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EXPLICIT ESTIMATES ON INTEGRAL INEQUALITIES WITH TIME SCALE

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ABSTRACT. The main objective of this paper is to obtain explicit estimates on some integral inequalities on time scale. The obtained inequalities can be used as tools in the study of certain classes of dynamic equations on time scale.

Key words and phrases: Explicit estimates, Time scale, Gronwall inequality, Bihari's Inequality.

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1. INTRODUCTION

In 1988 Stefan Hilger [4] first introduced in the literature calculus on time scales, which unifies continuous and discrete analysis. Motivated by the above paper [4], many authors have extended some fundamental inequalities used in analysis on time scales, see [1] – [3], [5], [9], [10]. In [3], [4], [9], [10] the authors have extended some fundamental integral inequalities used in the theory of differential and integral equations on time scales. The main purpose of this paper is to obtain time scale versions of some more fundamental integral inequalities used in the theory of differential and integral equations. The obtained inequalities can be used as tools in the study of certain properties of dynamic equations on time scales. Some applications are also given to illustrate the usefulness of some of our results.

2. PRELIMINARIES

Let $\mathbb{T}$ be a time scale and σ and ρ be two jump operators as $\sigma, \rho : \mathbb{T} \to \mathbb{R}$ satisfying

$$\sigma(t) = \inf\{s \in \mathbb{T} | s > t\} \qquad \text{and} \qquad \rho(t) = \sup\{s \in \mathbb{T} | s < t\}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right dense point and if the left sided limit exists at every left dense point. The set of all rd-continuous functions

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is denoted by $C_{rd}[\mathbb{T},\mathbb{R}]$. Let

$$\mathbb{T}^k := \begin{cases} \mathbb{T}-m & \text{if } \mathbb{T} \text{ has left scattered point in M} \\ \mathbb{T} & \text{otherwise} \end{cases}$$

Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$ then we define $f^{\Delta}(t)$ as: for $\epsilon>0$ there exists a neighbourhood $\mathbb{N}$ of t with

$$\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t)-s)\right| \le \epsilon\,|\sigma(t)-s|$$

for all $s\in\mathbb{N}$ and f is called delta-differentiable on $\mathbb{T}$. A function $F:\mathbb{T}\to\mathbb{R}$ is called an antiderivative of $f:\mathbb{T}\to\mathbb{R}$ provided $F^{\Delta}=f(t)$ holds for all $t\in\mathbb{T}^k$. In this case we define the integral of f by

$$\int_s^t f(\tau)\,\Delta\tau = F(t)-F(s) \quad \text{where } s,t\in\mathbb{T}.$$

We need the following two lemmas proved in [3].

Lemma 2.1. *Let $u,g\in C_{rd}(\mathbb{T},\mathbb{R})$ and $f\in\mathbb{R}^+$. If*

$$u^{\Delta}(t) \le f(t)\,u(t)+g(t) \tag{2.1}$$

for all $t\in\mathbb{T}^k$, then

$$u(t) \le u(a)\,e_f(t,a) + \int_a^t e_f(t,\sigma(s))\,g(t)\,\Delta s, \tag{2.2}$$

for all $t\in\mathbb{T}^k$, where $e_f(t,a)$ is a solution of the initial value problem (IVP)

$$u^{\Delta}(t) = f(t)\,u(t), \qquad u(a)=1 \tag{2.3}$$

Lemma 2.2. *Let $u,f,g,p\in C_{rd}(\mathbb{T},\mathbb{R})$ and assume $g,p\ge 0$ and f is nondecreasing on $\mathbb{T}$*

$$u(t) \le f(t) + p(t)\int_a^t g(\tau)u(\tau)\,\Delta\tau, \tag{2.4}$$

for all $t\in\mathbb{T}^k$ then

$$u(t) \le f(t)\left[1+p(t)\int_a^t g(\tau)\,e_{gp}(t,\sigma(\tau))\,\Delta\tau\right] \tag{2.5}$$

for all $t\in\mathbb{T}^k$ where $e_{gp}(t,\cdot)$ is a solution of IVP (2.3) when f is replaced by gp.

3. Statement of Results

Our main results are given in the following theorems.

Theorem 3.1. *Let $u,n,f\in C_{rd}(\mathbb{T},\mathbb{R}^+)$ and n be a nondecreasing function on $\mathbb{T}$. If*

$$u(t) \le n(t) + \int_a^t f(s)u(s)\,\Delta s \tag{3.1}$$

for all $t\in\mathbb{T}^k$, then

$$u(t) \le n(t)\,e_f(t,a) \tag{3.2}$$

for all $t\in\mathbb{T}^k$, where $e_f(t,a)$ is the solution of the initial value problem (2.2).

Remark 3.2. We note that Theorem 3.1 is a further extension of the inequality first given by Bellman see [6, p. 12]. In the special case if $n(t)$ is a constant say u_0, then the bound obtained in (3.2) reduces to the bound obtained in Corollary 2.10 given by Bohner, Bohner and Akin in [3].

We next establish the following generalization of the inequality given in Corollary 2.10 of [3] which may be useful in certain new applications.

Theorem 3.3. *Let $u, f, p, q \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ and $c \geq 0$ be a constant. If*

$$u(t) \leq c + \int_a^t f(s)\,[p(s)\,u(s) + q(s)]\,\Delta s, \tag{3.3}$$

for all $t \in \mathbb{T}^k$, then

$$u(t) \leq \left[c + \int_a^t f(s) q(s)\,\Delta s\right] e_{pf}(t, a), \tag{3.4}$$

for all $t \in \mathbb{T}^k$, where $e_{pf}(t, a)$ is the solution of IVP (2.3) when $f(t)$ is replaced by pf.

Remark 3.4. By taking $q = 0$ in Theorem 3.3, it is easy to observe that the bound obtained in (3.4) reduces to the bound obtained in Corrollary 2.10 given in [3].

The next theorem deals with the time scale version of the inequality due to Sansone and Conti, see [6, p. 86].

Theorem 3.5. *Let $u, f, p \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ and f be delta-differentiable on $\mathbb{T}$ and $f^{\Delta}(t) \geq 0$. If*

$$u(t) \leq f(t) + \int_a^t p(s)\,u(s)\,\Delta s \tag{3.5}$$

for all $t \in \mathbb{T}^k$, then

$$u(t) \leq f(a)\,e_p(t, a) + \int_a^t f^{\Delta}(s) e_p(t, \sigma(s))\,\Delta s \tag{3.6}$$

for all $t \in \mathbb{T}^k$, where $e_p(t, a)$ is a solution of the IVP (2.3) when f is replaced by p.

The following theorem combines both Gronwall and Bihari's inequalities and can be used in more general situations.

Theorem 3.6. *Let $u, g, f, h \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, $u_0 \geq 0$ is a constant. Let $W(u)$ be a continous, non-decreasing and submultiplicative function defined on $\mathbb{R}^+$ and $W(u) > 0$ for $u > 0$. If*

$$u(t) \leq u_0 + g(t) \int_a^t f(s)\,u(s)\,\Delta s + \int_a^t h(s)\,W(u(s))\,\Delta s, \tag{3.7}$$

for all $t \in \mathbb{T}^k$, then

$$u(t) \leq a(t)\,G^{-1}\left[G(u_0) + \int_a^t h(s)\,W(a(s))\,\Delta s\right], \tag{3.8}$$

for $t \in \mathbb{T}^k$, where

$$a(t) = 1 + g(t) \int_a^t f(s) e_{fg}(t, \sigma(s))\,\Delta s, \tag{3.9}$$

for $t \in \mathbb{T}^k$ and G is a solution of

$$G^{\Delta}(u(t)) = \frac{u^{\Delta}(t)}{W(u(t))}, \tag{3.10}$$

G^{-1} is the inverse function of G and $G(u_0) + \int_a^t h(s)\,W(a(s))\Delta s$ is in the domain of G^{-1} for $t \in \mathbb{T}^k$.

The following theorem deals with a time scale version of the inequality recently established by Pachpatte in [8].

Theorem 3.7. *Let $u, f \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ and $h(t,s) : \mathbb{T} \times \mathbb{T} \to \mathbb{R}^+$ for $0 \le s \le t < \infty$ and $c \ge 0$, $p > 1$ are real constants. Let $g(u)$ be a continuous nondecreasing function of $\mathbb{R}^+$ and $g(u) > 0$ for $u > 0$. If*

$$u^p(t) \le c + \int_a^t \left[f(s) g(u(s)) + \int_a^s h(s,\tau) g(u(\tau)) \Delta\tau \right] \Delta s, \tag{3.11}$$

for $t \in \mathbb{T}^k$, then

$$u(t) \le \left[G^{-1} \left[G(c) + A(t) \right] \right]^{\frac{1}{p}}, \tag{3.12}$$

where

$$A(t) = \int_a^t \left[f(s) + \int_a^s h(s,\tau) \Delta\tau \right] \Delta s, \tag{3.13}$$

for $t \in \mathbb{T}^k$, G is a solution of

$$G^{\Delta}(u(t)) = \frac{u^{\Delta}(t)}{g(u(t))^{\frac{1}{p}}}, \tag{3.14}$$

and G^{-1} is the inverse function on G with $G(c) + A(t)$ in the domain of G^{-1} for $t \in \mathbb{T}^k$.

4. PROOFS OF THEOREMS 3.1 – 3.5

Let $\epsilon > 0$ be a small constant. From (3.1) we observe that

$$u(t) \le (n(t) + \epsilon) + \int_a^t f(s) u(s) \Delta s. \tag{4.1}$$

Define a function $z(t)$ by

$$z(t) = \frac{u(t)}{n(t) + \epsilon}.$$

From (4.1) we have

$$\begin{aligned} z(t) &\le 1 + \int_a^t \left(f(s) \frac{u(s)}{n(t) + \epsilon} \right) \Delta s \\ &\le 1 + \int_a^t f(s) \frac{1}{n(s) + \epsilon} u(s) \Delta s \end{aligned}$$

i.e

$$z(t) \le 1 + \int_a^t f(s) z(s) \Delta s. \tag{4.2}$$

Define $m(t) = 1 + \int_a^t f(s) z(s) \Delta s$, then $m(a) = 1$, $z(t) \le m(t)$ and

$$\begin{aligned} m^{\Delta}(t) &= f(t) z(t) \\ &\le f(t) m(t). \end{aligned} \tag{4.3}$$

Now a suitable application of Lemma 2.1 to (4.3) yields

$$m(t) \le e_f(t,a). \tag{4.4}$$

Using the fact that $z(t) \le m(t)$ we get

$$\frac{u(t)}{n(t) + \epsilon} \le e_f(t,a),$$

$$\text{i.e} \qquad u(t) \le (n(t) + \epsilon) e_f(t,a). \tag{4.5}$$

Letting $\epsilon \to 0$ in (4.5), we get the required inequality in (3.2).

In order to prove Theorem 3.3, we rewrite (3.3) as

$$u(t) \le \left[c + \int_a^t f(s)\,q(s)\,\Delta s\right] + \int_a^t f(s)p(s)\,u(s)\,\Delta s. \tag{4.6}$$

Define $n(t) = c + \int_a^t f(s)q(s)\,\Delta s$, then (4.6) can be restated as

$$u(t) \le n(t) + \int_a^t f(s)p(s)\,u(s)\,\Delta s. \tag{4.7}$$

Clearly $n \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, $n(t)$ is nonnegative and nondecreasing . Now an application of Theorem 3.1 yields the required inequality in (3.4). This completes the proof of Theorem 3.3.

In order to prove Theorem 3.5, define a function $z(t)$ by

$$z(t) = f(t) + \int_a^t p(s)u(s)\,\Delta s, \tag{4.8}$$

then $z(a) = f(a)$, $u(t) \le z(t)$ for $t \in \mathbb{T}^k$ and

$$z^{\Delta}(t) = f^{\Delta}(t) + p(t)\,u(t) \tag{4.9}$$

$$\le f^{\Delta}(t) + p(t)\,z(t). \tag{4.10}$$

Now a suitable application of Lemma 2.1 to (4.8) yields

$$z(t) \le z(a)\,e_p(t,a) + \int_a^t e_p(t,\sigma(s))f^{\Delta}(s)\,\Delta s \tag{4.11}$$

for $t \in \mathbb{T}^k$. Using (4.11) in $u(t) \le z(t)$ we get the desired inequality in (3.6).

5. PROOFS OF THEOREMS 3.6 AND 3.7

To prove Theorem 3.6, we define

$$n(t) = u_0 + \int_a^t h(s)W(u(s))\,\Delta s. \tag{5.1}$$

Then (3.7) can be restated as

$$u(t) \le n(t) + g(t)\int_a^t f(s)u(s)\,\Delta s. \tag{5.2}$$

Clearly $n(t)$ is a nondecreasing function on $\mathbb{T}$. Applying Lemma 2.2 to (5.2) we have

$$u(t) \le a(t)\,n(t), \tag{5.3}$$

for $t \in \mathbb{T}^k$, where $a(t)$ is given by (3.9). From (5.1), (5.3) and using the assumptions on W, we have

$$\begin{aligned} n^{\Delta}(t) &= h(t)\,W(u(t)) \\ &\le h(t)\,W(a(t)\,n(t)) \\ &\le h(t)\,W(a(t)\,W(n(t))). \end{aligned} \tag{5.4}$$

From (3.10) and (5.4) we have

$$G^{\Delta}(n(t)) = \frac{n^{\Delta}(t)}{W(n(t))} \le h(t)\,W(a(t)). \tag{5.5}$$

Integrating (5.5) from a to $t \in \mathbb{T}^k$ we obtain

$$G(n(t)) - G(u_0) \leq \int_a^t h(t) W(a(t))\,\Delta s, \tag{5.6}$$

from (5.6) we observe that

$$n(t) \leq G^{-1}\left[G(u_0) + \int_a^t h(t) W(a(t))\,\Delta s\right]. \tag{5.7}$$

Using (5.7) in (5.3) we get the desired inequality in (3.8).

In order to prove Theorem 3.7, we first assume that $c > 0$ and define a function $z(t)$ by the right side of (3.11) . Then $z(t) > 0$, $z(a) = c$, $u(t) \leq (z(t))^{\frac{1}{p}}$ and

$$\begin{aligned} z^{\Delta}(t) &= f(t)\, g(u(t)) + \int_a^t h(t,\tau) g(u(\tau))\,\Delta\tau \\ &\leq f(t)\, g\left((z(t))^{\frac{1}{p}}\right) + \int_a^t h(t,\tau)\, g\left((z(t))^{\frac{1}{p}}\right)\Delta\tau \\ &\leq g\left((z(t))^{\frac{1}{p}}\right)\left[f(t) + \int_a^t h(t,\tau)\,\Delta\tau\right]. \end{aligned} \tag{5.8}$$

From (3.14) and (5.8) we have

$$\begin{aligned} G^{\Delta}(z(t)) &= \frac{z^{\Delta}(t)}{g\left((z(t))^{\frac{1}{p}}\right)} \\ &\leq \left[f(t) + \int_a^t h(t,\tau)\,\Delta\tau\right]. \end{aligned} \tag{5.9}$$

Integrating (5.9) from a to $t \in \mathbb{T}^k$ we have

$$G(z(t)) \leq G(c) + A(t). \tag{5.10}$$

From (5.10) we get

$$z(t) \leq G^{-1}\left[G(c) + A(t)\right]. \tag{5.11}$$

Using (5.11) in $u(t) \leq \left((z(t))^{\frac{1}{p}}\right)$ we have the desired inequality in (3.12). If c is nonnegative we carry out the above procedure with $c + \epsilon$ instead of c, where $\epsilon > 0$ is an arbitrary small constant and by letting $\epsilon \to 0$ we obtain (3.12).

6. APPLICATIONS

In this section we present some applications of Theorems 3.5 and 3.7 to obtain the explicit estimates on the solutions of certain dynamic equations.

First we consider the following intial value problem

$$x^{\Delta\Delta}(t) = f(t, x(t)), \qquad x(a) = A, \qquad x^{\Delta}(a) = B, \tag{6.1}$$

where $f \in C_{rd}(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ and A, B are given constants.

The following result gives the bound on the solution of IVP (6.1).

Theorem 6.1. *Suppose that the function f satisfies*

$$|(t-s) f(s, x(s))| \leq p(s)\,|x(s)|, \tag{6.2}$$

where $p \in C_{rd}\left(\mathbb{T}^k, \mathbb{R}^+\right)$, and assume that

$$|A + B(t-a)| \leq m(t), \tag{6.3}$$

$m \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, *$m$ is delta differentiable on* $\mathbb{T}^k$ *and* $m^\Delta(t) \geq 0$. *Then*

$$|x(t)| \leq m(a)\, e_p(t,a) + \int_a^t m^\Delta(s) e_p(t, \sigma(s))\, \Delta s, \tag{6.4}$$

for $t \in \mathbb{T}^k$, *where* $e_p(t,a)$ *is as in Theorem 3.5.*

Proof. Let $x(t)$ be a solution of the IVP (6.1). Then it is easy to see that $x(t)$ satisfies the equivalent integral equation

$$x(t) = A + B(t-a) + \int_a^t (t-s) f(s, x(s))\, \Delta s. \tag{6.5}$$

From (6.5) and using (6.2), (6.3), we have

$$\begin{aligned} |x(t)| &\leq |A + B(t-a)| + \int_a^t |(t-s)\, f(s, a(s))| \Delta s \\ &\leq m(t) + \int_a^t g(t)\, p(s)\, |x(s)| \Delta s. \end{aligned} \tag{6.6}$$

Now applying Theorem 3.5 to (6.6) we get

$$|x(t)| \leq m(a)\, e_p(t,a) + \int_a^t m^\Delta(s) e_p(t, \sigma(s))\, \Delta s.$$

This is the required estimate in (6.4). ☐

Next we consider the following intial value problem

$$\left(r(t)\, x^p(t)\right)^\Delta = f(t, x(t)), \qquad x(a) = c, \tag{6.7}$$

where $r(t) > 0$ is rd-continous for $t \in \mathbb{T}^k$, $f \in C_{rd}(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ and $c, p > 1$ are constants.
As an application of the special version of Theorem 3.7 we have the following.

Theorem 6.2. *Suppose that the function f satisfies*

$$|f(t, x(t))| \leq q(t)\, g(|x(t)|), \tag{6.8}$$

where $q \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$ *and g is as in Theorem 3.7 and assume that*

$$\left|\frac{1}{r(t)}\right| \leq d, \tag{6.9}$$

where $d \geq 0$ *is a constant. Then*

$$|x(t)| \leq \left[G^{-1}\left[G(|r(a)\, c|\, d) + \int_a^t q(s)\, \Delta s\right]\right]^{\frac{1}{p}}, \tag{6.10}$$

where G, G^{-1} *are as in Theorem 3.7.*

Proof. Let $x(t)$ be a solution of IVP (6.7). It is easy to see that $x(t)$ satisfies the equivalent integral equation

$$x^p(t) = \frac{r(a)}{r(t)} c + \frac{1}{r(t)} \int_a^t f(s, x(s)) \Delta s. \tag{6.11}$$

From (6.11) and using (6.8), (6.9) we get

$$|x(t)|^p \leq |r(a)\, c|\, d + \int_a^t dq(s) g(|x(s)|)\, \Delta s. \tag{6.12}$$

Now by applying Theorem 3.7 when $h = 0$ to (6.12) we get the required estimates in (6.10) ☐

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