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ON A CONJECTURE OF QI-TYPE INTEGRAL INEQUALITIES

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Abstract

A conjecture by Chen and Kimball on Qi-type integral inequalities is proven to be true.

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Recently, Chen and Kimball [1], studied a very interesting Qi-type integral inequality and proved the following result.

Theorem 1. *Let n belong to $\mathbb{Z}^+$. Suppose $f(x)$ has a derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) = 0$ for $i = 0,1,2,\dots,n-1$. If $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ and $f^{(n)}(x)$ is increasing, then*

$$\int_a^b [f(x)]^{n+2}dx \geq \left[\int_a^b f(x)dx\right]^{n+1}. \tag{1}$$

If $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ and $f^{(n)}(x)$ is decreasing, then the inequality (1) is reversed.

In this theorem and in the sequel, $f^{(0)}(a)$ stands for $f(a)$.

In [1], Chen and Kimball conjectured that the additional hypothesis on monotonicity in Theorem 1 could be dropped:

Theorem 2 (Conjecture). *Let n belong to $\mathbb{Z}^+$. Suppose $f(x)$ has derivative of the n-th order on the interval $[a,b]$ such that $f^{(i)}(a) = 0$ for $i = 0,1,2,\dots,n-1$.*

(i) If $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$, then the inequality (1) holds.

(ii) If $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$, then the inequality (1) is reversed.

In this article, we prove by mathematical induction that the conjecture is true. As a matter of fact, Theorem 2 holds under slightly weaker assumptions (existence of $f^{(n)}(x)$ at the endpoints $x = a, x = b$ is not needed). We start by applying Cauchy's mean value theorem (CMVT) (that is, the statement that for f, g differentiable on (a,b) and continuous on $[a,b]$ there exists a $\xi \in (a,b)$ such that

$$f'(\xi)(g(b) - g(a)) = g'(\xi)(f(b) - f(a)))$$

to prove the following lemma, which will in turn be used to prove Theorem 2.

Lemma 3. *Let n belong to $\mathbb{Z}^+$. Suppose $f(x)$ has a derivative of the n-th order on the interval (a,b) and $f^{(n-1)}(x)$ is continuous on $[a,b]$ such that $f^{(i)}(a) = 0$ for $i = 0,1,2,\dots,n-1$.*

(i) If $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$, then

$$(f(x))^{n+1} \geq (n+1)\left(\int_a^x f(s)ds\right)^n \quad \text{for} \quad x \in [a,b].$$

(ii) If $0 \le f^{(n)}(x) \le \frac{n!}{(n+1)^{(n-1)}}$ *for* $x \in (a,b)$*, then*

$$(f(x))^{n+1} \le (n+1)\left(\int_a^x f(s)ds\right)^n \quad \textit{for} \quad x \in [a,b].$$

Proof. First notice that if f is identically 0, then the statement is trivially true. Suppose that f is not identically 0 on $[a,b]$. Then the assumption implies that $f(x) \ge 0$ for $x \in [a,b]$. If $\int_a^x f(s)ds = 0$ for some $x \in (a,b]$ then $f(s) = 0$ for all $s \in [a,x]$. So we can assume that $\int_a^x f(s)ds > 0$ for all $x \in (a,b]$. Otherwise, we can find $a_1 \in (a,b)$ such that $\int_a^x f(s)ds = 0$ for $x \in [a,a_1]$ and $\int_a^x f(s)ds > 0$ for $x \in (a_1,b)$ and hence we only need to consider f on $[a_1,b]$.
(i) Suppose that $f^{(n)}(x) \ge \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$.

1. $n = 1$. By CMVT, for every $x \in (a,b]$, there exists a $b_1 \in (a,x)$ such that

$$\frac{(f(x))^2}{2\int_a^x f(s)ds} = \frac{2f(b_1)f'(b_1)}{2f(b_1)} = f'(b_1) \ge 1.$$

   So (i) is true for $n = 1$.

2. Suppose that (i) is true for $n = k > 1$. We prove that (i) is true for $n = k+1$. It then follows by mathematical induction that (i) is true for $n = 1, 2, \ldots$.

   By CMVT, for every $x \in (a,b]$, there exists a $b_1 \in (a,x)$ such that

$$\frac{(f(x))^{k+2}}{(k+2)\left(\int_a^x f(s)ds\right)^{k+1}} = \frac{1}{(k+2)}\left(\frac{(f(x))^{\frac{k+2}{k+1}}}{\int_a^x f(s)ds}\right)^{k+1}$$

$$= \frac{1}{(k+2)} \left(\frac{\frac{k+2}{k+1} (f(b_1))^{\frac{1}{k+1}} f'(b_1)}{f(b_1)} \right)^{k+1}$$

$$= \frac{(k+2)^k}{(k+1)^{k+1}} \frac{(f'(b_1))^{k+1}}{(f(b_1))^k}$$

$$= \frac{\left(\left(\frac{k+2}{k+1}\right)^k f'(b_1) \right)^{k+1}}{(k+1) \left(\int_a^{b_1} \left(\frac{k+2}{k+1}\right)^k f'(s) ds \right)^k} \geq 1.$$

Since

$$\frac{d^k}{dx^k} \left[\left(\frac{k+2}{k+1} \right)^k f'(x) \right] = \left(\frac{k+2}{k+1} \right)^k f^{(k+1)}(x)$$

$$\geq \left(\frac{k+2}{k+1} \right)^k \frac{(k+1)!}{(k+2)^k} = \frac{k!}{(k+1)^{k-1}}$$

for $x \in (a, b)$, by the induction assumption that (i) is true for $n = k$. So (i) is true for $n = 1, 2, \ldots$.

(ii) The proof of the second part is similar so we leave out the details. This completes the proof of the lemma. □

Now we are in a position to prove the conjecture (Theorem 2).

Proof of Conjecture (Theorem 2). As in the proof of Lemma 3, we can assume that $\int_a^x f(s) ds > 0$ for any $x \in (a, b]$.

(i) Suppose that $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$. By CMVT and Lemma 3, in case (i), there exists a $b_1 \in (a,x)$ such that

$$\frac{\int_a^b [f(x)]^{n+2}dx}{\left[\int_a^b f(x)dx\right]^{n+1}} = \frac{[f(b_1)]^{n+1}}{(n+1)\left[\int_a^{b_1} f(x)dx\right]^n} \geq 1.$$

This proves (i).

(ii) Suppose that $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$.

By CMVT and Lemma 3, in case (ii), there exists a $b_1 \in (a,x)$ such that

$$\frac{\int_a^b [f(x)]^{n+2}dx}{\left[\int_a^b f(x)dx\right]^{n+1}} = \frac{[f(b_1)]^{n+1}}{(n+1)\left[\int_a^{b_1} f(x)dx\right]^n} \leq 1.$$

This completes the proof of the conjecture. □

As the proofs show, we actually have the following slightly stronger result which is a generalization of Proposition 1.1 in [2] and Theorem 4 and Theorem 5 in [1].

Theorem 4. *Let n belong to $\mathbb{Z}^+$. Suppose $f(x)$ has derivative of the n-th order on the interval (a,b) and $f^{(n-1)}(x)$ is continuous on $[a,b]$ such that $f^{(i)}(a) = 0$ for $i = 0,1,2,\ldots,n-1$.*

(i) If $f^{(n)}(x) \geq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$, then the inequality (1) holds.

(ii) If $0 \leq f^{(n)}(x) \leq \frac{n!}{(n+1)^{(n-1)}}$ for $x \in (a,b)$, then the inequality (1) is reversed.

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