to the direct sum

$$\bar{T}_p \oplus \tilde{H}_{p+1} \oplus \left[\begin{array}{cc} 0 & I_s \\ I_s & 0 \end{array}\right],$$

where

$$\bar{T}_p = \left[\begin{array}{ccccc} H_1 & X_{12} & & & \\ X_{12}^* & H_2 & X_{23} & & \\ & X_{23}^* & \ddots & \ddots & \\ & & \ddots & \ddots & \tilde{X}_{p-1,p} \\ & & & \tilde{X}_{p-1,p}^* & \tilde{H}_p - \tilde{X}_{p,p+1}\tilde{H}_{p+1}^{-1}\tilde{X}_{p,p+1}^* \end{array}\right].$$

We only have to apply now the induction hypotheses to $\bar{T}_p$, taking in account the variation of the rank $\tilde{X}_{p-1,p}$ which is estimated in the Claim of [3]. The set of inertias of $\tilde{H}_p - \tilde{X}_{p-1,p}\tilde{H}_{p+1}^{-1}\tilde{X}_{p-1,p}^*$ is characterized by the Corollary 2.2 of [11].

Remark 2.2. We point out that in the first two inequalities of the Theorem 2.1, the indices of r_{ij}'s in the summation are always disjoint. By convention, $R_{p,p+1}$, $R_{0,1}$, $r_{p,p+1} - \nu_p$, $r_{0,1} - \nu_1$, $r_{p,p+1} - \pi_p$ and $r_{0,1} - \pi_1$ are zero. Also, the product $I \times I$ is defined as the set $\{(i,j) \mid i < j \in I\}$. Notice that some of the inequalities will be redundant. For example, in the case $p = 2$ or 3 the first summation in (2.6) is redundant. Also, we may take J_{nc} in (2.8) and (2.9) as a maximal set of non-consecutive elements in $\{1, \ldots, p\}$.

If we make all the main diagonal blocks equal to zero in the last theorem, then we have the following proposition:

Corollary 2.3. *Let us assume that* $n_i \geq 0$*, for* $i = 1, \ldots, p$ *and*

$$0 \leq r_{i,i+1} \leq R_{i,i+1} \leq \min\left\{n_i, n_{i+1}\right\}, \quad i = 1, \ldots, p-1\ .$$

Then the following conditions are equivalent:

(I) *For* $j = 1, \ldots, p-1$*, there exist* $n_j \times n_{j+1}$ *matrices* $X_{j,j+1}$ *such that* $r_{j,j+1} \leq \operatorname{rank} X_{j,j+1} \leq R_{j,j+1}$ *and*

$$T = \begin{bmatrix} 0 & X_{12} & & & \\ X_{12}^* & 0 & X_{23} & & \\ & X_{23}^* & \ddots & \ddots & \\ & & \ddots & \ddots & X_{p-1,p} \\ & & & X_{p-1,p}^* & 0 \end{bmatrix}.$$

has inertia $(\pi, \nu, *)$.

(II) *Let* J *be any subset of non-consecutive elements of* $\{1, \ldots, p-1\}$*. Let* $W_{j,j+1}$ *be any fixed* $n_j \times n_{j+1}$ *matrix with* $r_{j,j+1} \leq \operatorname{rank} W_{j,j+1} \leq R_{j,j+1}$*, for* $j \in J$*. (I) holds with* $X_{j,j+1} = W_{j,j+1}$*.*

(III) *The following inequalities hold:*

$$\pi = \nu \geq \max\left\{\sum_{i \in I_{nc}} r_{i,i+1} \mid I_{nc} \subset \{1, \ldots, p-1\}\right\}$$

and

$$\pi = \nu \leq \min\left\{\sum_{i \in I_{nc}} n_i + \sum_{(i,j) \in I_{nc}^C \times I_{nc}^C} R_{ij} \mid I_{nc} \subset \{1, \ldots, p\}\right\}\ .$$

We can find a general characterization of the set of inertias of a Hermitian matrix in [1]. In fact, given an $n_i \times n_i$ Hermitian matrix H_i with inertia $\operatorname{In}(H_i) = (\pi_i, \nu_i, \delta_i)$, for $i = 1, \cdots, m$, Cain characterized in terms of the π_i, ν_i, δ_i the range of $\operatorname{In}(H)$, where H varies over all Hermitian matrices which have a block decomposition $H = (X_{ij})_{i,j=1,\cdots,m}$ in which X_{ij} is $n_i \times n_j$ and $X_{ii} = H_i$.

3. An Application to Symmetric Sign Pattern Matrices

Several authors have been studied properties of matrices based on combinatorial and qualitative information such as the signs of the entries (cf. [9, 10, 13, 14]). A matrix whose entries are from the set $\{+, -, 0\}$ is called a *sign pattern matrix* (or simply, a *pattern*). For each $n \times n$ pattern A, there is a natural class of real matrices whose entries have the signs indicated by A, i.e., the *sign pattern class* of a pattern A is defined by

$$Q(A) = \{B \mid \operatorname{sign} B = A\}\ .$$

We say the pattern A *requires unique inertia* and is *sign nonsingular* if every real matrix in $Q(A)$ has the same inertia and is nonsingular, respectively. We shall be interested on symmetric matrices.

Example 3.1 ([14])**.** Let us consider the pattern

$$A = \left[\begin{array}{cc|cc} + & 0 & + & + \\ 0 & + & + & + \\ \hline + & + & - & 0 \\ + & + & 0 & - \end{array}\right].$$

Since the inertia of the diagonal blocks are always $(2,0,0)$ and $(0,2,0)$, respectively, and the rank of the off-diagonal block varies between 1 and 2, according to the Theorem 1.2 (also [3, cf. Theorem 2.1]), $\pi = \nu = 2$ and, therefore, A requires a unique inertia and is nonsingular.

As an immediate consequence of the Corollary 2.3, we have the following result:

Proposition 3.1 ([13])**.** *For the $n \times n$ symmetric tridiagonal pattern*

$$A_0 = \begin{bmatrix} 0 & + & & & \\ + & 0 & + & & \\ & + & \ddots & \ddots & \\ & & \ddots & \ddots & + \\ & & & + & 0 \end{bmatrix},$$

(a) *if n is even, then A_0 is sign nonsingular and* $\mathrm{In}(A_0) = \left(\frac{n}{2}, \frac{n}{2}, 0\right)$,
(b) *if n is odd, then A_0 is sign singular and* $\mathrm{In}(A_0) = \left(\frac{n-1}{2}, \frac{n-1}{2}, 1\right)$.

We observe that the result above is still true when the sign of any nonzero entry is "$-$". The same observation can be made for the off-diagonals of the patterns in the propositions below. Notice also that Proposition 3.1 is true if the even diagonal entries are possibly nonzero.

Let $\lfloor x \rfloor$ denotes the greater integer less or equal to the real number x.

Proposition 3.2. *If*

$$A_+ = \begin{bmatrix} + & \pm & & & \\ \pm & + & \pm & & \\ & \pm & \ddots & \ddots & \\ & & \ddots & \ddots & \pm \\ & & & \pm & + \end{bmatrix}$$

is an $n \times n$ symmetric tridiagonal pattern, then $\mathrm{In}(A_+)$ *has the form*

$$(n-k, k, 0),\ 0 \le k \le \left\lfloor \frac{n}{2} \right\rfloor, \quad \text{or} \quad (n-k, k-1, 1),\ 1 \le k \le \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. From the Theorem 2.1, if $\mathrm{In}(A_+) = (\pi, \nu, *)$, then $n - 1 \le \pi + \nu \le n$ and $0 \le \nu \le \left\lfloor \frac{n}{2} \right\rfloor$. ◻

The diagonal entries a_{ii} and a_{jj} are said in *ascending positions* when $i < j$.

We may state now a generalization which includes some results of [13, 14].

Proposition 3.3. *For the symmetric tridiagonal pattern*

$$A_* = \begin{bmatrix} * & \pm & & & \\ \pm & * & \pm & & \\ & \pm & \ddots & \ddots & \\ & & \ddots & \ddots & \pm \\ & & & \pm & * \end{bmatrix},$$

where each diagonal entry is 0, $+$ or $-$,

(a) *if n is even, then A_* is sign nonsingular if and only if neither two $+$ nor two $-$ diagonal entries in A_* are in odd-even ascending positions, respectively. In this case* $\operatorname{In}(A_*) = \left(\frac{n}{2}, \frac{n}{2}, 0\right)$,

(b) *if n is odd, then A_* is sign nonsingular if and only if there is at least one $+$ or one $-$ diagonal entry is in an odd position, but not $+$ and $-$ in odd positions at same time, and neither three $+$ nor three $-$ diagonal entries are in odd-even-odd ascending positions, respectively. In this case* $\operatorname{In}(A_*) = \left(\frac{n+1}{2}, \frac{n-1}{2}, 0\right)$ *or* $\operatorname{In}(A_*) = \left(\frac{n-1}{2}, \frac{n+1}{2}, 0\right)$,

(c) *if n is odd and neither $+$ nor $-$ diagonal entries are in odd positions, then A_* requires the unique inertia* $\left(\frac{n-1}{2}, \frac{n-1}{2}, 1\right)$.

Proof. Remind that if A is in the sign pattern class of A_* and $\operatorname{In}(A) = (\pi, \nu, \delta)$, then $0 \leq \delta \leq 1$. Also, according to (2.2) and (2.3), since $R_{i,i+1} = r_{i,i+1} = 1$, for $i = 1, \ldots, n-1$, the minima values of π and ν are obtained in maximal sets of nonconsecutive elements of I_n.

Suppose that n is even. If there are two $+$ in odd-even ascending positions, then $\nu \geq m$ such that $m < n/2$ and $\nu \geq n/2$, i.e., A_* does not require unique inertia and is not sign nonsingular. Otherwise, without loss of generality, suppose that the first nonzero main diagonal element in an odd $(2i+1)-$position is a $+$ (if the main diagonal is zero, the result follows from Proposition 3.1). Then

$$\pi \geq r_{12} - \nu_1 + \cdots + r_{2i-1,2i} - \nu_{2i-1} + \pi_{2i+1} = i + 1 \ , \tag{3.1}$$

$$\nu \geq r_{12} - \pi_1 + \cdots + r_{2i-1,2i} - \pi_{2i-1} + r_{2i+1,2i+2} - \pi_{2i+2} = i + 1 \ . \tag{3.2}$$

If the element in $(2i+3)-$position is a $+$, $-$ or 0, then we add to the right side of (3.1) $\pi_{2i+3} = 1$, $r_{2i+3,2i+4} - \nu_{2i+4} = 1$ and $r_{2i+3,2i+4} - \nu_{2i+3} = 1$, respectively, and to right side of (3.2) $r_{2i+3,2i+4} - \pi_{2i+4} = 1$, $\nu_{2i+3} = 1$ and $r_{2i+3,2i+4} - \pi_{2i+3} = 1$, respectively. Following this procedure we get $\pi, \nu \geq n/2$, i.e., $\operatorname{In}(A_*) = (\frac{n}{2}, \frac{n}{2}, 0)$.

If n is odd, suppose the first diagonal entry is $+$. Then, by (2.2),

$$\pi \geq \pi_1 + r_{23} - \nu_3 + \cdots + r_{n-1,n} - \nu_n \ ,$$

i.e., $\pi \geq (n+1)/2$. On the other hand, by (2.3), $\nu \geq (n-1)/2$. Therefore $\operatorname{In}(A_*) = (\frac{n+1}{2}, \frac{n-1}{2}, 0)$.

Suppose now n is odd and neither $+$ nor $-$ diagonal entries are in odd positions. From the Theorem 2.1, making $I = \{1, 3, 5, \ldots, n-2\}$ in (2.4) and in (2.5) we get $\pi, \nu \leq \frac{n-1}{2}$, and $I_{nc} = \{2, 4, \ldots, n-1\}$ in (2.2) and in (2.3) we get $\pi, \nu \geq \frac{n-1}{2}$. Then A_* requires the unique inertia $(\frac{n-1}{2}, \frac{n-1}{2}, 1)$. □

REFERENCES

[1] B.E. CAIN, The inertia of a Hermitian matrix having prescribed diagonal blocks, *Linear Algebra Appl.*, **37** (1981), 173–180.

[2] B.E. CAIN AND E. MARQUES DE SÁ, The inertia of a Hermitian matrix having prescribed complementary principal submatrices, *Linear Algebra Appl.*, **37** (1981), 161–171.

[3] B.E. CAIN AND E. MARQUES DE SÁ, The inertia of Hermitian matrices with a prescribed 2×2 block decomposition, *Linear and Multilinear Algebra*, **31** (1992), 119–130.

[4] B.E. CAIN AND E. MARQUES DE SÁ, The inertia of certain skew-triangular block matrices, *Linear Algebra Appl.*, **160** (1992), 75–85.

[5] N. COHEN AND J. DANCIS, Inertias of block band matrix completions, *SIAM J. Matrix Anal. Appl.*, **19** (1998), 583–612.

[6] J. DANCIS, The possible inertias for a Hermitian matrix and its principal submatrices, *Linear Algebra Appl.*, **85** (1987), 121–151.

[7] J. DANCIS, Several consequences of an inertia theorem, *Linear Algebra Appl.*, **136** (1990), 43–61.

[8] J. DANCIS, Ranks and inertias of Hermitian block Toeplitz matrices, *Linear Algebra Appl.*, **353** (2002), 21–32.

[9] C. ESCHENBACH AND C. JOHNSON, A Combinatorial Converse to the Perron Frobenius Theorem, *Linear Algebra Appl.*, **136** (1990), 173–180.

[10] C. ESCHENBACH AND C. JOHNSON, Sign Patterns that Require Real, Nonreal or Pure Imaginary Eigenvalues, *Linear and Multilinear Algebra*, **29** (1991), 299–311.

[11] C.M. DA FONSECA, The inertia of certain Hermitian block matrices, *Linear Algebra Appl.*,, **274** (1998), 193–210.

[12] C.M. DA FONSECA, The inertia of Hermitian block matrices with zero main diagonal, *Linear Algebra Appl.*, **311** (2000), 153–160.

[13] F.J. HALL AND Z. LI, Inertia sets of symmetric sign pattern matrices, *Numer. Math. J. Chinese Univ. (English Ser.)*, **10** (2001), 226–240.

[14] F.J. HALL, Z. LI AND D. WANG, Symmetric sign pattern matrices that require unique inertia, *Linear Algebra Appl.*, **338** (2001) 153–169.

[15] E.V. HAYNSWORTH, Determination of the inertia of some partitioned Hermitian matrices, *Linear Algebra Appl.*, **1** (1968), 73–81.

[16] E.V. HAYNSWORTH AND A.M. OSTROWSKI, On the inertia of some classes of partitioned matrices, *Linear Algebra Appl.*, **1** (1968), 299–316.

[17] A.M. OSTROWSKI AND HANS SCHNEIDER, Some theorems on the inertia of general matrices, *J. Math. Anal. Appl.*, **4** (1962), 72–84.

[18] E. MARQUES DE SÁ, On the inertia of sums of Hermitian matrices, *Linear Algebra Appl.*, **37** (1981), 143–159.