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# THREE MAPPINGS RELATED TO CHEBYSHEV-TYPE INEQUALITIES

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ABSTRACT. In this paper, by the Chebyshev-type inequalities we define three mappings, investigate their main properties, give some refinements for Chebyshev-type inequalities, obtain some applications.

*Key words and phrases:* Chebyshev-type inequality, Monotonicity, Refinement.

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## 1. INTRODUCTION

Let $n(\geq 2)$ be a given positive integer, $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be known as sequences of real numbers. Also, let $p_i > 0$ and $q_i > 0$ ($i = 1, 2, \ldots, n$), $P_j = p_1 + p_2 + \cdots + p_j$ and $Q_j = q_1 + q_2 + \cdots + q_j$ ($j = 1, 2, \ldots, n$).

If $A$ and $B$ are both increasing or both decreasing, then

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \leq n \sum_{i=1}^{n} a_i b_i. \tag{1.1}$$

If one of the sequences $A$ or $B$ is increasing and the other decreasing, then the inequality (1.1) is reversed.

The inequality (1.1) is called the Chebyshev's inequality, see [1, 2].

For $A$ and $B$ both increasing or both decreasing, Behdzet in [3] extended inequality (1.1) to

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$$\left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} q_i b_i\right) + \left(\sum_{i=1}^{n} q_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right) \le P_n \sum_{i=1}^{n} q_i a_i b_i + Q_n \sum_{i=1}^{n} p_i a_i b_i. \tag{1.2}$$

If one of the sequences $A$ or $B$ is increasing and the other decreasing, then the inequality (1.2) is reversed.

For $p_i = q_i$, $i = 1, 2, \dots, n$, the inequality (1.2) reduces to

$$\left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right) \le P_n \sum_{i=1}^{n} p_i a_i b_i, \tag{1.3}$$

where, $A$ and $B$ are both increasing or both decreasing. If one of the sequences $A$ or $B$ is increasing and the other decreasing, then the inequality (1.3) is reversed.

Let $r, s : [a, b] \to \mathbb{R}$ be integrable functions, either both increasing or both decreasing. Furthermore, let $p, q : [a, b] \to [0, +\infty)$ be the integrable functions. Then

$$\begin{aligned} &\int_a^b p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_a^b q(t)r(t)dt \int_a^b p(t)s(t)dt \\ &\qquad \le \int_a^b p(t)dt \int_a^b q(t)r(t)s(t)dt + \int_a^b q(t)dt \int_a^b p(t)r(t)s(t)dt. \end{aligned} \tag{1.4}$$

If one of the functions $r$ or $s$ is increasing and the other decreasing, then the inequality (1.4) is reversed.

When $p(t) = q(t)$, $t \in [a, b]$, the inequality (1.4) reduces to

$$\int_a^b p(t)r(t)dt \int_a^b p(t)s(t)dt \le \int_a^b p(t)dt \int_a^b p(t)r(t)s(t)dt, \tag{1.5}$$

where $r$ and $s$ are both increasing or both decreasing. If one of the functions $r$ or $s$ is increasing and the other decreasing, then the inequality (1.5) is reversed.

Inequalities (1.4) and (1.5) are the integral forms of inequalities (1.2) and (1.3), respectively (see [1, 2]).

The results from other inequalities connected with (1.1) to (1.5) can be seen in [1], [3] – [8] and [2, pp. 61–65].

We define three mappings $c$, $C$ and $\widetilde{C}$ by $c : \mathbb{N}_+ \times \mathbb{N}_+ \to \mathbb{R}$,

$$\begin{aligned} c(k, n; p_i, q_i) &= P_k \sum_{i=1}^{k} q_i a_i b_i + Q_k \sum_{i=1}^{k} p_i a_i b_i \\ &\quad + \left(\sum_{i=k+1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} q_i b_i\right) + \left(\sum_{i=1}^{k} p_i a_i\right)\left(\sum_{i=k+1}^{n} q_i b_i\right) \\ &\quad + \left(\sum_{i=k+1}^{n} q_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right) + \left(\sum_{i=1}^{k} q_i a_i\right)\left(\sum_{i=k+1}^{n} p_i b_i\right), \end{aligned} \tag{1.6}$$

where $k = 1, 2, \dots, n$, and

$$\sum_{i=n+1}^{n} q_i a_i = \sum_{i=n+1}^{n} p_i b_i = \sum_{i=n+1}^{n} p_i a_i = \sum_{i=n+1}^{n} q_i b_i = 0$$

is assumed.

For $C : [a,b] \to \mathbb{R}$,

$$\begin{aligned} C(x;p,q;r,s) &= \int_a^x p(t)dt \int_a^x q(t)r(t)s(t)dt + \int_a^x q(t)dt \int_a^x p(t)r(t)s(t)dt \\ &\quad + \int_x^b p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_a^x p(t)r(t)dt \int_x^b q(t)s(t)dt \\ &\quad + \int_x^b q(t)r(t)dt \int_a^b p(t)s(t)dt + \int_a^x q(t)r(t)dt \int_x^b p(t)s(t)dt \end{aligned} \tag{1.7}$$

and for $\widetilde{C} : [a,b] \to \mathbb{R}$,

$$\begin{aligned} \widetilde{C}(y;p,q;r,s) &= \int_y^b p(t)dt \int_y^b q(t)r(t)s(t)dt + \int_y^b q(t)dt \int_y^b p(t)r(t)s(t)dt \\ &\quad + \int_a^y p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_y^b p(t)r(t)dt \int_a^y q(t)s(t)dt \\ &\quad + \int_a^y q(t)r(t)dt \int_a^b p(t)s(t)dt + \int_y^b q(t)r(t)dt \int_a^y p(t)s(t)dt. \end{aligned} \tag{1.8}$$

We write

$$\begin{aligned} c_1(k,n;p_i) &= \frac{1}{2}c(k,n;p_i,p_i) \\ &= P_k \sum_{i=1}^{k} p_i a_i b_i + \left(\sum_{i=k+1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right) + \left(\sum_{i=1}^{k} p_i a_i\right)\left(\sum_{i=k+1}^{n} p_i b_i\right), \end{aligned} \tag{1.9}$$

$$\begin{aligned} c_2(k,n) &= c_1(k,n;1) \\ &= k\sum_{i=1}^{k} a_i b_i + \left(\sum_{i=k+1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) + \left(\sum_{i=1}^{k} a_i\right)\left(\sum_{i=k+1}^{n} b_i\right), \end{aligned} \tag{1.10}$$

$$\begin{aligned} C_0(x;p;r,s) &= \frac{1}{2}C(x;p,p;r,s) \\ &= \int_a^x p(t)dt \int_a^x p(t)r(t)s(t)dt + \int_x^b p(t)r(t)dt \int_a^b p(t)s(t)dt \\ &\quad + \int_a^x p(t)r(t)dt \int_x^b p(t)s(t)dt \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} \widetilde{C}_0(y;p;r,s) &= \frac{1}{2}\widetilde{C}(y;p,p;r,s) \\ &= \int_y^b p(t)dt \int_y^b p(t)r(t)s(t)dt + \int_a^y p(t)r(t)dt \int_a^b p(t)s(t)dt \\ &\quad + \int_y^b p(t)r(t)dt \int_a^y p(t)s(t)dt. \end{aligned} \tag{1.12}$$

(1.10), (1.6), (1.9), (1.7) and (1.8), (1.11) and (1.12) are generated by the inequalities (1.1) to (1.5), respectively.

The aim of this paper is to study the monotonicity properties of $c$, $C$ and $\widetilde{C}$, and obtain some refinements of (1.1) to (1.5) using these monotonicity properties. Some applications are given.

## 2. MAIN RESULTS

The monotonicity properties of the mapping $c$, $c_1$ and $c_2$ are embodied in the following theorem.

**Theorem 2.1.** *Let $c$, $c_1$ and $c_2$ be defined as in the first section. If $A$ and $B$ are both increasing or both decreasing, then we have the following refinements of (1.2), (1.3) and (1.1)*

$$\begin{aligned}(2.1)\qquad &\left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} q_i b_i\right) + \left(\sum_{i=1}^{n} q_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right)\\ &= c(1,n;p_i,q_i) \le \cdots \le c(k,n;p_i,q_i) \le c(k+1,n;p_i,q_i) \le \cdots\\ &\le c(n,n;p_i,q_i) = P_n \sum_{i=1}^{n} q_i a_i b_i + Q_n \sum_{i=1}^{n} p_i a_i b_i,\end{aligned}$$

$$\begin{aligned}(2.2)\qquad \left(\sum_{i=1}^{n} p_i a_i\right)\left(\sum_{i=1}^{n} p_i b_i\right) &= c_1(1,n;p_i) \le \cdots \le c_1(k,n;p_i)\\ &\le c_1(k+1,n;p_i) \le \cdots \le c_1(n,n;p_i)\\ &= P_n \sum_{i=1}^{n} p_i a_i b_i\end{aligned}$$

*and*

$$\begin{aligned}(2.3)\qquad \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) &= c_2(1,n) \le \cdots \le c_2(k,n)\\ &\le c_2(k+1,n) \le \cdots \le c_2(n,n)\\ &= n \sum_{i=1}^{n} a_i b_i,\end{aligned}$$

*respectively. If one of the sequences $A$ or $B$ is increasing and the other decreasing, then inequalities in (2.1)–(2.3) are reversed.*

The monotonicity properties of the mappings $C$ and $C_0$ are given in the following theorem.

**Theorem 2.2.** *Let $C$ and $C_0$ be defined as in the first section. If $r$ and $s$ are both increasing or both decreasing, then $C(x;p,q;r,s)$ and $C_0(x;p;r,s)$ are increasing on $[a,b]$ with $x$, and for $x \in [a,b]$ we have the following refinements of (1.4) and (1.5)*

$$\begin{aligned}(2.4)\qquad &\int_a^b p(t)r(t)dt \int_a^b q(t)s(t)dt + \int_a^b q(t)r(t)dt \int_a^b p(t)s(t)dt\\ &= C(a;p,q;r,s) \le C(x;p,q;r,s) \le C(b;p,q;r,s)\\ &= \int_a^b p(t)dt \int_a^b q(t)r(t)s(t)dt + \int_a^b q(t)dt \int_a^b p(t)r(t)s(t)dt\end{aligned}$$

*and*

$$\begin{aligned}(2.5)\qquad \int_a^b p(t)r(t)dt \int_a^b p(t)s(t)dt &= C_0(a;p;r,s)\\ &\le C_0(x;p;r,s) \le C_0(b;p;r,s)\\ &= \int_a^b p(t)dt \int_a^b p(t)r(t)s(t)dt,\end{aligned}$$

*respectively. If one of the functions $r$ or $s$ is increasing and the other decreasing, then $C(x;p,q;r,s)$ and $C_0(x;p;r,s)$ are decreasing on $[a,b]$ with $x$, and inequalities in (2.4) and (2.5) are reversed.*

The monotonicity properties of $\widetilde{C}$ and $\widetilde{C_0}$ are given in the following theorem.

**Theorem 2.3.** *Let $\widetilde{C}$ and $\widetilde{C_0}$ be defined as in the first section. If $r$ and $s$ are both increasing or both decreasing, then $\widetilde{C}(y;p,q;r,s)$ and $\widetilde{C}_0(y;p;r,s)$ are decreasing on $[a,b]$ with $y$, and for $y \in [a,b]$ we have the following refinements of (1.4) and (1.5)*

$$\begin{aligned}\int_a^b p(t)r(t)dt\int_a^b q(t)s(t)dt + \int_a^b q(t)r(t)dt\int_a^b p(t)s(t)dt \\ = \widetilde{C}(b;p,q;r,s) \le \widetilde{C}(y;p,q;r,s) \le \widetilde{C}(a;p,q;r,s) \\ = \int_a^b p(t)dt\int_a^b q(t)r(t)s(t)dt + \int_a^b q(t)dt\int_a^b p(t)r(t)s(t)dt\end{aligned} \tag{2.6}$$

*and*

$$\begin{aligned}\int_a^b p(t)r(t)dt\int_a^b p(t)s(t)dt = \widetilde{C}_0(b;p;r,s) \le \widetilde{C}_0(y;p;r,s) \le \widetilde{C}_0(a;p;r,s) \\ = \int_a^b p(t)dt\int_a^b p(t)r(t)s(t)dt,\end{aligned} \tag{2.7}$$

*respectively. If one of the functions $r$ or $s$ is increasing and the other decreasing, then $\widetilde{C}(y;p,q;r,s)$ and $\widetilde{C_0}(y;p;r,s)$ are increasing on $[a,b]$ with $y$, and the inequalities in (2.6) and (2.7) are reversed.*

## 3. PROOF OF THEOREMS

*Proof of Theorem 2.1.* For $k = 2,3,\dots,n$, we have

$$\begin{aligned}
&c(k,n;p_i,q_i) - c(k-1,n;p_i,q_i) \\
&= (P_{k-1}+p_k)\left(\sum_{i=1}^{k-1} q_ia_ib_i + q_ka_kb_k\right) + (Q_{k-1}+q_k)\left(\sum_{i=1}^{k-1} p_ia_ib_i + p_ka_kb_k\right) \\
&\quad - \left[P_{k-1}\sum_{i=1}^{k-1} q_ia_ib_i + Q_{k-1}\sum_{i=1}^{k-1} p_ia_ib_i\right] + \sum_{i=k+1}^{n} p_ia_i\sum_{i=1}^{n} q_ib_i \\
&\quad + \left(p_ka_k + \sum_{i=1}^{k-1} p_ia_i\right)\sum_{i=k+1}^{n} q_ib_i - \left(p_ka_k + \sum_{i=k+1}^{n} p_ia_i\right)\sum_{i=1}^{n} q_ib_i \\
&\quad - \sum_{i=1}^{k-1} p_ia_i\left(q_kb_k + \sum_{i=k+1}^{n} q_ib_i\right) + \sum_{i=k+1}^{n} q_ia_i\sum_{i=1}^{n} p_ib_i \\
&\quad + \left(q_ka_k + \sum_{i=1}^{k-1} q_ia_i\right)\sum_{i=k+1}^{n} p_ib_i - \left(q_ka_k + \sum_{i=k+1}^{n} q_ia_i\right)\sum_{i=1}^{n} p_ib_i \\
&\quad - \sum_{i=1}^{k-1} q_ia_i\left(p_kb_k + \sum_{i=k+1}^{n} p_ib_i\right) \\
&= \left[p_k\sum_{i=1}^{k-1} q_ia_ib_i + p_ka_kb_k\sum_{i=1}^{k-1} q_i - p_ka_k\sum_{i=1}^{k-1} q_ib_i - p_kb_k\sum_{i=1}^{k-1} q_ia_i\right]
\end{aligned} \tag{3.1}$$

$$+\left[q_k\sum_{i=1}^{k-1}p_ia_ib_i+q_ka_kb_k\sum_{i=1}^{k-1}p_i-q_ka_k\sum_{i=1}^{k-1}p_ib_i-q_kb_k\sum_{i=1}^{k-1}p_ia_i\right]$$
$$=p_k\sum_{i=1}^{k-1}q_i\left(a_k-a_i\right)\left(b_k-b_i\right)+q_k\sum_{i=1}^{k-1}p_i\left(a_k-a_i\right)\left(b_k-b_i\right).$$

If $A$ and $B$ are both increasing or both decreasing, then

$$\Big(a_k-a_i\Big)\Big(b_k-b_i\Big)\geq 0,\quad (i=1,2,\dots,k-1). \tag{3.2}$$

Using (1.6), (3.1) and (3.2), we obtain (2.1).

If one of the sequences $A$ or $B$ is increasing and the other decreasing, then (3.2) is reversed, which implies that the inequalities in (2.1) are reversed.

For $i=1,2,\dots,n$, replacing $q_i$ in (2.1) with $p_i$ and replacing $p_i$ in (2.2) with $1$, we obtain (2.2) and (2.3), respectively. This completes the proof of Theorem 2.1. □

*Proof of Theorem 2.2.* For any $x_1,x_2\in[a,b]$, $x_1<x_2$, we write

$$I_1=\int_{x_1}^{x_2}p(t)dt\int_{x_1}^{x_2}q(t)r(t)s(t)dt+\int_{x_1}^{x_2}q(t)dt\int_{x_1}^{x_2}p(t)r(t)s(t)dt$$
$$-\int_{x_1}^{x_2}p(t)r(t)dt\int_{x_1}^{x_2}q(t)s(t)dt-\int_{x_1}^{x_2}q(t)r(t)dt\int_{x_1}^{x_2}p(t)s(t)dt.$$

For $t\in[a,x_1]$, $u\in[x_1,x_2]$, using the properties of double integrals, we get

$$I_2=\iint_{[a,x_1]\times[x_1,x_2]}p(t)q(u)\Big(r(t)-r(u)\Big)\Big(s(t)-s(u)\Big)dtdu$$
$$=\int_a^{x_1}p(t)dt\int_{x_1}^{x_2}q(t)r(t)s(t)dt+\int_a^{x_1}p(t)r(t)s(t)dt\int_{x_1}^{x_2}q(t)dt$$
$$-\int_a^{x_1}p(t)r(t)dt\int_{x_1}^{x_2}q(t)s(t)dt-\int_a^{x_1}p(t)s(t)dt\int_{x_1}^{x_2}q(t)r(t)dt$$

and

$$I_3=\iint_{[a,x_1]\times[x_1,x_2]}p(u)q(t)\Big(r(t)-r(u)\Big)\Big(s(t)-s(u)\Big)dtdu$$
$$=\int_a^{x_1}q(t)dt\int_{x_1}^{x_2}p(t)r(t)s(t)dt+\int_a^{x_1}q(t)r(t)s(t)dt\int_{x_1}^{x_2}p(t)dt$$
$$-\int_a^{x_1}q(t)r(t)dt\int_{x_1}^{x_2}p(t)s(t)dt-\int_a^{x_1}q(t)s(t)dt\int_{x_1}^{x_2}p(t)r(t)dt.$$

When $x_1=a$, from (1.7), we get

$$C(x_2;p,q;r,s)-C(x_1;p,q;r,s) \tag{3.3}$$
$$=\int_{x_1}^{x_2}p(t)dt\int_{x_1}^{x_2}q(t)r(t)s(t)dt+\int_{x_1}^{x_2}q(t)dt\int_{x_1}^{x_2}p(t)r(t)s(t)dt$$
$$-\int_{x_1}^{x_2}p(t)r(t)dt\int_{x_1}^{x_2}q(t)s(t)dt-\int_{x_1}^{x_2}q(t)r(t)dt\int_{x_1}^{x_2}p(t)s(t)dt$$
$$=I_1.$$

When $x_1 > a$, from (1.7), we have

$$
\begin{aligned}
(3.4)\qquad & C(x_2;p,q;r,s) - C(x_1;p,q;r,s) \\
&= \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) p(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) q(t)r(t)s(t)dt \\
&\quad + \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) q(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) p(t)r(t)s(t)dt \\
&\quad - \int_a^{x_1} p(t)dt \int_a^{x_1} q(t)r(t)s(t)dt - \int_a^{x_1} q(t)dt \int_a^{x_1} p(t)r(t)s(t)dt \\
&\quad + \int_{x_2}^{b} p(t)r(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{b}\right) q(t)s(t)dt \\
&\quad + \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) p(t)r(t)dt \int_{x_2}^{b} q(t)s(t)dt \\
&\quad - \left(\int_{x_1}^{x_2} + \int_{x_2}^{b}\right) p(t)r(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{b}\right) q(t)s(t)dt \\
&\quad - \int_a^{x_1} p(t)r(t)dt \left(\int_{x_1}^{x_2} + \int_{x_2}^{b}\right) q(t)s(t)dt \\
&\quad + \int_{x_2}^{b} q(t)r(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{b}\right) p(t)s(t)dt \\
&\quad + \left(\int_a^{x_1} + \int_{x_1}^{x_2}\right) q(t)r(t)dt \int_{x_2}^{b} p(t)s(t)dt \\
&\quad - \left(\int_{x_1}^{x_2} + \int_{x_2}^{b}\right) q(t)r(t)dt \left(\int_a^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^{b}\right) p(t)s(t)dt \\
&\quad - \int_a^{x_1} q(t)r(t)dt \left(\int_{x_1}^{x_2} + \int_{x_2}^{b}\right) p(t)s(t)dt \\
&= \Bigg[\int_{x_1}^{x_2} p(t)dt \int_{x_1}^{x_2} q(t)r(t)s(t)dt + \int_{x_1}^{x_2} q(t)dt \int_{x_1}^{x_2} p(t)r(t)s(t)dt \\
&\quad - \int_{x_1}^{x_2} p(t)r(t)dt \int_{x_1}^{x_2} q(t)s(t)dt - \int_{x_1}^{x_2} q(t)r(t)dt \int_{x_1}^{x_2} p(t)s(t)dt\Bigg] \\
&\quad + \Bigg[\int_a^{x_1} p(t)dt \int_{x_1}^{x_2} q(t)r(t)s(t)dt + \int_a^{x_1} p(t)r(t)s(t)dt \int_{x_1}^{x_2} q(t)dt \\
&\quad - \int_a^{x_1} p(t)r(t)dt \int_{x_1}^{x_2} q(t)s(t)dt - \int_a^{x_1} p(t)s(t)dt \int_{x_1}^{x_2} q(t)r(t)dt\Bigg] \\
&\quad + \Bigg[\int_a^{x_1} q(t)dt \int_{x_1}^{x_2} p(t)r(t)s(t)dt + \int_a^{x_1} q(t)r(t)s(t)dt \int_{x_1}^{x_2} p(t)dt \\
&\quad - \int_a^{x_1} q(t)r(t)dt \int_{x_1}^{x_2} p(t)s(t)dt - \int_a^{x_1} q(t)s(t)dt \int_{x_1}^{x_2} p(t)r(t)dt\Bigg] \\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

(1) If $r$ and $s$ are both increasing or both decreasing, then we have

$$\text{(3.5)}\qquad \Big(r(t) - r(u)\Big)\Big(s(t) - s(u)\Big) \geq 0,$$

i.e., $I_2 \geq 0$ and $I_3 \geq 0$. By the inequality (1.4), $I_1 \geq 0$ holds . Using (3.3) and (3.4), we obtain that $C(x;p,q;r,s)$ is increasing on $[a,b]$ with $x$. Further, from (1.11), we get that $C_0(x;p;r,s)$ is increasing on $[a,b]$ with $x$.

From (1.7) and (1.11), using the increasing properties of $C(x;p,q;r,s)$ and $C_0(x;p;r,s)$, we obtain (2.4) and (2.5), respectively.

(2) If one of the functions $r$ or $s$ is increasing and the other decreasing, then the inequality in (3.5) is reversed, which implies that $I_2 \leq 0$ and $I_3 \leq 0$. By the reverse of (1.4), $I_1 \leq 0$ holds. From (3.3) and (3.4), (1.11), we obtain that $C(x;p,q;r,s)$, $C_0(x;p;r,s)$ are decreasing on $[a,b]$ with $x$, respectively.

From (1.7) and (1.11), using the decreasing properties of $C(x;p,q;r,s)$ and $C_0(x;p;r,s)$, we obtain the reverse of (2.4) and (2.5), respectively.

This completes the proof of Theorem 2.2. □

*Proof of Theorem 2.3.* Using the same arguments as those in the proof of Theorem 2.2, we can prove Theorem 2.3. □

## 4. APPLICATIONS

Let $\mathbf{I}$ be a real interval and $u, v, w : \mathbf{I} \to [0, +\infty)$. For any $\alpha, \beta \in \mathbb{R}$ and any $x_i \in \mathbf{I}$ $(i = 1, 2, \ldots, n, n \geq 2)$ satisfying $x_1 \leq x_2 \leq \cdots \leq x_n$, we define

$$\begin{aligned} K(k,n) &= \sum_{i=1}^{k} v(x_i)w^{\beta}(x_i) \sum_{i=1}^{k} u(x_i)w^{-\beta}(x_i) \\ &\quad + \sum_{i=1}^{k} v(x_i)w^{-\alpha}(x_i) \sum_{i=1}^{k} u(x_i)w^{\alpha}(x_i) + \sum_{i=k+1}^{n} v(x_i)w^{\alpha}(x_i) \sum_{i=1}^{n} u(x_i)w^{-\alpha}(x_i) \\ &\quad + \sum_{i=1}^{k} v(x_i)w^{\alpha}(x_i) \sum_{i=k+1}^{n} u(x_i)w^{-\alpha}(x_i) + \sum_{i=k+1}^{n} v(x_i)w^{-\beta}(x_i) \sum_{i=1}^{n} u(x_i)w^{\beta}(x_i) \\ &\quad + \sum_{i=1}^{k} v(x_i)w^{-\beta}(x_i) \sum_{i=k+1}^{n} u(x_i)w^{\beta}(x_i), \end{aligned}$$

and

$$\begin{aligned} L(k,n) &= \sum_{i=1}^{k} v(x_i)w^{\beta}(x_i) \sum_{i=1}^{k} u(x_i)w^{\alpha}(x_i) \\ &\quad + \sum_{i=k+1}^{n} v(x_i)w^{\alpha}(x_i) \sum_{i=1}^{n} u(x_i)w^{\beta}(x_i) \\ &\quad + \sum_{i=1}^{k} v(x_i)w^{\alpha}(x_i) \sum_{i=k+1}^{n} u(x_i)w^{\beta}(x_i), \end{aligned}$$

where, $k = 1, 2, \ldots, n$,

$$\sum_{i=n+1}^{n} u(x_i)w^{\beta}(x_i) = \sum_{i=n+1}^{n} v(x_i)w^{\alpha}(x_i) = 0,$$

$$\sum_{i=n+1}^{n} u(x_i)w^{-\alpha}(x_i) = \sum_{i=n+1}^{n} v(x_i)w^{-\beta}(x_i) = 0.$$

**Proposition 4.1.** *Let $w$ and $u/v$ be both increasing or both decreasing. If $\alpha > \beta$, then we have*

$$\begin{aligned}(4.1)\qquad &\sum_{i=1}^{n} v(x_i)w^{\alpha}(x_i)\sum_{i=1}^{n} u(x_i)w^{-\alpha}(x_i)+\sum_{i=1}^{n} v(x_i)w^{-\beta}(x_i)\sum_{i=1}^{n} u(x_i)w^{\beta}(x_i)\\ &= K(1,n)\le\cdots\le K(k,n)\le K(k+1,n)\le\cdots\le K(n,n)\\ &=\sum_{i=1}^{n} v(x_i)w^{\beta}(x_i)\sum_{i=1}^{n} u(x_i)w^{-\beta}(x_i)+\sum_{i=1}^{n} v(x_i)w^{-\alpha}(x_i)\sum_{i=1}^{n} u(x_i)w^{\alpha}(x_i)\end{aligned}$$

*and*

$$\begin{aligned}(4.2)\qquad &\sum_{i=1}^{n} v(x_i)w^{\alpha}(x_i)\sum_{i=1}^{n} u(x_i)w^{\beta}(x_i)\\ &= L(1,n)\le\cdots\le L(k,n)\le L(k+1,n)\le\cdots\le L(n,n)\\ &=\sum_{i=1}^{n} v(x_i)w^{\beta}(x_i)\sum_{i=1}^{n} u(x_i)w^{\alpha}(x_i).\end{aligned}$$

*If $\alpha < \beta$, then the inequalities in (4.1) and (4.2) are reversed.*

*Proof.* Replacing $p_i$, $q_i$, $a_i$ and $b_i$ in (2.1) (or the reverse of (2.1)) with $v(x_i)w^{\beta}(x_i)$, $v(x_i)w^{-\alpha}(x_i)$, $w^{\alpha-\beta}(x_i)$ and $u(x_i)/v(x_i)$, respectively, we obtain (4.1) (or the reverse of (4.1)). Replacing $p_i$, $a_i$ and $b_i$ in (2.2) (or the reverse of (2.2)) with $v(x_i)w^{\beta}(x_i)$, $w^{\alpha-\beta}(x_i)$ and $u(x_i)/v(x_i)$, respectively, we obtain (4.2) (or the reverse of (4.2)).

This completes the proof of Proposition 4.1. □

**Remark 1.** (4.1) and (4.2) are generated by Proposition 1 in [4].

Let $f : [a,b] \to \mathbb{R}$ be a continuous convex function with $f'_{+}(a)$ $(= f'_{-}(a)$ is assumed) and $f'_{-}(b)$, $\{f(x)|x\in[a,b]\} = [d,e]$. Also, let $h : [d,e] \to (0,+\infty)$ be an integrable function, and $g : [d,e] \to \mathbb{R}$ be a strict monotonic function. We define

$$(4.3)\qquad E(g;f,h) = g^{-1}\left[\left(\int_a^b h(f(t))f'_{-}(t)dt\right)^{-1}\int_a^b h(f(t))g(f(t))f'_{-}(t)dt\right],$$

$$(4.4)\qquad M(g;f,h) = g^{-1}\left[\left(\int_a^b h(f(t))dt\right)^{-1}\int_a^b h(f(t))g(f(t))dt\right],$$

$$(4.5)\qquad R(x;g;f,h) = g^{-1}\left[\left(\int_a^b h(f(t))dt\int_a^b h(f(t))f'_{-}(t)dt\right)^{-1} C_0\Big(x;h(f);g(f),f'_{-}\Big)\right]$$

and

$$(4.6)\qquad \widetilde{R}(y;g;f,h) = g^{-1}\left[\left(\int_a^b h(f(t))dt\int_a^b h(f(t))f'_{-}(t)dt\right)^{-1} \widetilde{C}_0\Big(y;h(f);g(f),f'_{-}\Big)\right].$$

**Proposition 4.2.** *If $f$ is monotone, Then we have*

(1) $R(x;g;f,h)$ *is increasing on* $[a,b]$ *with* $x$. *For* $x\in[a,b]$ *we have*

$$(4.7)\qquad M(g;f,h) = R(a;g;f,h) \le R(x;g;f,h) \le R(b;g;f,h) = E(g;f,h).$$

(2) $\tilde{R}(y;g;f,h)$ *is decreasing on* $[a,b]$ *with* $y$. *For* $y\in[a,b]$ *we have*

$$(4.8)\qquad M(g;f,h) = \tilde{R}(b;g;f,h) \le \tilde{R}(y;g;f,h) \le \tilde{R}(a;g;f,h) = E(g;f,h).$$

*Proof.* From the convexity of $f$, we get that $f'_-(t)$ is increasing on $[a,b]$ and the integrals in $E(g;f,h)$, $R(x;g;f,h)$ and $\widetilde{R}(y;g;f,h)$ are valid (see [5]). From $h(x)>0$, $x\in[d,e]$, we have

$$\int_a^b h(f(t))dt > 0. \tag{4.9}$$

From the convexity of $f$, when $f(a)<f(b)$ or $f(a)>f(b)$, Wang in [5] proved that

$$\int_a^b h(f(t))f'_-(t)dt > 0 \tag{4.10}$$

or

$$\int_a^b h(f(t))f'_-(t)dt < 0. \tag{4.11}$$

(1) Let us first assume that $g$ is a strictly increasing function.
*Case 1.* From the increasing properties of $f$, we have $f(a)<f(b)$. Further, (4.10) holds. To prove that $R(x;g;f,h)$ is increasing, from (4.5), (4.9) and (4.10), we only need to prove that

$$C_0\Big(x;h(f);g(f),f'_-\Big) = \left(\int_a^b h(f(t))dt\int_a^b h(f(t))f'_-(t)dt\right) g\Big(R(x;g;f,h)\Big) \tag{4.12}$$

is increasing on $[a,b]$ with $x$.
Indeed, since $f$ is increasing on $[a,b]$, we have that $g(f(t))$ is increasing on $[a,b]$. By Theorem 2.2, $C_0(x;h(f);g(f),f'_-)$ is monotonically increasing with $x\in[a,b]$.
For $x\in[a,b]$, from (4.3), (4.4), (4.5), (4.9) and (4.10), then (4.7) is equivalent to

$$\begin{aligned} &\int_a^b h(f(t))g(f(t))dt\int_a^b h(f(t))f'_-(t)dt \\ &= C_0(a;h(f);g(f),f'_-) \le C_0(x;h(f);g(f),f'_-) \le C_0(b;h(f);g(f),f'_-) \\ &= \int_a^b h(f(t))dt\int_a^b h(f(t))g(f(t))f'_-(t)dt. \end{aligned} \tag{4.13}$$

Replacing $p(t)$, $r(t)$ and $s(t)$ in (2.5) with $h(f(t))$, $g(f(t))$ and $f'_-(t)$, respectively, we obtain (4.13).
*Case 2.* If $f$ is decreasing on $[a,b]$, then we have $f(a)>f(b)$, i.e. (4.11) holds. To prove that $R(x;g;f,h)$ is increasing, from (4.5), (4.9) and (4.11), we only need to prove that $C_0(x;h(f);g(f),f'_-)$ (see (4.12)) is decreasing on $[a,b]$ with $x$.
Indeed, since $f$ is decreasing on $[a,b]$, then $g(f(t))$ is decreasing on $[a,b]$. By Theorem 2.2, $C_0(x;h(f);g(f),f'_-)$ is decreasing with $x\in[a,b]$.
For $x\in[a,b]$, from (4.3), (4.4), (4.5), (4.9) and (4.11), then (4.7) is equivalent to the reverse of (4.13). Replacing $p(t)$, $r(t)$ and $s(t)$ in the reverse of (2.5) with $h(f(t))$, $g(f(t))$ and $f'_-(t)$, respectively, we obtain the reverse of (4.13).
The second case: $g$ is a strictly decreasing function. Using the same arguments for $g$ as a strictly increasing function, we can also prove (1).

(2) Using the same arguments as those for (1), with (2.6) and (2.7), we can prove that $\tilde{R}(x;g;f,h)$ is decreasing on $[a,b]$ with $x$, and (4.8) holds.

This completes the proof of Proposition 4.2. □

**Remark 2.** (4.7)–(4.8) can be generated by $(*)$ in [6] or Proposition 8.1 in [5].

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